SOLUTIONS

OF THE

CAMBRIDGE PROBLEMS,

FROM 1800 TO 1820.

BY I. M. F. WRIGHT, B. A.,

LATE SCHOLAR OF TRINITY COLLEGE, CAMBRIDGE

IN TWO VOLUMES.

VOL. II.

LONDON:

PRINTED FOR BLACK, YOUNG AND YOUNG,

TAVISTOCK-STREET, COVENT-GARDEN.

MDCCCXXV.

LONDON:
Printed by WILLIAM CLOWES,
Northumberland-court.

INDEX

REFERRING

TO THE PROBLEMS.

N. B. The column marked N gives the number of the solution. Those which are headed by p and n, give the page and number in the page of the volume containing the enunciations. Consequently the three columns give the enunciation of any problem corresponding to its solution.

CURVES.

MECHANICS.

HYDROSTATICS.

HYDRODYNAMICS.

PNEUMATICS.

OPTICS.

ASTRONOMY.

MISCELLANIES.

SOLUTIONS.

THE THEORY OF CURVES.

THE NATURE AND CONSTRUCTION OF CURVES.

1. To describe the curve whose equation is $x^4 - a^2x^2 + a^2y^2 = 0$, we have, by differentiating

$$(2x^3 - a^2x)\,dx + a^2y\,dy = 0 \;....\; (a)$$

And $(6x^2 - a^2)dx^2 + a^2dy^2 + a^2yd^2y = 0 ....(b)$

First let us determine the limits or *maxima* and *minima* of y;

$$\frac{dy}{dx} = \frac{a^2x - 2x^3}{a^2y} = 0$$

gives $a^2x - 2x^3 = 0$

or $x = 0$, and $x = \pm \dfrac{a}{\sqrt{2}}$.

whence by substituting in the given equation we get,

$y = 0$ for a minimum value at the origin A, (See Fig. 1).

and $y = \pm \dfrac{a}{2}$ indicates two pairs of maximum values PM, PM′; pm, pm' corresponding to the two values of x.

To find the limits of x we have

$$\frac{dx}{dy} = \frac{a^2y}{a^2x - 2x^3} = 0$$

$\therefore a^2y = 0$, or $y = 0$

And substituting in the given equation there will result
$x = \pm a$, which indicates the points B, b.

Again, to find the multiple points (if any) we have

$$\frac{dy}{dx} = \frac{ax - 2x^3 = 0}{a^2 y = 0}$$, and it being found, by equation (b),

that $x = 0$, $y = 0$, are the only values deduced from the numerator and denominator, which satisfy the given equation, we have a *double point* at A.

Also, since when x and y each $= 0$,

$\cdot\ \frac{dy}{dx} = \pm \sqrt{\frac{a^2}{a^2}} = \pm 1 = \pm \tan. \frac{\pi}{4}$ the tangents at the point A are inclined to Bb at an angle of 45 degrees.

Again, to find the points of inflexion, we have

$$\frac{d^2y}{dx^2} = \frac{a^2 - 6x^2 - a^2 \frac{dy^2}{dx^2}}{a^2 y} = 0 \text{ or } = \frac{1}{0}$$

$$\therefore a^2 y = 0, \text{ or } y = 0$$

$$\text{And } a^2 - 6x^2 - a^2 \frac{dy^2}{dx^2} = 0, \text{ and } \therefore x = 0,$$

$\therefore$ we have a *point of inflexion* at the origin A.

Since there are no infinite branches, the curve cannot have an asymptote.

This curve which has been called the *lemniscata*, is the *locus of the extremity of that part of the ordinate of a circle whose radius is* (a), *which is constantly taken* $=$ *a sin.* θ. *cos.* θ, θ *being the angle at the centre subtended by that ordinate.*

For from the given equation

$$y = \pm \frac{x}{a} \sqrt{a^2 - x^2}.$$ See Fig. 2.

To find the whole area

$$\int y dx = \pm \int \frac{x dx}{a} \sqrt{a^2 - x^2} = C \mp \frac{1}{3a} (a^2 - x^2)^{\frac{3}{2}}$$

$$= \pm \left\{ \frac{a^2}{3} - \frac{1}{3a} (a^2 - x^2)^{\frac{3}{2}} \right\}.$$

Let $x = \pm a$

Then the area ABM or Abm $= \frac{a^2}{3}$.

And AM′B and A$m'b$ are equal to either of these.

$\therefore$ the whole area A M B M′ Ambm'A $= \frac{4a^2}{3}$.

N. B. Unless the branches of an oval intersect, it is not perfectly quadrable. See *Waring*, *Vince*, &c.

(2) Let E F (Fig. 3) $= y$, AE $= x$

AD, the diameter of the generating ⊙ A H D, $= a$. Then AH being joined, by the property of the cycloid, we have

$y = 2\,\text{AH} = 2\sqrt{\text{AE. AD}}$, by similar Δ,

$= 2\sqrt{ax}$.

$\therefore y^2 = 4ax$, the equation to the curve, which is therefore a *parabola*, whose *vertex* is A, and *focus* D.

Now the area of a cycloid is known to be triple that of its generating ⊙, and the area of a parabola $\frac{2}{3}$ of its circumscribing rectangle; hence

AFGD : ABCD :: $\frac{2}{3}$AD $\times$ DG : 3A H D

But if $\pi =$ the circumference of a circle whose diameter is 1, $\frac{\pi}{2} \times \frac{1}{2} = \frac{\pi}{4} =$ its area.

$\therefore$ AHD : $\frac{\pi}{4} \times \frac{1}{2}$ :: AD^2 : 1

And DG $=$ 2AD

Hence AFGD : ABCD :: $\frac{4}{3}$AD^2 : $\frac{3\pi}{8}$AD^2

:: 32 : 9π.

(3) Let CB or CD or CR (Fig. 4) $= r$, C$b = a$, PM $= y$, CP $= x$.

Then since M bisects Rr in M, we have

$$y = \text{PM} = \frac{\text{RN} + rn}{2} = \frac{\text{RN}}{2} + \frac{a}{2}$$

But RN : y :: r : CM $(= \sqrt{x^2+y^2})$

$$\therefore \text{RN} = \frac{ry}{\sqrt{x^2+y^2}}.$$

Hence $y = \dfrac{ry}{2\sqrt{x^2+y^2}} + \dfrac{a}{2}$ and by involution, &c.

$x = \dfrac{y}{2y-a}\sqrt{r^2-a^2-4y^2+4ay}$ which is the equation to the curve referred to *rectangular* co-ordinates.

To find the equation referred to *polar co-ordinates,* we have (putting the $\angle$MCP $= \theta$ and CM $= \varrho$)

$$\text{PM} = \varrho \sin. \theta = \frac{\text{RN} + rn}{2} = \frac{r. \sin. \theta}{2} + \frac{a}{2}$$

$\therefore \varrho = \dfrac{r}{2} + \dfrac{a}{2 \sin. \theta}$ ..... (*b*) which is more simple than the former.

To find the asymptotes, let

$x = \infty$

Then $2y - a = 0$, or $y = \dfrac{a}{2}$

$\therefore$ if C*b* be bisected in *v*, and *vt* be drawn parallel to AD, it will be an asymptote to the branch xM.

Let $y = a$

Then $x = \sqrt{r^2 - a^2} =$ F*b*, or the curve passes through F, the intersection of *br* with the circle.

Similar properties belong to each of the branches generated by the other quadrants.

(4) To construct $a^2y - x^2y - a^3 = 0$

or $y = \dfrac{a^3}{a^2-x^2}$

Let the origin of abscissæ be at A, (Fig. 5), x being then $= 0$, and $\therefore y = a =$ AC.

Again, let $x = \pm a =$ AB or A*b*.

Then $y = \infty =$ BN or BN, which are $\therefore$ asymptotes to the branches CR, C*r* respectively.

When $\pm x$ is $> a$, the values of y are negative, and the corresponding branches are S*s*, T*t*; which, since when $\pm x$ is infinite,

$y = -\infty$, are placed between the asymptotes BD, BM and bd, bm.

5. Since the subtangent of the curve $= n$ times the abscissa, and the subtangent $= \frac{ydx}{dy}$ (x and y being the co-ordinates), we have

$$nx = \frac{ydx}{dy}$$

$$\therefore \frac{ndy}{y} = \frac{dx}{x}$$

$$\text{And } ly^n = lx + \text{const.}$$
$$= lx + lc$$
$$= l.\,cx$$
$$\therefore y^n = cx$$

The curve is, therefore, *parabolic*.

The locus of the vertex of a $\triangle$ of given base, when the sum of its angles at the base are also given, is evidently a circular arc described upon the base, so as to contain an angle $=$ the supplement of the given sum of the $\angle$ at the base.

For since the sum of the $\angle$ of a $\triangle = 2$ right $\angle$, the angle at the vertex $=$ the supplement of the $\angle$ at the base, and is $\therefore$ given.

6. Again, let BC, the given base of the $\triangle$ (Fig. 6) $= a$, and let P be any point in the required locus. Make C the origin of abscissæ, or PM $= y$ and CM $= x$. Also put the angle PBC $= \theta$.

Then by the question PCB $= 2\theta$, and we have

$$y = x.\tan.2\theta = (a - x)\tan.\theta$$

$$\therefore \frac{2x.\tan.\theta}{1-\tan.^2\theta} = (a - x).\tan.\theta, \text{ whence}$$

$$\tan.\theta = \sqrt{\frac{a - 3x}{a-x}}$$

$$\text{And } \therefore y = (a - x)\tan.\theta = \sqrt{(a-x).(a-3x)}$$
$$= \sqrt{a^2 - 4ax + 3x^2}\text{ which indicates a conic section.}$$

Let $y = 0$

Then $x = a$ or $\frac{a}{3}$. Take $\therefore$ CA $= \frac{a}{3}$, and making A the origin of co-ordinates, by putting $x' = \frac{a}{3} - x$, we have

$$y = \sqrt{ax' + 3x'^2} = \frac{\sqrt{3}}{1}. \sqrt{2.\frac{a}{3}x' + x'^2}$$

$$= \frac{\frac{a}{\sqrt{3}}}{\frac{a}{3}} \sqrt{2.\frac{a}{3}x' + x'^2}$$ which being of the form

$y = \frac{b}{a}\sqrt{2ax' + x'^2}$, demonstrates that the locus is an *hyperbola* whose semi-axes are $\frac{a}{3}$ and $\frac{a}{\sqrt{3}}$.

The branch Q A q corresponds to the positive values of x'. From $x = 0$ to $-\frac{2a}{3} =$ AB we have y imaginary; but when x' is taken negatively $> \frac{2a}{3}$, y again becomes real, and describes the branch Q$'$ Bq'.

(7). Let the given area be denoted by a, the given $\angle$ at A (Fig. 7) by A, and the given ratio of BF : FC by $n : 1$.

Draw FG, Fg parallel to AB, AC respectively, and CN, FM $\perp$ AB.

Then GF : AB :: FC : BF :: $1 : n + 1$

$$\therefore \text{ GF} = \frac{\text{AB}}{n+1}.$$

Also CN : FM :: CB : FB :: $n + 1 : n$

$$\therefore \text{ FM} = \frac{n\text{CN}}{n+1} = \text{also F}g.\sin.\text{ A}$$

$$\text{Hence GF} \times \text{F}g = \frac{\text{AB}}{n+1} \times \frac{n\text{CN}}{(n+1)\sin.\text{ A}} = \frac{2an}{(n+1)^2\sin.\text{A}}.$$

Or making GF $=$ A$g = x$

and gF $= y$, we have

$xy = \dfrac{2an}{(n+1)^2 \sin.A} = m^2$ a constant quantity, which is the equation to an *hyperbola whose asymptotes* are AD, AE.

To find its principal diameters, bisect the $\angle$A by AV, make $x = y$, or $y^2 = m^2$, and $\therefore y = m$; and we have $Ah = hV = VH = HA$. Hence $AV = \dfrac{\sin.\ AhV}{\sin.\ AVh} \times Ah = \dfrac{\sin.\ A}{\sin.\ \frac{A}{2}}.m = 2\cos.\dfrac{A}{2}$

$$\times \frac{\sqrt{2an}}{(n+1)\sqrt{\sin A}} = \frac{2\sqrt{\cos.\frac{A}{2}} \times \sqrt{an}}{(n+1).\sqrt{\sin.\frac{A}{2}}} = \frac{2}{n+1}\sqrt{an \cot.\frac{A}{2}},$$

which is the *semi-axis major*. The *semi-axis minor* $\therefore$ (= VK)

$$= AV.\ \tan.\frac{A}{2} = \frac{2}{n+1}\sqrt{an \tan.^2\frac{A}{2}.\cot.\frac{A}{2}} = \frac{2}{n+1}\times$$

$$\sqrt{an\ \tan.\frac{A}{2}}.$$

8. Let PM $= y$ (see Fig. p. 84 in the Problem) AM $= x$, and AB $= a$.

Then $y^2 = AZ^2 = x^2 + ZM^2 = x^2 + x.(a-x)$
$= ax$, the equation to a *parabola* whose Latus-Rectum $= a$ the diameter of the circle.

Again, the area $= \int y dx = \int \dfrac{2y^2 dy}{a} = \dfrac{2}{3}.\dfrac{y^3}{a} + C = \dfrac{2}{3}xy.$

Let $x = a$. Then $y = a$, and the area required $= \dfrac{2}{3}a^2$.

9. Let ABCD (Fig. 8) be any position of the generating square, QM the axis of the solid and PQp the circular section $\perp$ to AD and BC. Let also P′Qp' be a section inclined to AD, BC at any given $\angle$ (α), and passing through the axis QM; and put QM $= x$ P′M $= y$, and the radius of PQp $= a$.

$$\text{Then } y = \frac{PM}{\sin.\alpha} = \frac{\sqrt{2ax - x^2}}{\sin.\alpha} = \frac{\left(\frac{a}{\sin.\alpha}\right)\sqrt{2ax - x^2}}{a}$$

which is the equation to an *ellipse* whose axes are $\frac{a}{\sin.\alpha}$ and a; the origin of abscissæ being at the extremity of the axis-minor, because $\frac{a}{\sin.\alpha}$ is necessarily $> a$.

10. To trace the curve whose equation is $a^2 - x^2 + (x - b)^2 = x^2 y^2$, we first reduce it to $a^2 + b^2 - 2bx = x^2 y^2$; then we have

$$y = \pm \frac{\sqrt{a^2 + b^2 - 2bx}}{x}$$

and putting $x = 0$, we get $y = \pm \infty$ which are represented by AB, Ab (fig. 9). Again, making $x = \frac{a^2 + b^2}{2b} = AC$, $y = \pm 0$, and we have a *cusp of the first species* at C. No positive value of x can exceed $\frac{a^2 + b^2}{2b}$.

Let $x = -\infty$. Then $y = \pm 0$, and Ac is an asymptote to the branches FE, fe. Also Bb is an asymptote to each of the branches.

11. To prove that every section of a conoid is a conic section.

We will here exhibit a method of ascertaining the nature of the section made by a plane with any *solid of revolution* whatever.

Let DAd (fig. 10) be the solid formed by the revolution of DAQ about AQ as an axis, and let BPbp be a circle described by any point B. The plane of this circle will evidently be $\perp$ plane DAd, and their intersection Bb will be $\perp$ axis AQ.

Again, let CPcp be the section made by a plane, passing any how through the solid, and intersecting the circle by the line Pp; which latter also cuts Bb in M. Join CM. Then since the points C, M are both in the planes CAd, CPc, CM being produced will meet the curve in c, and cut the axis AQ in R.

Now if the circle BPb move parallel to itself, the intersection

Pp will also move parallel to itself, and we, therefore, put the constant angle BMP $= \alpha$.

Also let $\angle$RMN $= \beta$, cR $= b$, AR $= a$, cM $= x$, and PM $= y$.

By the property of the circle, we have

$$Pm^2 = BN^2 - Nm^2 = BN^2 - (NM \pm Mm)^2$$
$$= BN^2 - (RM \,.\, \cos.\beta + y \cos.\alpha)^2$$
$$= BN^2 - \{(x-b)\cos.\beta + y\cos.\alpha\}^2.$$

But $Pm^2 = y^2 \sin.^2 \alpha$, and supposing $y'^2 = f(x')$ to be equation of the generating curve, we have

$BN^2 = f.\,(AN) = f(a + RN) = f.\,(a + RM.\sin.\beta) = f.\,\{a + (x-b)\sin.\beta\}$.

$\therefore$ by substitution

$y^2 \sin.^2 \alpha = f.\,(a + \overline{x-b}\,.\sin.\beta) - (\overline{x-b}\,.\cos.\beta + y\cos.\alpha)^2$;

whence by reduction we get

$y^2 - 2\cos.\alpha\,.\cos.\beta\,.\,(x-b)\,y = f.\,(a + \overline{x-b}\,.\sin.\beta) - (x-b)^2\cos.^2\beta$, which is the general equation of the section of a solid of revolution.

Now, since in the conic sections $y'^2 = f.\,(x')$ is always of two dimensions, the equation of the section of a conoid is likewise of two dimensions, and $\therefore$ the section itself is a conic section Q.E.D.

Again, let $y'^2 \quad px'$, or the *conoid* be a *paraboloid*. Also let vcw, the cutting plane, be parallel to the axis AQ or $\beta = 90°$. Then $\alpha = 90°$, $a = \infty$, and $b = \infty$, and PM becomes $\perp$ Cc.

Hence substituting in the general equation, we have

$$y^2 = px.$$

The section is $\therefore$ a parabola, similar to, because of equal parameter with, the generating one Q E.D.

If in other applications it should be necessary to ascertain the angle at which PM is inclined to Cc, in order to transform the co-ordinates of the section to rectangular or otherwise, the following process will serve that purpose.

Take MA$'$ $=$ MB$'$ $=$ MC$'$ $= 1$, and with M, as centre, describe the arcs A$'$B$'$, B$'$C$'$, C$'$A$'$ forming a spherical $\triangle$ A$'$B$'$C$'$.

Then since the angle A$'$ measures the inclination of the planes CPc, BPb, A$'$B$'$ measures the $\angle\,\alpha$, and B$'$C$'$ measures β, and are

therefore given, we have two sides, and an angle subtended one of them to find the third side A′C′ which will give the inclination required. Thus, by common forms

$$\text{Sin.}^2 \frac{A'}{2} = \frac{\sin. \frac{a' + b' - c'}{2} \cdot \sin. \frac{a' + c' - b'}{2}}{\sin. b' . \sin. c'}$$

a', b', c', being the sides opposite to A′, B′, C′. Hence by proper reductions and the solution of a quadratic, we get

$$\tan. \frac{b'}{2} = \frac{\cos. A' \sin. c' \pm \sqrt{\sin.^2 a' - \sin. c' \sin.^2 A'}}{2 \cos. \frac{a' - c'}{2} . \cos. \frac{a' + c'}{2}}$$

$$\text{or } \tan. \frac{\angle PMC}{2} = \frac{\cos. A' . \sin. \alpha \pm \sqrt{\sin.^2 \beta - \sin.^2 \alpha . \sin.^2 A'}}{2 \cos. \frac{\beta - \alpha}{2} . \cos. \frac{\beta + \alpha}{2}}$$

which, when adapted to logarithmic computation, will give ∠PMC or the angle of co-ordinates.

The following process will apply still more generally. It will give us the co-ordinates of the intersection of a plane, and the surface of any solid whatever.

Let A′X′, A′Y′, A′Z′, (fig. 10 . A.) denote the rectangular co-ordinates of the surface, BC the intersection of the cutting plane with the plane Y′A′X′, and P be any point in the curve required. Draw PN ⊥ plane Y′A′X′, and meeting it in N, and NM ⊥ BC, and join PM. Also draw MN′ ⊥ AX′, and A′A ⊥ BC.

Then x', y', z' ; x, y, z, denoting the co-ordinates of the surface, and of the intersection respectively, let us suppose, for the greater brevity, that x is measured from A along CB, and y in the plane BPC, ⊥ CB. Also put $\theta = \angle$ PMN, which = the inclination of the cutting plane with that of x', y'; a = the given line AE, and β = the given angle REM′.

Now PN is parallel to A′Z′.

$\therefore z' = PN = PM \times \sin. \theta = y . \sin. \theta.$

Also $x' = A'E + EM' = a + ER . \cos. \beta = a + (AM - RM - AE) \cos. \beta = a + x \cos. \beta - NM \times \tan. \beta \times \cos. \beta - a \cos.^2\beta = a . \sin.^2\beta + x \cos. \beta - y . \cos. \theta \sin. \beta.$

In like manner we get

$y' = NM' = y \,.\, \cos.\, \theta \,.\, \cos.\, \beta + x \,.\, \sin.\, \beta - a \,.\, \cos.\, \beta \,.\, \sin.\, \beta.$

The three equations

$$x' = a \,.\, \sin.^2\beta + x \cos.\, \beta - y \cos.\, \theta \,.\, \sin.\, \beta$$
$$y' = -a \,.\, \cos.\, \beta \,.\, \sin.\, \beta + x \sin.\, \beta + y \,.\, \cos.\, \theta \,.\, \cos.\, \beta$$
$$z' = y \,.\, \sin.\, \theta$$

expressing the co-ordinates of the surface in terms of x and y, the given equation of the surface will enable us by elimination to find y in terms of x; *i. e.* the nature of the *section* required.

Ex. Let the given surface be that of a *right cone*, whose angle at the vertex, put $= A$. Also let its co-ordinates x', y', z', be measured from the vertex, and x' coincide with the axis. Then its equation will be

$$z'^2 + y'^2 = x'^2 \tan.^2 \frac{A}{2}$$

in which the above values of x', y', z', being substituted, there results an equation of two dimensions. The *curve of intersection* will, therefore, be an *ellipse*, *circle*, *hyperbola*, or *parabola*, according to the values of the constants. We leave the discussion of these separate cases, and the application of the above equations to other surfaces, to the reader.

12. To trace the curve whose equation is $y = \dfrac{x}{1 + x^2}$ ×

Let $x = 0$, then $y = 0$, or the curve meets the axis in the origin of co-ordinates A, (fig. 11.)

Let $x = \pm \infty$; then $y = 0$, or AB, Ab are asymptotes to the curve.

To find the maximum or minimum values of y, we have

$$\frac{dy}{dx} = \frac{1 - x^2}{(1 + x^2)^2} = 0$$
$$\therefore x = \pm 1,$$

which may be represented by AM and Am, the abscissæ corresponding to the maxima (PM) and (pm) of y, PM being $= \dfrac{1}{1+1} = \dfrac{1}{2}$, and $pm = -\dfrac{1}{2}$.

Again, since when $x = 0$

$\frac{dy}{dx} = 1 = \tan.$ TAM, we have the angle TAM at which the curve cuts the axis $= 45^\circ$.

To find the points of inflexion

$$\frac{d^2y}{dx^2} = \frac{2x^3 - 6x}{(1 + x^2)^3} = 0$$

$\therefore x = 0$ and $x^2 - 3 = 0$,

or $x = 0$, and $x = \pm\sqrt{3}$

and the corresponding values of y are

$$y = 0, \text{ and } y = \frac{\pm\sqrt{3}}{4}$$

Take, therefore, AM′, Am' $= \sqrt{3}, \ -\sqrt{3}$, and P′M′, $p'm' = \frac{\sqrt{3}}{4}, - \frac{\sqrt{3}}{4}$, and the points P′, A, p', will be those of contrary flexure.

The curve p'AP′ is concave to the axis, because $\frac{d^2y}{dx^2} = \frac{2x^3 - 6x}{(1 + x^2)^3}$ is negative between the values of x, 0 and $\sqrt{3}$; and positive, with negative ordinates between 0 and $-\sqrt{3}$. Hence P′D, and $p'd$, are convex to the axis.

13. To find the equation to the curve of which subnormal : (abscissa)^3 :: $2a^2$: 1.

Let $y =$ its ordinate, x its abscissa,

Then $\frac{ydy}{dx} : x^3 :: 2a^2 : 1$

$$\therefore ydy = 2a^2x^3dx$$

$$\text{and } \frac{y^2}{2} = \frac{a^2x^4}{2}$$

$\therefore y = ax^2$, or the curve is the common parabola, the line of abscissæ issuing from the vertex at right angles to the axis, and its parameter being $\frac{1}{a}$.

$$\text{The subtangent} = \frac{ydx}{dy} = \frac{ax^2}{2ax} = \frac{x}{2}.$$

$$\text{The area} = \int ydx = \int ax^2dx = \frac{ax^3}{3} + C$$

$$= \frac{x \times y}{3}.$$

14. To find the equation of the curve of which, $(\text{ordinate})^2 = \sqrt{b \times area}$.

Let the ordinate $= y$, and abscissa $= x$.

$$\begin{aligned} \text{Then } y^2 &= \sqrt{b\,.\int ydx} \\ \therefore\ y^4 &= b\int ydx \\ \text{and } 4y^3dy &= b\,ydx \\ \therefore\ 4y^2dy &= b\,dx, \text{ and integrating} \\ y^3 &= \frac{3b}{4}x + C \end{aligned}$$

which is a particular case of the *cubical parabola.*

Fig. 12, will represent the course of this curve, Dd being the line of abscissæ, and A their origin.

Since $\frac{d^2y}{dx^2} = \frac{-b^2}{8.\left(\frac{3b}{4}x + c\right)^{\frac{5}{3}}}$, there is a point of inflexion when $x = \frac{-4c}{3b}$, which is also the point where the curves cut the axis, or where $y = 0$. This point is $\therefore$ C. Tt, the tangent at C, is $\perp$ axis Dd.

15. Supposing the curve EAF (fig. 13,) to be described by the uniform motion of the point A along CB, and an uniform angular motion round C, it can evidently make no difference whether this angular motion be that of the plane EAC, whilst CB is at rest, or that of CB, in an opposite direction, whilst the plane remains at rest. The curve may, therefore, be conceived as described by the uniform motion of A along the *radius vector* CA,

which also moves with an uniform angular motion. Hence C will be the pole of the curve; and calling the radius vector ρ, and the $\angle$ between it, and any radius vector given in position, as CE, θ, we have

$$\rho \propto \theta$$
$$\therefore \rho = m\theta,$$

which is the equation of the *Spiral of Archimedes.*

The applicability of teeth of this form in the construction of pumps, steam engines, and other machines requiring uniform motion in a given direction, is pretty evident.

Thus, supposing F, F′, F″, three equal and similar teeth made by the union of the equal and similar spirals EF, E′F; E′F′, E″F′; E″F″, EF″, similarly posited around the circumference of the wheel; then if the piston AB be constrained to move in the vertical direction AB by means of a tube, the uniform motion of the wheel round its axis will cause the piston to move upwards and downwards also with an uniform motion, and the greatest altitude through which the piston will be raised is GF.

16. To construct the spiral whose areas are the measures of the ratios between the ordinates which terminate them, we have

$$d \,.\, \text{Area} = \frac{\pm \rho^2 d\theta}{2} = d \,.\, \log.\, \rho$$

$$= \text{M} \times \frac{d\rho}{\rho} \text{ (M being the modulus of the sys-}$$

tem, and ρ the radius vector, and θ the angle described by it)

$$\therefore \theta = \pm \int \frac{2\,\text{M}\,d\rho}{\rho^3} = \text{C} \mp \frac{\text{M}}{\rho^2}$$

Let $\rho = \infty$. Then $\theta = 0$, and $\text{C} = 0$.

$\therefore \theta = \mp \dfrac{\text{M}}{\rho^2}$, which is an equation to the *Lituus.* (See Cotes' *Harmonia Mensurarum,* Pars III., p. 85.)

The *prime radius vector* Ab (Fig. 14,) is evidently an *asymptote* to the curve.

To find the point of *contrary flexure*, we have

$$p = \frac{\rho^2 d\theta}{\sqrt{\rho^2 d\theta^2 + d\rho^2}} \text{ (p being the $\perp$ upon the tangent).}$$

Hence substituting

$$p = \frac{2M\rho}{\sqrt{4M^2 + \rho^4}}$$

$$\therefore \frac{dp}{d\rho} = \frac{8M^3 - 2M\rho^4}{(4M^2 + \rho^4)^{\frac{3}{2}}} = 0 \text{ at the point of inflexion, (see}$$

Simpson, new edit.),

$$\therefore \rho^4 - 4M^2 = 0$$

$$\left.\begin{aligned} &\text{or } \rho = \pm\sqrt{2M} \\ &\text{Hence } \theta = \frac{M}{\rho^2} = \frac{M}{2M} = \frac{1}{2} \end{aligned}\right\} \text{ which equations will give the}$$

position of the point of *contrary flexure*.

17. To find that curve whose normal has the same relation to the distance between the origin of abscissæ, and the intersection of the normal with the axis, that the ordinate of a parabola has to its corresponding abscissa, we have

$y'^2 = px'$ in the parabola, and

$\therefore$ (Normal)^2 = p . (abscissa + subnormal) in the curve required.

$$\therefore \left(y \cdot \sqrt{1 + \frac{dy^2}{dx^2}}\right)^2 = p \cdot \left(x + \frac{ydy}{dx}\right).$$

$$\text{Hence } \frac{pdx - 2ydy}{2\sqrt{\frac{p^2}{4} + px - y^2}} = dx$$

$$\text{and } \therefore \sqrt{\frac{p^2}{4} + px - y^2} = x + c$$

$$\therefore y^2 = \frac{p^2}{4} - c^2 + (p - 2c)\,x - x^2$$

Now if r be the radius of a $\odot$, and the origin of abscissæ be distant from its circumference by the interval a, we have

$$y_1^2 = -(2r + a)\,a + 2 \cdot (r + a)\,x_1 - x^2.$$

The curve is therefore a circle.

18. To trace the curve whose equation is $(x^2 + y^2)^2 = x^2 - y^2$.

By the solution of a quadratic, we have

$$y^2 = \frac{\pm\sqrt{8x^2+1} - 2x^2 - 1}{2}$$

$$\therefore y = \pm\left(\frac{\pm\sqrt{8x^2+1} - 2x - 1}{2}\right)^{\frac{1}{2}}.$$

Let $x = 0$.

Then $y = 0$, or the curve passes through the origin of abscissæ A, (see Fig. 1.)

Let $x = \pm 1 = \text{AB}$ or Ab,

Then $y = 0$, or the curve passes through B and b, and y is imaginary for every greater value of x.

Also since the pairs of values of y, $(\pm y)$ are the same for the positive as for the equal negative values of x, the curve will consist of two equal and similar ovals AMBA, AmbA, intersecting, (the curve is therefore quadrable) in A.

$$\text{Again } \frac{dy}{dx} = \sqrt{2}\,.\,\frac{2x - x\sqrt{8x^2+1}}{\sqrt{8x^2+1}\times(\sqrt{8x^2+1}-2x^2-1)^{\frac{1}{2}}}$$

$= \tan\theta$ (θ being the inclination of the tangent at any point to the line of abscissæ.)

Hence when the curve is parallel to the axis, $\tan.\ \theta = \tan.\ 0 = 0$, or

$$2x - \sqrt{8x^2 + 1} = 0$$

$$\text{or } x = \pm\frac{1}{2}\sqrt{\frac{3}{2}}.$$

Again, since $\frac{dy}{dx} = \frac{0}{0}$ when $x = 0$, the point at A is multiple, and it will be found, by the usual methods, that this point is *double;* that there is at A for each branch a point of *single inflexion,* and that the curve there cuts the axis at an angle of 45°.

To find at what angle the curve cuts the axis when $x =$ a maximum, or when $x = \pm 1$, we have

$$\tan.\ \theta = \frac{dy}{dx} = \frac{1}{0} = \infty = \tan.\ 90°.$$

$\therefore$ the angle required $= 90°$

Making AM $= \rho$, and $\angle$ MAB $= \varphi$,

$$\left.\begin{array}{l} y = PM = \rho . \sin. \varphi \\ x = AP = \rho . \cos. \varphi \end{array}\right\}$$ and substituting in the given equation

we get the polar equation

$$\rho^2 = \cos. 2\,\varphi \ldots\ldots\ldots\ldots\ldots \quad (a)$$

If BM $= z$, then we get

$$dz = \frac{d\rho}{\sqrt{1-\rho^4}},$$ and the arcs of the curve will be found

$=$ those of the *elastic curve*, whose equation is $y = \int \frac{x^2 dx}{\sqrt{1-x^4}}$

the abscissa x corresponding to ρ.

This curve is called the *Lemniscata of James Bernoulli.*

It is the *locus* of the intersection of the perpendicular from the centre upon the tangent of an equilateral hyperbola, whose semi-axes are unity.

For in the equilateral hyperbola

$$\rho^2 = \frac{a^2 b^2}{b^2 \cos.^2 \varphi - a^2 \sin.^2 \varphi}$$

$$= \frac{1}{\cos. 2\varphi}, \text{ when } a = b = 1$$

and p^2 (the perpendicular) $= \dfrac{a^2 b^2}{\rho^2 - a^2 + b^2} = \dfrac{1}{\rho^2}$

$\therefore p^2 = \cos. 2\varphi$, (See equat. a).

See Prob. 1, Vol. II.

19. To find the equation of the curve whose subtangent $= b + y$, we have

$$\frac{ydx}{dy} = b + y$$

$$\therefore x = \int \frac{bdy}{y} + \int dy$$

$$= b\,l.y + y + c$$

$$\therefore l.\,(y^b e^y) = x - c$$

and $y^b e^y = e^{x-c} = \dfrac{1}{e^c} . e^x = c' e^x$, the equation required.

20. The subcontrary section of an oblique cone is a circle.

Let $a\,m\,b$ (Fig. 14,) be the section made by a plane passing through the cone parallel to the circular base AB, AVB being the triangular section of the cone made by a plane passing through the vertex and centre of the base. Also let $A'm\,B'$ be the subcontrary section made by a plane $\perp$ plane AVB, and inclined to BV by the angle $A'B'V = A$, and intersecting amb in mp.

Join Vm, Vp and produce them to M and P respectively; join PM.

Then since the planes amb, $A'm\,B'$, are $\perp$ plane AVB, their intersection mp is $\perp$ ab. Also because PM is parallel to pm, (being the intersections of a plane with two parallel planes,) it is $\perp$ plane AVB, and $\therefore$ $\perp$ AB.

Again, from similar triangles,

$$AP : ap :: VP : vp :: MP : mp$$
$$\text{and } BP : bp :: VP : vp :: MP : mp$$
$$\therefore AP \,.\, BP : ap \,.\, bp :: MP^2 : mp^2$$
$$\text{But } AP \,.\, BP = MP^2 \text{ (by prop.}^y \text{ of } \odot)$$
$$\therefore ap \,.\, bp = mp^2$$

Also since $\angle\, pB'b = \angle\, A = \angle\, a$, and the vertical $\angle$s are equal, the $\triangle$ $A'ap$, $B'bp$ are similar,

$$\therefore A'p : ap :: pb : B'p$$
$$\therefore A'p \,.\, B'p = ap \,.\, bp = mp^2,$$

and $\therefore$ the subcontrary section $A'm\,B'$ is a circle.

21. To find the curve by whose revolution round its line of abscissæ, a solid is generated $= \frac{3}{5}$ of its circumscribing cylinder.

Let x and y be its co-ordinates, x beginning at the vertex.

Then, by the question

$$\int \pi y^2 dx = \frac{3}{5} \,.\, \pi y^2 \times x \left(\frac{3}{5} \text{ base of solid} \times \text{alt.}\right)$$

$$\therefore y^2 dx = \frac{6}{5}\, ydy \times x + \frac{3}{5} \times y^2 dx$$

$$\therefore \frac{3dy}{y} = \frac{dx}{x}$$

and $l.\ y^3 = l.\ x + l.\ c = l.\ cx$

$\therefore y^3 = cx$ is the equation to the curve, which is, therefore, the *cubic parabola.*

22. To construct the spiral whose arc is the measure of the ratio between the ordinates which intercept it.

If $CAP = \theta$, $AP = \rho$, and $CP = z$,

AC being $= 1$ (Fig. 15)

$$\text{Then } z = l.\ \frac{1}{\rho}$$

$$\text{and } dz = -\frac{d\rho}{\rho}$$

But $dz = \sqrt{d\rho^2 + \rho^2\, d\theta^2}$, as we readily learn from the figure.

$$\therefore d\rho^2 + \rho^2\, d\theta^2 = \frac{d\rho^2}{\rho^2}$$

$$\therefore d\theta = \frac{d\rho}{\rho^2} \cdot \sqrt{1 - \rho^2}$$

$$\text{Let } \frac{1}{\rho} = u$$

$$\text{Then } d\theta = -\frac{du}{u} \cdot \sqrt{u^2 - 1}$$

$$= -\frac{udu}{\sqrt{u^2 - 1}} + \frac{du}{u\sqrt{u^2 - 1}}$$

$$\therefore \theta = -\sqrt{u^2 - 1} + \sec^{-1} u + c$$

$$= -\frac{\sqrt{1 - \rho^2}}{\rho} + \sec.^{-1} \frac{1}{\rho} + c$$

Let $\theta = 0$, then $\rho = 1$, and $c = 0$

and $\therefore \theta = \sec.^{-1} \dfrac{1}{\rho} - \dfrac{\sqrt{1 - \rho^2}}{\rho}$, the equation of the curve; which being analogous to that of the involute of a circle whose radius is unit, we will attempt the construction by the aid of that curve.

Thus, CP′ being the involute of the ⊙ CQD, whose radius is unit, let the ordinates be represented by ρ'. Let also $P'y$ be a tangent at P′ and Ay a ⊥ upon it. Then Dp' being the unwound line it must be ⊥ curve, or ⊥ $p'y$. It is also ⊥ AD. ∴ $p'y =$ AD.

Now from similar △, we have

$$p'r' : P'r' :: AD : AY$$

$$\text{or } d\rho' : \rho' d\theta :: 1 : \sqrt{\rho'^2 - 1}$$

$$\therefore d\theta = \frac{d\rho'}{\rho'} \cdot \sqrt{\rho'^2 - 1} \text{ which is the same form as}$$

$$d\theta = \frac{du}{u} \cdot \sqrt{u^2 - 1}$$

$$\text{where } u = \frac{1}{\rho}.$$

Hence it appears that ρ always $= \frac{1}{\rho'}$, and the construction is ∴ obtained by unwinding the thread CD so as to describe the *involute*, and taking AP always $= \frac{AC^2}{AP'} = \frac{1}{AP'}$. The spiral is called the *Complicated Tractrix.*

23. Let AB $= x$ (See Fig. C. P. p. 209)
BP $= y$, the radius of the ⊙ $= 1$, and BC $= y'$.

Then by the question

$$\int y dx = \text{area AGC} = \text{AGCB} - \text{ACB}$$

$$= \int y' dx - \frac{y'x}{2}$$

$$= \int dx \sqrt{2x - x^2} - \frac{x \sqrt{2x - x^2}}{2}$$

$$\therefore y dx = dx \sqrt{2x - x^2} - \frac{dx \sqrt{2x - x^2}}{2} - \frac{x dx (1 - x)}{2 \sqrt{2x - x^2}}.$$

Hence $y = \dfrac{x}{2 \sqrt{2x - x^2}}$, the equation of the curve APF.

The curve cuts the semicircle when $y' = y$, or when

$$\sqrt{2x-x^2} = \frac{x}{2\sqrt{2x-x^2}} \text{ or when}$$

$$x^2 - \frac{3}{2}x = 0, \text{ or when}$$

$$x = 0 \text{ and } x = \frac{3}{2}.$$

Hence FL bisects the radius in L.

Again

$$ABP = \int y dx = \int \frac{x dx}{2\sqrt{2x-x^2}}$$

$$= -\frac{1}{2}\int \frac{(dx - x dx) - dx}{\sqrt{2x - x^2}}$$

$$= -\frac{1}{2}\sqrt{2x-x^2} + \frac{1}{2}\text{vers.}^{-1} x, \text{ there being}$$

no correction.

Let $x = AM = 2$,

Then the whole area AFM $= \frac{1}{2} \times$ ACFM $= \frac{1}{2} \odot$ ACFM.

Hence the part of the area without the $\frac{1}{2} \odot =$ whole area $-$ APFMA $=$ ACFM $-$ APFM $=$ AGCFPA.

The $\perp$ at M is evidently an *asymptote* to the curve.

24. Let NM (Fig. 16,) the line given in length $= m$, pass through the point A given in position, and meet the straight line BC given in position. Required the locus of the extremity M.

Let DAa be that position of NM which is $\perp$ BC, and since AD is given, let it $= n$.

Also let PM $= y$ and AP $= x$.

Then from similar $\triangle$s

$$x : n :: AM : m - AM$$

$$\therefore x : n + x :: AM = \sqrt{x^2+y^2} : m.$$

Hence $y = \pm \frac{x}{n+x}\sqrt{m^2-(n+x)^2}$ which is the equation of the curve. It is an oval similar and equal on each side the diameter.

25. Given two *radii vectores* of a *logarithmic spiral*, and the angle between them, to construct the spiral.

The property which distinguishes the logarithmic spiral is that it cuts all its radii vectores at the same angle. Let that angle PQR (AB, AB′ are the given radii vectores, and AP, AQ are indefinitely near) $= \alpha$, (Fig. 17). Also let $AB = r$, $AB' = r'$, $AP = \rho$, $\angle BAB' = \beta$, and $\angle BAP = \theta$, and AC, the radius of the $\odot$ $Cpr = 1$.

Then $PR = QR \,.\, \tan. Q = d\rho \,.\, \tan. \alpha$ also $= \rho \,.\, d\theta$.

$$\therefore d\theta = \tan. \alpha \,.\, \frac{d\rho}{\rho}$$

$$\text{and } \theta = \tan. \alpha \,.\, l. \rho + c$$

Let $\theta = 0$. Then $\rho = AB = r$, and

$$\theta = \tan. \alpha \,.\, l. \frac{\rho}{r} \ldots\ldots\ldots (a)$$

Hence $\beta = \tan. \alpha \,.\, l. \dfrac{r'}{r}$ which gives $\tan. \alpha = \dfrac{\beta}{l. r' - lr}$

$$\left.\begin{aligned} \text{Hence } \theta &= \frac{\beta}{lr' - lr} \times l. \frac{\rho}{r} \\ \text{or } &= \log. \left(\frac{\rho}{r}\right) \text{ to modulus } \frac{\beta}{lr' - lr} \end{aligned}\right\} \text{the equation of}$$

the curve expressed in terms of θ, and given quantities.

The points of the curve corresponding to every possible value of θ, may therefore be found, by reference to the Tables; or the curve may be constructed.

26. The locus of the intersections of the tangents at corresponding points of the common cycloid and its generating circle, is the *involute* of the generating circle.

Let PT, QT (Fig. 18,) the tangents at P and Q intersect in T.

Join AQ, OQ, &c. &c.

Then, it is well known that

TP is parallel to AQ, and

TQ is $\perp$ OQ the radius.

Hence $\angle AQM = R \,.\, \angle - \angle O\,AQ = R \,.\, \angle - \angle AQO = \angle AQT = \angle QTP$,

$\therefore\ \angle QTP = \angle AQM = \angle TPQ$

$\therefore\ TQ = QP = \text{arc } AQ$

or T is a point in the *involute* AT.

27. To find the locus of the intersections of the tangents to a circle with the perpendiculars to them, let fall from a given point in the circumference.

Let A (Fig. 19,) be the given point, M the intersection of the tangent at R and its $\perp$ from A, and let MR meet the diameter of the $\odot$ produced in T.

Join CR (C being the centre). Then referring the locus BMA to A as a pole, and $\therefore$ putting $AM = \rho$ and $\angle MAB = \theta$ (B being the point where $\rho =$ the diameter of the circle), since CR is parallel to AM we have $\angle MAR = \angle ARC = \angle RAN$

$\therefore\ AM = AN = AC \pm CN = r\,.\,(1 + \cos.\,\theta.)$

Hence $\rho = r\,.\,(1 + \cos.\,\theta)$....... (a) the polar equation of the curve.

Again, referring the curve to rectangular co-ordinates, by putting

$Ap = x$, $PM = y$, we get

$\sqrt{x^2+y^2} = AM = AN\ (\angle RAN = \angle ARC = \angle MAR)$ $= r \pm CN = r + r\,.\,\cos.\,\theta.$

$$\text{But } \frac{y}{x} = \tan.\,\theta = \frac{\sin.\,\theta}{\cos.\,\theta} = \frac{\sqrt{1-\cos.^2\theta}}{\cos.\,\theta}$$

$$\therefore\ \cos.\,\theta = \frac{x}{\sqrt{x^2+y^2}}$$

$$\therefore\ \sqrt{x^2+y^2} = r\,.\,\left(1 + \frac{x}{\sqrt{x^2+y^2}}\right)$$

Hence

$y^4 + (2x^2 - 2rx - r^2)\,y^2 = 2rx^3 - x^4$ the equation between the *rectangular co-ordinates.*

Let $y = 0$.

Then $x = 0$ and $= 2r$, $\therefore$ the curve passes through the points A and B.

To find the area of the curve, we have

$$\text{Area} = \int \frac{\rho^2 d\theta}{2} \text{ (See } \textit{Lacroix.})$$

$$= \frac{r^2}{2}\left(\frac{3}{2}\,\theta + 2 \sin.\,\theta + \frac{\sin.\,2\theta}{4}.\right)$$

Let $\theta = \pi$

Then the whole area AMB

$$= \frac{r}{2} \cdot \frac{3}{2} \cdot r\theta = \frac{3}{2} \cdot \frac{r}{2} \text{ ARB}$$

$$= \frac{3}{2} \cdot \frac{1}{2} \odot \text{ ABR} = \frac{3}{4} \cdot \odot.$$

To find the length of the curve we have

$$z = \int \sqrt{d\rho^2 + \rho^2 d\theta^2} = \int d\theta \sqrt{\frac{d\rho^2}{d\theta^2} + \rho^2}$$

$$= r \sqrt{2} \int d\theta \sqrt{1 + \cos.\,\theta} = r\sqrt{2} \int d\theta \sin.\theta \times$$

$$\sqrt{\frac{1 + \cos.\,\theta}{1 - \cos.^2\theta}} = r \sqrt{2} \int \frac{d\theta \sin.\,\theta}{\sqrt{1 - \cos.\,\theta}}$$

$$\therefore z = 2 \sqrt{2} \,.\, r \sqrt{1 - \cos.\,\theta}.$$

Let $\theta = \pi$

Then $z = 2\sqrt{2} \,.\, r \times \sqrt{2} = 4r.$

$\therefore$ the whole length of the arc AMB $= 4r = 4$ times the radius of the circle.

The greatest ordinate is most easily found thus

$$y = \rho \,.\, \sin.\,\theta = r \sin.\,\theta \,.\, (1 + \cos.\,\theta)$$

$$\therefore \frac{dy}{d\theta} = r \cos.\,\theta \,.\, (1 + \cos.\,\theta) - r \sin.^2\theta.$$

Hence when $y =$ max. or min.

$$\cos.\,\theta + \cos^2\theta - \sin.^2\theta = 0$$

$$\text{or } \cos^2\theta + \frac{\cos.\,\theta}{2} = \frac{1}{2}$$

$$\therefore \cos.\,\theta = -\frac{1}{4} \pm \sqrt{\frac{1}{16} + \frac{1}{2}} = \frac{-1 \pm 3}{4}$$

$$= \frac{1}{2} \text{ or } -1.$$

And the corresponding values of y are

$$\left.\begin{array}{l} r \cdot \sqrt{1 - \frac{1}{4}} \times \frac{3}{2} = \pm r \cdot \frac{3\sqrt{3}}{4} \\ \text{and } r \times 0 = 0 \end{array}\right\} \text{the former}$$

maxima; the latter a *minimum;* the position of the maximum value being found by taking $\theta = 60°$.

Since in the *Trisectrix* the equation is

$$\left.\begin{array}{l} \rho = r \cdot (1 + 2 \cos. \theta) \\ \text{and here} \\ \rho' = r \cdot (1 + \cos. \theta) \end{array}\right\}$$

the analogy between the curves is very evident. We may, in fact, derive the *Trisectrix* from the latter by constantly taking $Mm =$ CN. They will intersect when $\theta = 90°$ scil. in M′. The *Trisectrix* will pass through the centre C; for at A, CN becomes negatively = radius, and $\rho = 0$. When $\rho = 2r$, $\rho' = Ab = 3r$.

Multiplying the two equations together, &c., the result $\rho = -\frac{\cos. \theta}{2} \pm \frac{\sqrt{4 + 12 \cos. \theta + 9 \cos.^2 \theta}}{2}$ will represent the compound curve.

For a complete discussion of the Trisectrix, see *Trisection de l'Angle*, par *Azemar* et *Garnier*, Paris, 1809.

The problem may be generalized, by *requiring the* LOCUS *of the intersection of the tangent and perpendicular let fall upon it, from any point given in position, of any curve whose polar equation is*

$$r = f(p)$$

r being the radius vector, and $f(p)$ any function of the corresponding ⊥ upon the tangent.

First, let (A) the given point be also the pole of the given curve.

Let P and Q (Fig. 21) be two points indefinitely near in the given curve BPQ, and corresponding tangents Pp, Qq. Also let p, q, be the intersections of these tangents with the perpendiculars Ap, Aq, and suppose pqc the *locus* required.

Since P, Q, and ∴ p, q, are indefinitely near, $p\,q$ being joined and produced will touch the *locus* in p, and $p\,q$ will be a straight

line. Draw $Ay \perp pq$. Again, because the $\angle$ at p, and q, are right angles, and Q and P may be considered as coincident, a semicircle may be described passing through A Ppq.

Hence $\angle\, ypA = \angle\, pPA$, and the $\angle\, p$ and y are right $\angle$.

$\therefore\ \triangle\, Apy$ is similar to $\triangle\, ApP$, and we have

$$Ay : Ap :: Ap : AP$$

Let $Ay = \varpi$, $Ap\ (= p) = \rho$, and $AP = r$,

Then $\varpi = \dfrac{\rho^2}{r} = \dfrac{\rho^2}{f\cdot(p)} = \dfrac{\rho^2}{f(\rho)}$ .... (a) the equation between the radius vector ρ, and the perpendicular upon the tangent of the required locus. Hence $\therefore$ the *locus* may be found.

Again, let $pAB = \theta$. Then by similar $\triangle\, Ayp$, qpr

$$Ay : py :: qr : pr$$

$$\therefore\ Ay^2 : Ap^2 :: qr^2 : qr^2 + pr^2$$

But $qr = Aq\,.\,d\theta = \rho\,.\,d\theta$

and $pr = d\rho$

$$\therefore\ \varpi^2 = \frac{\rho^4\, d\theta^2}{\rho^2\, d\theta^2 + d\rho^2} = \frac{\rho^4}{\rho^2 + \dfrac{d\rho^2}{d\theta^2}}$$

$$\therefore\ \varpi = \frac{\rho^2}{\sqrt{\rho^2 + \dfrac{d\rho^2}{d\theta^2}}} = \frac{\rho^2}{f(\rho)},\ \text{by } (a)$$

$$\text{Hence } \theta = \int \frac{d\rho}{\sqrt{(f\rho\cdot)^2 + \rho^2}}\ \ldots\ldots (b)$$

the equation between the radius vector, and polar angle of the *Locus*.

Equations (a) and (b) will each determine the *locus* when its pole is coincident with that of the given curve. But when its pole (A′) does not coincide with A, let $p'q'$, &c., be the required locus,

and put $A'p' = \rho'$, $\angle\, p'A'B = \theta'$,
the given distance $AA' = a$,
and the given $\angle AA'B = \beta$. Draw $AM \perp A'p'$.

Then $\because$ Aq, $A'q'$ are $\perp$ Qq, and $A'p$, $A'p'$ are $\perp Pp$, the $\angle q'A'p'$ is evidently $= \angle\, qAp$,

$$\therefore\ d\theta' = d\theta.$$

$$\begin{aligned} \text{Also } \rho' &= p'A' = p'M \pm MA' \\ &= p'M + AA' \,.\, \cos.\, AA'M \\ &= \rho + a \,.\, \cos.\, (\alpha - \theta') \end{aligned}$$

and substituting in equation (b) we get

$$d\theta' = \frac{d \,.\, (\rho' - a \,.\, \cos.\, \overline{\alpha - \theta'})}{\sqrt{\{f \,.\, \overline{\rho' - a \,.\, \cos.(\alpha - \theta')}\}^2 - (\rho' - a \,.\, \cos.\, \overline{\alpha - \theta'})^2}} \;\ldots\; (b').$$

by the integration of which we may find the *locus* wherever the given point, and the pole of the given curve, may be situated.

The formula (b), which will generally be most commodious in practice, may readily be exemplified by applying it to the solution of our problem.

Since RM (Fig. 20,) touches the ⊙ in R, the ∠ MRA, = ∠ RBA, and ∴ the right-angled △ ARM, ARB, are similar.

Hence AM . AB = AR², or making (as before) AC = 1, and AM $= p = \rho$, and AR $= r$,

$$2p = r^2$$

$$\therefore r = \sqrt{2p} = \sqrt{2\rho} = f(\rho)$$

Hence by substituting in equation (b), the negative sign being taken, because $d\rho$ and $d\theta$ have different signs,

$$\theta = -\int \frac{d\rho}{\sqrt{2\rho - \rho^2}} = -\text{vers.}^{-1}\,\rho + C$$

Let $\theta = 0$. Then $\rho = 2$, and $C = \text{vers.}^{-1}\,2$.

$$\therefore \theta = \text{vers.}^{-1}\,(2 - \rho)$$

$$\therefore 2 - \rho = \text{vers.}\,\theta = 1 - \cos.\,\theta$$

$\therefore \rho = 1 + \cos.\,\theta$, the same as before determined.

Ex. 2. Let the given curve be the common parabola referred to its focus.

The latus rectum being called (l) it is known that

$$\frac{l}{4} \,.\, r = p^2,$$

$$\therefore r = \frac{4p^2}{l} = \frac{4\rho^2}{l} = f \,.\, (\rho)$$

$\therefore$ by substitution in equat. (b) we get, (measuring θ from the vertex)

$$\theta = \int \frac{d\rho}{\rho} \cdot \frac{1}{\sqrt{\frac{16\rho^2}{l^2} - 1}}$$

Put $u = \frac{l}{4\rho}$, substitute, &c.

$$\text{and } \theta = \int \frac{-du}{\sqrt{1 - u^2}} = \cos.^{-1} u + c = \cos.^{-1} \frac{l}{4\rho}$$

$$\therefore \frac{l}{4\rho} = \cos.\ \theta$$

and $\rho = \frac{l}{4} . \sec.\ \theta$, which is the equation of a straight line $\perp$ at the vertex to the line joining the focus and vertex of the parabola.

By equation (a) we get immediately

$\varpi = \frac{\rho^2}{f(\rho)} = \rho^2 \times \frac{l}{4\rho^2} = \frac{l}{4}$ a constant quantity; $\therefore$ the locus is a straight line, &c.

Ex. 3. Let the given curve be the *common cycloid*; required the locus of the intersection of the tangents and perpendiculars upon them from the centre of the generating circle.

Since PT (Fig. 19,) touches the cycloid, it is parallel to AB (PM being $\perp$ AO, &c. &c.)

Then Op being $\perp$ AQ is also $\perp$ TP.

Let AOD $= \theta$, AO $= 1$, Op, the radius vector of the *locus* $= \rho$.

Then CG being joined and produced to G since ABG $=$ R . $\angle$, CG is parallel Op, and G is a R . $\angle$.

Hence, the $\triangle$ PBG, MBC, are similar, and

$\therefore$ Dp $=$ BG : BP :: BM : BC

But BP $=$ arc AB (by property of cycloid)

$= \angle$ AOB $= 2\theta$

BM $=$ OB . sin. $2\theta =$ sin. 2θ

and BC $=$ OD $\times \frac{\text{CA}}{\text{OA}} = 2$OD $= 2 \cos.\ \theta$

$$\therefore \ Dp = \frac{2\theta \ . \sin. 2\theta}{2 \cos. \theta} = 2\theta \ . \sin. \theta$$

Hence $\rho = \mathrm{OD} + \mathrm{D}p = \cos. \theta + 2\theta \ . \sin. \theta$, the equation of the *locus*.

After due attention to the above discussion (which will be useful hereafter in the theory of the Aberration of Light), the student may proceed to find the *Locus* of the intersections of the tangents, and perpendiculars of the first locus of a given curve; the *locus* of the intersections of tangents and perpendiculars of the second locus, and so on to any number of *Loci*. He will thus obtain many curious and interesting results.

28. To trace and construct the curve whose equation is, $y^2 = \dfrac{x^3 + bx^2}{c - x}$.

In the equation

$y = \pm x \ . \sqrt{\dfrac{b+x}{c-x}}$... (*a*) let $x = 0, c$, and $-b$. The corresponding values of y are $\pm 0, \pm \dfrac{1}{0} = \infty, 0$. Hence, if from A (Fig. 22,) the origin of abscissæ, we take $\mathrm{AC} = c$ $\mathrm{AB} = -b$, and draw CD, C*d* $\perp$ BC, the curve will pass through A, and B, and CD, C*d* will be asymptotes to the infinite branches AE, A*e*. Also for all values of $x > c$, or $> (-b)$ (the sign not being considered), the values of y are imaginary.

Again, putting $\theta =$ inclination the tangent to the axis, we have

$$\tan. \theta = \frac{dy}{dx} = \pm \sqrt{\frac{b+x}{c-x}} \pm \frac{b+c}{2} \cdot \frac{x}{(c-x)^{\frac{3}{2}} \cdot \sqrt{b+x}}$$

Let $x = 0$, and $-b$, and the corresponding values of $\tan. \theta$ are $\pm \sqrt{\dfrac{b}{c}}, \dfrac{1}{0} = \infty$, respectively.

$$\left.\begin{array}{l} \text{Hence at A, } \pm \theta = \tan.^{-1} \sqrt{\dfrac{b}{c}} = \angle \mathrm{TAC} \text{ or } \angle t\mathrm{AC} \\ \quad\text{at B, } \pm \theta = \tan.^{-1} \infty = 90^\circ. \end{array}\right\}$$

To find the convexities and concavities, multiple points, &c., at any given point in the curve it is frequently advisable to adopt this method.

$$y' - y = \frac{dy}{dx}\,(x' - x)$$

is the equation to the tangent at any point in the curve, x', y' being the co-ordinates of the tangent, and x, y those of the curve at the point touched.

Now at the point A, $y = 0$, and $x = 0$ and $\frac{dy}{dx} = \pm\sqrt{\frac{b}{c}}$,

$\therefore$ the equation to the tangent at A, is

$$y' = \pm\sqrt{\frac{b}{c}} \times x' = \text{MQ}$$

But the values of y corresponding to x', MP are $\pm\sqrt{\frac{b+x'}{c-x}} \times x'$

which are greater or less than $\pm\sqrt{\frac{b}{c}} \times x'$ according as x'

is positive, or negative and not greater than b. that is, AE, Ae are convex; AP′B, Ap'B are concave to the axis. The point A is evidently therefore double.

29. To find the equation to the spiral in which $\theta \propto \frac{1}{\rho^n}$, θ being the angle at the pole, and ρ the radius vector.

Let $\theta = \alpha$, when $\rho =$ SA (Fig. 23) $a = 1$.

Then $\theta : \alpha :: \frac{1}{\rho^n} : \frac{1}{1^n}$

And $\theta = \frac{\alpha}{\rho^n}$ ..... (a) the equation required.

Again, the subtangent ST

$$= \frac{-\rho^2\, d\theta}{d\rho} = \frac{\rho^2 \times n\,\alpha}{\rho^{n+1}} = \frac{n\,\alpha}{\rho^{n-1}} \text{(by equation } a) = n\rho \times \frac{\alpha}{\rho^n} = n\,\rho\,\theta.$$

Now whatever line, issuing from S, θ begins from, the circular

arc described from that line to the curve with the radius SP $= \frac{SP \times \theta}{SA} = \rho . \theta$.

Hence the *subtangent at any point* $P = n \times$ *circular arc subtending* θ *at the distance SP;* which is more general than the enunciation of the latter part of the problem.

The enunciation does not hold good, indeed, in the case of $n = 1$; for then the subtangent, corresponding to the infinite value of ρ, being $= \frac{n a}{\rho^{1-1}} = n\, a =$ ST, the asymptote TL $\perp$ ST (see *Simpson's* Flux.) does not pass through the centre.

30. To trace the curve whose ordinates are equal to the vers.-sines of twice the corresponding abscissæ; we have

$$y = \text{vers. } 2x \;\ldots\ldots\; (a)$$

And putting $y = 0$, the corresponding values of x, are 0, C, 2C, 3C, 4C, &c. (C being the semi-circumference of the circle) which being represented by 0, A_1A_2, A_1A_3, A_1A_4, &c. (Fig. 24.) the curve will meet the axis in the several points A_1, A_2, A_3, &c.

Again, when $x = \frac{C}{2}, \frac{3C}{2}, \frac{5C}{2}$, &c. y is greatest, being for each of these values, $= 2r$ (where $r =$ radius of the circle). Hence if A_1M_1, A_1M_2, A_1M_3, &c. be taken $= \frac{C}{2}, \frac{3C}{2}, \frac{5C}{2}$ &c. and M_1P_1, M_2P_2, M_3P_3 &c. each $= 2r$ be drawn $\perp A_1A_3$, the curve will pass through P_1, P_2, P_3, &c.

To find the inclination of the tangents at A_1, P_1, A_2, P_2, &c., we have tan. $\theta = \frac{dy}{dx} = 2 \sin. 2x$, which equalling zero for each of the values $x = 0$, C, 2C, 3C, &c. $\frac{C}{2}$, $\frac{3}{2}$ C, &c. gives $\theta = 0$, at each of those points.

Hence the tangents at A_1, A_2, &c. P_1, P_2, &c. are parallel to the axis, and therefore the axis touches the curve in A_1, A_2, A_3,

&c. Hence also at A_1, A_2, &c. the curve is convex towards the axis.

To find the points of *contrary flexure*, we have

$\frac{d^2y}{dx^2} = 4 \cos. 2x$, which being put, according to rule, $= 0$, gives

$$x = \frac{C}{4}, \quad \frac{3C}{4}, \quad \frac{5C}{4}, \text{ \&c. and for each of these values of } x$$

$$y = \text{vers. } 2x = r - \cos. 2x, \text{ becomes } = r$$

Therefore, bisecting $A_1 M_1$, $M_1 A_2$, $A_2 M_2$, &c. in N_1, n_1, N_2, n_2, &c., and drawing $N_1 Q_1$, $n_1 q_1$, $N_2 Q_2$, $n_2 q_2$, &c. each $= r$, and $\perp$ A_1M_2, Q_1, Q_2, Q_3, &c. q_1, q_2, &c. will be the points of contrary flexure.

Hence $A_1 Q_1$, $q_1 A_2 Q_2$, $q_2 A_3 Q_3$, &c. will be convex, and the remaining parts concave towards the axis.

To find the area of the part $A_1 P_1 M_1$, whose abscissa is a quadrant, we have

$$\text{Area} = \int y dx = \int \text{vers. } 2x \times dx$$
$$= \int r dx - \int dx \cos. 2x = rx - \frac{\sin. 2x}{2}.$$

Let $x = \frac{C}{2}$.

Then the area of $AQ_1 P_1M_1 = \frac{r \times C}{2} =$ the semi-circle whose radius is r.

Hence $AP_1A_2 =$ the whole $\odot$.

31. To find the equation to the curve, the part of whose tangent comprised between the curve and its axis, is a constant quantity.

Let this constant quantity $= a$.

Then x and y being the co-ordinates of the curve, we have

$$a = \sqrt{y^2 + \frac{y^2 dx^2}{dy^2}}, \quad \frac{ydx}{dy} \text{ being the subtangent;}$$

$$\therefore a^2 = y^2 + \frac{y^2 dx^2}{dy^2}$$

$$\text{and } dx = \frac{dy}{y}\sqrt{a^2-y^2} = \frac{a^2dy}{y\sqrt{a^2-y^2}} - \frac{ydy}{\sqrt{a^2-y^2}}$$

$$\therefore\ x = \int\frac{a^2dy}{y\sqrt{a^2-y^2}} + \sqrt{a^2-y^2}$$

$$\text{Let } \frac{a}{y} = u.$$

$$\text{Then } \int\frac{a^2dy}{y\sqrt{a^2-y^2}} = -\int\frac{adu}{\sqrt{u^2-1}} = -a\,.\,l.\,(u+\sqrt{u^2-1})$$

$$= -a\,.\,l.\,\frac{a}{y} + \frac{\sqrt{a^2-y^2}}{y}$$

Hence $x = \sqrt{a^2-y^2} - a\,.\,l.\,a + \dfrac{\sqrt{a^2-y^2}}{y} + C$, the equation required.

The greatest ordinate of the curve is $= a$.

Also, since $x = \infty$, when $y = 0$, the line of abscissæ is an asymptote. The curve is convex to the axis throughout.

32. To find the greatest ordinate in the curve whose equation is

$$y^3 - axy + x^3 = 0.$$

$\dfrac{dy}{dx} = \dfrac{ay - 3x^2}{3y^2 - ax}$ and when y is a maximum or minimum

$$ay - 3x^2 = 0$$

$$\text{or } x = \sqrt{\frac{ay}{3}}.$$

Hence by substitution in the given equation we get,

$y^6 - \dfrac{4a^3}{27}\,.\,y^3 = 0$, which gives

of y the real values 0 and $\dfrac{4^{\frac{1}{3}}}{3}\,.\,a$.

To find which is a *maximum* and which a *minimum*, we have

$$\frac{d^2y}{dx^2} = \frac{(3ay^2 - 6y - a^2x)\dfrac{dy}{dx} - 6x(3y^2 - ax) + a}{(3y^2 - ax)^2}$$

$\therefore$ when $y =$ max. or min. $\frac{dy}{dx}$ being $= 0$, we have $\frac{d^2y}{dx^2} =$

$\frac{a - 6x \,.\, (3y^2 - ax)}{(3y^2 - ax)^2}$,

which is positive or negative, according as $y = 0$, or $\frac{4\frac{1}{3}}{3} \,.\, a$, (as it is easily found by substitution and the proper reductions), and therefore

$$y = 0, \text{ is a } \textit{minimum} \text{ value,}$$

and $y = \frac{4\frac{1}{3}}{3} \,.\, a$, is a *maximum* value of the ordinate, which is indeed very evident.

33. To find the equation to the curve, whose nature is such, that if from a given point B in its axis (Fig. 25.) BT, $\perp$ axis BM, be drawn meeting the tangent PT in T, then BT + TP = the arc AP.

Draw TN $\perp$ PM or parallel to BM, and tn (indefinitely near P), also $\perp$ PM. Then, putting MP $= y$, A′M $= x$, and A′B $= a$, we get

$$TB = PM - PN = PM - TN \,.\, \frac{Pn}{tn}$$

$$= y - (a + x)\frac{dy}{dx}.$$

Also $PT = \sqrt{TN^2 + PN^2} = \sqrt{(x + a)^2 + (a + x)^2\frac{dy^2}{dx^2}}$

$$= (x + a)\sqrt{1 + \frac{dy^2}{dx^2}}$$

and $AP = \int\sqrt{dx^2 + dy^2} = \int dx \sqrt{1 + \frac{dy^2}{dx^2}}$

Hence, by the question,

$$y - (a+x)\frac{dy}{dx} + (x + a)\sqrt{1 + \frac{dy^2}{ax^2}} = 2\int dx \sqrt{1 + \frac{dy^2}{dx^2}},$$

which being like the formula of *Clairaut*, may be integrated by differentiation; for, putting $\frac{dy}{dx} = p$, substituting and differentiating, we get

$$dy - pdx - (x + a)\,dp + dx\,\sqrt{1 + p^2} + (x + a) \times \frac{pdp}{\sqrt{1 + p^2}} = 2\,dx\,\sqrt{1 + p^2}.$$

But $dy = pdx$.

$$\therefore \frac{pdp}{1 + p^2} - \frac{dp}{\sqrt{1 + p^2}} = \frac{dp}{x + a},$$

$$\therefore l.\,\sqrt{1 + p^2} - l.\,(p + \sqrt{1 + p^2}) = l.\,(x+a) + lc$$

$$\therefore \frac{\sqrt{1 + p^2}}{p + \sqrt{1 + p^2}} = c\,.\,(x + a)$$

$$\text{or } \frac{p}{\sqrt{1 + p^2}} = \frac{1}{c\,.\,(x+a)} - 1 = \frac{1 - ca - cx}{c\,(x + a)}$$

$$\therefore \frac{1}{p^2} = \frac{c^2\,.\,(x + a)^2}{(1 - ca - cx)^2} - 1 = \frac{2cx + 2ca - 1}{(cx + ca - 1)^2}$$

$$= \frac{2}{c}\,.\,\frac{x + \frac{2ca - 1}{2c}}{\left(x + \frac{ca - 1}{c}\right)^2}$$

Let $\frac{2ca - 1}{2c} = c'$.

Then, substituting and inverting the result,

$$p^2 = \frac{1}{4\,.\,(a - c')}\;\frac{(x + 2c' - a)^2}{x + c'}$$

$$\therefore dy = \frac{1}{2\sqrt{a - c'}} \times dx\,.\,\frac{(x + c' + c' - a)}{\sqrt{x + c'}}$$

$$= \frac{1}{2\sqrt{a - c'}}\,.\,dx\,\sqrt{x + c'} - \frac{(a - c')^{\frac{1}{2}}dx}{2\sqrt{x + c'}}.$$

$$\therefore y = \frac{1}{3\sqrt{a - c'}} \times (x + c')^{\frac{3}{2}} - .\,(a - c')^{\frac{1}{2}}\,.\,(x + c')^{\frac{1}{2}} + c''$$

$$= \frac{\sqrt{2cx + 2ca - 1}}{3c} \times (cx + ca - 2) + c'',$$ the equation required.

To simplify the construction, we may make B, instead of A′, the origin of abscissæ, by putting $a = 0$, in the above equation, which thence becomes,

$$y = \frac{\sqrt{2c}}{3}\sqrt{x - \frac{1}{2c}} \times \left(x - \frac{2}{c}\right) + c'' \;\ldots\ldots\; (a)$$

Now, since $\frac{1}{2c}$ is the least value of x which will give $y =$ a real quantity, and therefore denotes the abscissa of the vertex, in order to have the vertex in the line of abscissæ, according to the enunciation (which is by no means a necessary limitation), the arbitrary constant c'' must be zero.

Hence the equation (a) becomes

$$y = \frac{\sqrt{2a}}{3}\sqrt{x - \frac{1}{2c}} \times \left(x - \frac{2}{c}\right) \ldots\ldots (b)$$

The vertex A (see Fig. 25) is found by putting $x = \frac{1}{2c} = \text{AB}$; for then $y = 0$.

The curve also cuts the axis in C, when $x = \frac{2}{c} = 4 \times \frac{1}{2c} =$ 4 BA.

The pairs of values of y, different only in signs between A and C, are all possible; after which they increase with x to infinity. The branches AP C, Apc, therefore, form a node and CD, Cd are infinite; the former are concave, the latter convex, towards the axis.

Again, since.

$$\tan.\ \theta = \frac{dy}{dx} = p = \sqrt{\frac{c}{2}} \times \frac{x - \frac{1}{c}}{\sqrt{x - \frac{1}{2c}}},$$ at A, $\tan.\ \theta = \infty$,

$\therefore\ \theta = 90$, and AT_1, is $\perp$ BC; at C, $\tan.\ \theta = \sqrt{\frac{c}{2}} \times$

$$\frac{\frac{2}{c} - \frac{1}{c}}{\sqrt{\frac{2}{c} - \frac{1}{2c}}} = \frac{1}{\sqrt{3}} = \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{\sin. 30^\circ}{\cos. 30^\circ} = \tan\ 30^\circ,\ \therefore\ \theta =$$

30°; $\theta = 0$, or P_2T_2 is parallel to BC, when $x = \text{BM}_2 = \frac{1}{c}$.

N. B. When BT falls below BC, the point P being above that line, BT must be considered negative.

34. This problem may be generalized by stating it, *Required the nature of a curve by the revolution of which about its axis, a surface may be generated, the distance of whose centre of gravity from the vertex is $\frac{1}{n}$th of the abscissa, measured from that vertex.*

The general expression for the distance of the centre of gravity of any solid of revolution from the vertex is

$$\frac{\int y\, xdz}{\int y\, dz} = \frac{x}{n} \text{ by the question.}$$

Hence, we get

$$(n-1)\, xydz = dx \int ydz$$

$$\therefore \frac{ydz}{\int ydz} = \frac{1}{n-1} \cdot \frac{dx}{x}$$

$$\therefore l. \int ydz = \frac{1}{n-1}\, l. x + C = l. x^{\frac{1}{n-1}} + l. c = l. c\, x^{\frac{1}{n-1}}$$

$$\therefore \int ydz = cx^{\frac{1}{n-1}}$$

$$\therefore y\sqrt{dx^2+dy^2} = \frac{c}{n-1} \cdot x^{\frac{2-n}{n-1}}\, dx \; \ldots.\; (a)$$ the general differential equation, which is integrable, in finite terms, in some particular cases.

Let $n = 2$.

$$\text{Then } y\sqrt{dx^2+dy^2} = c \,.\, x^0 dx = cdx$$

$$\therefore y^2 + y^2 \frac{dy^2}{dx^2} = c^2$$

$$\text{Hence } \frac{ydy}{\sqrt{c^2 - y^2}} = dx$$

$$\therefore c' - \sqrt{c^2 - y^2} = x$$

But since, by the question, and the above expression for the distance of the centre of gravity, we suppose $x = 0$ at the vertex, where also $y = 0$, by putting $x = 0$, we get

$$c' = \sqrt{c^2} = c$$

$$\therefore (c - x)^2 = c^2 - y^2$$

$\therefore y^2 = 2cx - x^2$ ...... (*b*) the equation to a circle whose radius is c.

Again, let $n = \dfrac{3}{2}$, or the distance of the centre of gravity be $\dfrac{2}{3}$ of the abscissa,

Then $y\sqrt{dx^2 + dy^2} = 2c\, xdx$; which being homogeneous, put $y = ux$, substitute, &c., and we get

$$1 + \left(u + \frac{xdu}{dx}\right)^2 = \frac{4c^2}{u^2}$$

$$\therefore u + \frac{xdu}{dx} = \frac{\sqrt{4c^2 - u^2}}{u}$$

$$\therefore \frac{udu}{\sqrt{4c - u^2} - u^2} = \frac{dx}{x}$$

and putting $\sqrt{4c^2 - u^2} = v$, substituting, &c., we get

$$lc' + lx = -\frac{1}{2}\, l.\,(v^2 + v - 4c^2) + \frac{1}{2\sqrt{4c^2 + \frac{1}{4}}} \times$$

$$l\,\frac{v + \frac{1}{2} - \sqrt{4c^2 + \frac{1}{4}}}{v + \frac{1}{2} + \sqrt{4c^2 + \frac{1}{4}}}.$$

$$\therefore c'x\sqrt{v^2 + v - 4c^2} = \left(\frac{v + \frac{1}{2} - \sqrt{4c^2 + \frac{1}{4}}}{v + \frac{1}{2} + \sqrt{4c^2 + \frac{1}{4}}}\right)^{\frac{1}{\sqrt{16c^2 + 1}}}$$

But, when $x = 0$, $l.\,c' = -\infty = l.\,0$, $\therefore c' = 0$.

Hence $v + \dfrac{1}{2} - \sqrt{4c^2 + \frac{1}{4}} = 0$

And reducing the equation, we finally obtain

$y = \left(\dfrac{\sqrt{16c^2 + 1} - 1}{2}\right)^{\frac{1}{2}} x$ .... (c) the equation to a straight line.

The surface in this case is, therefore that of a *cone*.

35. To trace the curve whose equation is

$x^2 + y^2 = \frac{b^2 x^2}{2ax - x^2}$, we have $y = \pm \sqrt{x \cdot \frac{b^2 - 2ax + x^2}{2a - x}}$.

Let $y = 0$. Then $x = 0$, or $= a + \sqrt{a^2 - b^2}$ or $= a - \sqrt{a^2 - b^2}$. If $\therefore$ A (Fig. 26,) be the origin of abscissæ, AB $= a$, BC $= \sqrt{a^2 - b^2}$, BC′ $= - \sqrt{a^2 - b^2}$, the curve will pass through each of the points A, C, C′.

Again, $y = \pm \infty$, when $x = 2a$: take $\therefore$ AD $= 2a$, and draw E$e \perp$ AD. DE, De, are asymptotes to the curve.

x cannot be $> 2a$; for then y is imaginary. For the same reason it has no negative values.

Since

$\tan \theta = \frac{dy}{dx} = \frac{ab^2 - 4a^2x + 4ax^2 - x^3}{(2a-x)^{\frac{3}{2}} \sqrt{b^2x - 2ax^2 + x^3}}$, when $x = 0$, or $= a - \sqrt{a^2 - b^2}$, or $= a + \sqrt{a^2 - b^2}$, or at the points A, C′, C, the $\angle \theta = 90°$. $\therefore$ at each of these points the tangent to the curve is $\perp$ line of abscissæ. Hence, when $\sqrt{a^2 - b^2}$ is real, or $a > b$, the curve will be as described.

If $a = b$, the oval C′C will vanish, and the points C, C′, will unite in B.

If a be $< b$, the curve meets the axis in A only.

To find the *area* of the curve when $b = a$, we have

$$\frac{\text{Area}}{2} = \int y dx = \pm \int dx \cdot (a - x) \sqrt{\frac{x}{2a - x}} =$$

$$\pm \int \frac{ax - x^2}{\sqrt{2ax - x^2}} \cdot dx.$$

Let $u = x\sqrt{2ax - x^2}$

Then $du = 2 \cdot \frac{ax - x^2}{\sqrt{2ax - x^2}} dx - a \cdot \frac{a - x}{\sqrt{2ax - x^2}} dx + \frac{a^2 dx}{\sqrt{2ax - x^2}}$. Hence, we find $\pm \frac{\text{Area}}{2} = \frac{x + a}{2} \sqrt{2ax - x^2} - \frac{a}{2} \cdot \text{vers.}^{-1} x$ $\therefore \pm$ Area $= (x + a) \sqrt{2ax - x^2} - a \cdot \text{vers.}^{-1} x$.

Let $x = a$. Then

$Area = 2a^2 - a \,.\, \text{vers.}^{-1} a = 2a^2 - \frac{1}{2} \odot (\text{rad. } a) = \frac{a^2}{2} (4 - \pi)$, or *the difference between a* $\frac{1}{2} \odot$ (rad. $= a$) *and its circumscribing rectangle.*

Let $x = 2a$. Then

$Area = a \,.\, \text{vers.}^{-1} 2a = \frac{a}{2} \,.$ circumference of $\odot$ (rad. $= a$) $=$ Area of that $\odot$.

36. Let AB, Dd (Fig. 27,) be any two diameters of the $\odot$ ADBd $\perp$ one another, and let Dd be indefinitely produced both ways to E and e. Then, any chord AF being drawn intersecting Dd in P, and PM $=$ cos. FB being erected $\perp$ Dd, required the *locus* of the point M.

Let PC $= x$, PM $= y$, and AC $= r$. Then, from similar $\triangle$ we get

$$y = r \,.\, \frac{\text{PF}}{\text{AP}}$$

and the property of the $\odot$ gives

$$\text{PF} = \frac{\text{DP} \,.\, \text{P}d}{\text{AP}} = \frac{\text{CD}^2 - \text{CP}^2}{\text{AP}} = \frac{r^2 - x^2}{\text{AP}}$$

Hence $y = r \,.\, \dfrac{r^2 - x^2}{\text{AP}^2} = r \,.\, \dfrac{r^2 - x^2}{r^2 + x^2}$, the equation to the locus.

Let $x = 0$. Then $y = r$, or the curve passes through B.

Let $x = \pm r$. Then $y = 0$, or the curve passes through D and d.

Let $x = \pm \infty$. Then $y = - r$, or the tangents to the $\odot$ at A are asymptotes to the curve.

Again,

$$\tan.\, \theta = \frac{dy}{dx} = - \frac{4r^3x}{(r^2 + x^2)^2}$$

Let $x = 0$. Then $\tan.\, \theta = 0$, or $\theta = 0$, or the tangent at B is parallel to CE.

Let $x = \pm r$. Then tan. $\theta = \mp 1$, or $\theta = 135°$, or $= 45°$, $\therefore$ the tangent at D is inclined to CE at an angle of 135°, and at d at an angle of 45°.

Again, since

$\frac{d^2y}{dx^2} = - \frac{4r^3}{(r^2 + x^2)^3} \times (r^2 - 3x^2)$, at the points of inflexion

$$r^2 - 3x^2 = 0$$

$$\text{or } x = \pm \frac{r}{\sqrt{3}}$$

$$\text{or } y = \frac{r}{2}.$$

Hence, if CB be bisected in G, and Ff be drawn through G $\perp$ AB meeting the *locus* in M, m; M, m will be points of *contrary flexure*, and the *locus* will be as it is described in the figure.

37. To find the relation between the ordinate and abscissa of a curve (y, x) from the equation,

$$e^{\int\sqrt{dx^2+dy^2}} = e^x - e^{-x}.$$

Differentiating, we have

$$\sqrt{dx^2 + dy^2} \,.\, e^{\int\sqrt{dx^2+dy^2}} = dx \,.\, (e^x + e^{-x})$$

$$\therefore \sqrt{dx^2 + dy^2} = dx \,.\, \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

$$\text{Hence } \frac{dy^2}{dx^2} = \frac{4e^{2x}}{(e^{2x} - 1)^2}$$

$$\text{and } dy = \frac{2e^x dx}{e^{2x} - 1} = \frac{e^x dx}{e^x - 1} + \frac{e^x dx}{e^x + 1}$$

$$\therefore y = l.(e^x - 1) + l.(e^x + 1) + l.c$$

$$\text{or } y = l.c.(e^{2x} - 1,)$$

which expresses the relation required.

38. PN (Fig. 28,) is any ordinate of the ellipse ABa, and Q is a point in it such, that the line CQ joining it and the

centre of the ellipse C, shall always $=$ PN; required the locus of Q.

Let $CN = x$, $QN = y$, $AC = a$, and $BC = b$.

Then $y^2 + x^2 = QC^2 = PN^2 = \frac{b^2}{a^2}.(a^2 - x^2)$ by the equation to the ellipse.

Hence $y^2 = \frac{b^2}{\left(\frac{a^2 b^2}{a^2+b^2}\right)}.\left(\frac{a^2 b^2}{a^2+b^2} - x^2\right)$, which is the equation to an ellipse, whose semi-axes BC, CN′ are b and $\frac{ab}{\sqrt{a^2+b^2}}$ respectively.

The point N′ always falls within the given ellipse, and may be found *geometrically*, by bisecting the $\angle$ ACB by the line CP′, and drawing the ordinate P′N′.

39. The problem may be generalized thus; ACB (Fig. 29,) being any given curve, whose equation is $y' = f.x'$, and CD a straight line of given length, one of whose extremities is always in the line of abscissæ, and the other in the curve, then CD having every possible situation, required the locus of the point P which divides CD in a given ratio.

Let O be the origin of abscissæ of both the curve and the locus, $ON = x'$, $CN = y'$, $OM = x$, $PM = y$, $CD = a$, and $PD = \frac{CD}{m} = \frac{a}{m}$.

Then, from similar $\triangle$, we get

$$y = y' \times \frac{PD}{CD} = f(x').\frac{a}{ma}$$

$$= \frac{1}{m}.f.(ON)$$

But $ON = x - NM = x - (ND - MD) = x - (m-1) \times MD = x - (m-1)\sqrt{\frac{a^2}{m^2} - y^2}$.

$\therefore y = \frac{1}{m}.f.\left\{x - (m-1)\sqrt{\frac{a^2}{m^2} - y^2}\right\}$ ..... (a) which expresses the relation between x and y in the *locus* generally.

Now, in the problem, we have

$$\left.\begin{array}{l} y' = f.\, x' = \sqrt{r^2 - x'^2} \\ m = 2,\, a' = r\sqrt{2}, \end{array}\right\} \therefore \text{ by substitution, we here get}$$

$$y = \frac{1}{2}.\ \sqrt{r^2 - \left(x - \sqrt{\frac{r^2}{2} - y^2}\right)^2}$$

$$\text{Hence } x = \sqrt{r^2 - 4y^2} + \sqrt{\frac{r^2}{2} - y^2}$$

$$= 2\sqrt{\frac{r^2}{4} - y^2} + \sqrt{\frac{r^2}{2} - y^2}$$

the equation required.

To find the area of the curve, we have

$$\int y dx = \int \frac{-2y^2 dy}{\sqrt{\frac{r^2}{4} - y^2}} - \int \frac{y^2 dy}{\sqrt{\frac{r^2}{2} - y^2}}$$

$$\text{Let } y\sqrt{\frac{r^2}{4} - y^2} = u$$

$$\text{Then } du = dy\sqrt{\frac{r^2}{4} - y^2} - \frac{y^2 dy}{\sqrt{\frac{r^2}{4} - y^2}}$$

$$= \frac{dy}{\sqrt{\frac{r^2}{4} - y^2}} - \frac{2y^2 dy}{\sqrt{\frac{r^2}{4} - y^2}}$$

$$\therefore -\int \frac{2y^2 dy}{\sqrt{\frac{r^2}{4} - y^2}} = y\sqrt{\frac{r^2}{4} - y^2} - \int \frac{dy}{\sqrt{\frac{r^2}{4} - y^2}}$$

$$= y\sqrt{\frac{r^2}{4} - y^2} - \frac{2}{r}\int \frac{dy}{\sqrt{1 - \frac{4y^2}{r^2}}}$$

$$= y\sqrt{\frac{r^2}{4} - y^2} - \sin.^{-1}\frac{2y}{r}.$$

Similarly we find

$$-\int \frac{y^2 dy}{\sqrt{\frac{r^2}{2} - y^2}} = \frac{y}{2}\sqrt{\frac{r^2}{2} - y^2} - \frac{1}{2}\sin.^{-1}\frac{\sqrt{2}}{r}y$$

Hence $\int y dx = y \sqrt{\frac{r^2}{4} - y^2} + \frac{y}{2}\sqrt{\frac{r^2}{2} - y^2} - \sin.^{-1}\frac{2y}{r}$
$-\frac{1}{2}\sin.^{-1}\frac{\sqrt{2}.y}{r} + C$; and transferring the origin of abscissæ to (b) that of ordinates, by putting

$$u = r + \frac{r}{\sqrt{2}} - x, \text{ we get}$$

$$\int y du = \sin.^{-1}\frac{2y}{r} + \frac{1}{2}\sin.^{-1}\frac{\sqrt{2}.y}{r} - y\sqrt{\frac{r^2}{4} - y^2} - \frac{y}{2} \times \sqrt{\frac{r^2}{2} - y^2}$$
$=$ area bPM, the correction in this case being $= 0$.

Let $y = \frac{r}{2}$ its greatest value.

Then, the area

$$bao = \sin^{-1}. 1 + \frac{1}{2}\sin.^{-1}\frac{1}{\sqrt{2}} - \frac{r^2}{8}$$

$$= 1 \times \frac{\pi}{2} + \frac{1}{2}.\frac{\pi}{4} - \frac{r^2}{8}$$

$$= \frac{5\pi - r^2}{8} \text{ or } = \frac{5.\odot\,(\text{rad.} = 1) - r^2}{8}.$$

If ACB be a straight line, it will be found, in like manner that the locus is a conic section.

If our limits would permit, we might here resolve the inverse problem; given the *equation to the locus of* P, *to find that of the extremity* C; *or, more generally, given the equations to the* loci *of two points of a straight line, to find that of any other point.*

40. To trace $a. (y - b)^2 = x. (x - a)^2$, let $y - b = u$, and we get

$$u = \pm \frac{1}{\sqrt{a}}. (x - a)\sqrt{x}$$

Let $x = 0$. Then $u = \pm 0$, and the curve passes through the point A (Fig. 30), the origin of u.

Let $x = AB = a$. Then $u = 0$, and the curve passes through B, which is double.

Let $x = \infty$. Then $u = \pm \infty$, and the branches BZ, Bz, are infinite.

The curve is limited at A, because all negative values of x give imaginary values of u.

To find the position of the tangent at singular points, we have $\tan. \theta = \frac{dy}{dx} = \pm \frac{1}{2\sqrt{a}} \cdot \frac{3x-a}{\sqrt{x}}$, in which let

$x = 0$; then, $\tan. \theta = \pm \infty$, and the tangents at A (AT, At) are $\perp$ AC. $\therefore$ the curve at A is concave towards AC.

Let $x = AN = \frac{a}{3}$. Then $\tan. \theta = 0$, and the tangents at P and p are parallel to AC.

Let $x = a$. Then $\tan. \theta = \pm 1$, or the tangents at B are inclined to AC at an angle of 45°.

41. To find the nature of the curve AP (Fig. 31) such that AB being drawn from the vertex making $\angle$ BAC = 45°, and cutting the ordinate PM in Q,

$$PQ : \text{subtangent} = \frac{ydx}{dy} :: a : y.$$

First, we have $\frac{adx}{dy} = PQ = y - QM = y - AM = y - x$.

Let $y - x = u$.

Then, by differentiating and substitution, we get

$$\frac{dy}{a} = \frac{du}{a-u}$$

$$\therefore \frac{y}{a} = C - l.(a-u)$$

$$= la - l.(a+x-y) = l.\frac{a}{a+x-y}$$

Hence $x = y + ae^{-\frac{y}{a}} - a$ ..... (a), which express the relation between x and y.

$\left.\begin{array}{l}\text{Let } x = 0, \text{ then } y = 0 \\ \text{Let } x = \infty, \text{ then } y = \infty\end{array}\right\}$ ∴ the curve passes through A, and is infinite towards C.

Again, $\tan.\theta = \frac{dy}{dx} = \frac{1}{1 - e^{-\frac{y}{2}}}$

Let $x = 0$, or $y = 0$, and ∴ $\tan.\theta = \infty$

∴ the tangent at A is $\perp$ AC.

To find the area of the curve, we have

$$\int ydx = \int ydy - \int e^{-\frac{y}{2}} dy.y$$

$$= \frac{y^2}{2} + 2e^{-\frac{y}{2}}y + \int 2e^{-\frac{y}{2}}dy$$

$$= \frac{y^2}{2} + 2e^{-\frac{y}{2}}y - 4e^{-\frac{y}{2}} + C$$

$$= \frac{y^2}{2} + 2e^{-\frac{y}{2}} \times (y - 2) + 4,$$ which expresses the area comprehended between the vertex, and any ordinate y generally.

42. BM (Fig. 32) being any chord of the ⊙ whose centre is C, and CF any radius cutting the chord in E, and circumference in F; then if EP be drawn $\perp$ BM and $=$ EF, required the locus of the point P.

Draw CAD $\perp$ BM, and put $CF = r$, $CA = a$, $AE = x$, $EP = y$.

Then $y = EF = \pm (CF - CE) = \pm (r - \sqrt{a^2 + x^2})$, the equation to the curve.

The signs $\pm$ show whether E falls within or without the curve.

Let $x = 0$; then $y = r - a = AD$.

Let $y = 0$; then $x = \pm \sqrt{r^2 - a^2} = AM$, or AB.

Let $x = \pm \infty$; then $y = -\infty$.

Again, $\tan.\theta = \frac{dy}{dx} = \mp \frac{x}{\sqrt{a^2 + x^2}}$; ∴ at D the tangent is parallel to BM, and at M or B, $\tan.\theta = \mp \frac{\sqrt{r^2 - a}}{r}$. There are no points of contrary flexure.

If we transfer the origin of co-ordinates to D, and the line of abscissæ to DC, by putting

$$x' = \mathrm{DN} = r - \mathrm{CN} = r - y - a,$$
$$\text{and } y' = \mathrm{PN} = x,$$

which will give, by substitution in the above equation, and proper reductions,

$$y'^2 = 2ax' + x'^2,$$

we easily recognise the locus to be the *equilateral hyperbola*, whose axes are $= a$.

Here we may remark, that as most *known curves* are defined by co-ordinates originating in the vertex, or other points symmetrically placed with respect to the branches, in investigating the nature of a curve, it may frequently aid us to refer the co-ordinates to that point.

43. The problem may be generalized by stating it; *required the curve to which a straight line, cutting from two straight lines which meet in any angle two segments whose sum* = a, *is always a tangent.*

Let BC, B′C′ (Fig. 33,) be any two positions immediately consecutive of the line, cutting off

$\mathrm{AB} + \mathrm{AC} = \mathrm{AB}' + \mathrm{AC}' = a$, and intersecting in the point P.

Let the equation of the line AB, (x and y being measured from A along AX, and AY) be

$$y = \mathrm{M}x + \mathrm{N} \;..... (a)$$

$$\text{Then AC} = -\frac{\mathrm{N}}{\mathrm{M}}, \text{ and AB} = \mathrm{N}$$

$$\therefore -\frac{\mathrm{N}}{\mathrm{M}} = a - \mathrm{AB} = a - \mathrm{N}$$

$$\therefore \mathrm{M} = \frac{\mathrm{N}}{\mathrm{N}-a}, \text{ and the equation } (a) \text{ becomes}$$

$$y = \frac{\mathrm{N}}{\mathrm{N}-a} . x + \mathrm{N} \;..... (b)$$

Now at the point P, the co-ordinates of the two lines CB, C′B′, being precisely the same, the variation, due to their change of

position, that will take place in equation (b), must arise from N only; and differentiating on that supposition, we get

$$\left(\frac{dN}{N-a}-\frac{NdN}{(N-a)^2}\right)x+dN=0$$

Hence $N=a\pm\sqrt{ax}$

$$\therefore\ M=\frac{\sqrt{x}\pm\sqrt{a}}{\sqrt{x}}$$

And substituting in equation (a) we get, after proper reductions,

$$y=x\pm 2\sqrt{ax}+a=(\sqrt{x}\pm\sqrt{a})^2\ \ldots..\ (c)$$

which, since P is evidently a point in the curve, is the equation of the curve (viz. a Parabola) referred to the co-ordinates AX, AY.

This Theory, which is closely allied to that of *particular solutions* in the Integral Calculus, is worthy the attention of the Student.

The next problem affords another illustration of it.

44. Given the angle A (Fig. 33) and area cut off by CB, to find the curve to which CB is always a tangent.

Let the line C′B′ be the immediately consecutive position of CB, and let CB be denoted, as in the preceding problem, by

$$y=Mx+N\ \ldots..\ (d)$$

Let, also, the given area ABC $=a^2$.

Then, since $AC=-\frac{N}{M}$, $AB=N$

And $a^2=\triangle\,ABC=\frac{AC\times BG}{2}$

$$=\frac{AC\times AB\sin.\angle A}{2}=\frac{-N^2\sin.A}{2M},$$

And $\therefore\ M=-\frac{N^2\sin.A}{2a^2}$, by substituting in equation (d) we get

$$y=-\frac{N^2\sin.A}{2a^2}x+N\ \ldots..\ (f)$$

Now, supposing CB to undergo a minute variation in position, x and y, at its intersection with the curve, will remain the same,

and N (being a function of the angle PCX) alone will vary. Taking $\therefore$ the differential of (f) on that supposition, we get

$$- \frac{\mathrm{N}d\mathrm{N} \times \sin. \mathrm{A}}{a^2} x + d\mathrm{N} = 0$$

$$\therefore \mathrm{N} = \frac{a^2}{x \sin \mathrm{A}}$$

$$\text{and } \therefore \mathrm{M} = - \frac{a^2}{2x^2 \sin. \mathrm{A}};$$

Hence equation (d) becomes, by substitution,

$$y = - \frac{a^2}{2x^2 \sin. \mathrm{A}} x + \frac{a^2}{x \sin. \mathrm{A}}$$

$$\text{or } xy = \frac{a^2}{2 \sin. \mathrm{A}} \,\ldots\ldots\, (g)$$

the equation to the curve, which is therefore an *hyperbola* whose asymptotes are AX, AY.

If $\angle$ A = a right-angle, the curve is a rectangular hyperbola.

45. The section of a prolate spheroid made by a plane passing through the focus of the generating ellipse, is an ellipse having the same focus.

The truth of this proposition may be shewn by either of the methods explained in pages 8, &c., vol. ii. As the student will find little or no difficulty in using either of them, beyond that of actual computation in obtaining an equation of the second degree, and afterwards discussing it according to *Lacroix Traité de Trig.*, pp. 156, &c., and other elementary writers, we shall prefer giving a solution founded on principles different from those referred to, and tending to abridge the usual labour in such inquiries.

Let DPE (Fig. 34) be the spheroid generated by the revolution of the $\frac{1}{2}$. ellipse DE′E about its axis-major DE, and let DPE′QD′ be the intersection of a plane, passing through the focus S, with the surface of the spheroid; D′PE′QD′ is an ellipse whose focus is S.

For, supposing P to be the intersection of the curve D′PQ with the generating ellipse in any position DPE, SP will be a radius-vector, issuing from the pole S, to both curves; and D′E′ being the intersection of the plane of D′PQ with that of the ellipse DED′, given in position, ∠ PSE′ and ∠ PSE will be the corresponding angles. Take SB = SA = SC = 1, and in the planes D′PE′, DPE, DE′E describe the respective arcs BA = θ, BC = α, AC = β.

Now, by trig. we have

$$\cos. \alpha = \sin. \beta \cos. A. \sin. \theta + \cos. \beta. \cos. \theta,$$

And $\rho = SP = \frac{a.(1-e^2)}{1-e\cos.\alpha}$, where $a = \frac{\text{axis-major}}{2}$, and $e = \frac{\text{eccentricity}}{a}$ of the ellipse DPE (see any *good* work on conic sections.)

∴ substituting for cos. α, we have,

$$\rho = \frac{a.(1-e^2)}{1-e\sin.\beta\cos.A.\sin.\theta - e\cos.\beta.\cos.\theta}, \ldots (a),$$ an equation between ρ and θ, expressing the nature of the section D′PE′Q.

Now to recognise the precise form of the curve D′PE′Q, represented *in plano* by Fig. 34. *a*, we will transfer the origin of θ from Se' to the maximum value of SP *viz.*, Se, by putting $\theta' = \theta + s$, s being the interval between Se' and Se.

$$\text{Now } \frac{d\rho}{d\theta} = \frac{a(1-e^2)\,e\,(\cos. A \sin. \beta \cos. \theta - \cos. \beta \sin. \theta)}{(1 - e\sin.\beta\cos. A. \sin. \theta - e\cos. \beta\cos. \theta)^2} \mp 0,$$

gives $\tan. \theta = \cos. A. \tan. \beta = \tan. (-s) = -\tan. s.$

$$\therefore \sin. \theta = \sin. (\theta' - s) = \cos. s\,(\sin. \theta' - \cos. \theta' \tan. s)$$
$$= \cos. s\,(\sin. \theta' + \cos. A. \tan. \beta. \cos. \theta'),$$
$$\text{and } \cos. \theta = \cos. (\theta' - s) = \cos. s\,(\cos. \theta' + \sin. \theta' \tan. s)$$
$$= \cos. s\,(\cos. \theta' - \cos. A \tan. \beta. \sin. \theta').$$

Hence, substituting in equation (*a*), we get,

$$\rho = \frac{a.(1-e^2)}{1 - e\cos.\beta\cos.(s)(1+\cos.^2 A.\tan.^2\beta)\cos.\theta'}$$

$$\text{But } \tan.^2 s = \frac{1-\cos.^2 s}{\cos.^2 s} = \cos.^2 A. \tan.^2 \beta,$$

$$\therefore \cos. s = \frac{1}{\sqrt{1+\cos.^2 A. \tan.^2 \beta}}$$

$$\therefore \rho = \frac{a(1-e^2)}{1-e\cos.\beta\sqrt{1+\cos.^2 A.\tan.^2\beta}\times\cos.\theta'},$$ which being of the form

$\rho' = \dfrac{a'(1-e'^2)}{1-e'\cos.\theta'}$ shews that the curve D′PE′Q is an *ellipse* whose focus is S, whose $\dfrac{\text{eccentricity}}{\frac{1}{2}\,(\text{axis major})} = e\cos.\beta\sqrt{1+\cos.^2 A.\tan^2\beta}$

and $\dfrac{\text{axis major}}{2} = \dfrac{a.(1-e^2)}{1-e^2\cos.^2\beta\,(1+\cos.^2 A.\tan.^2\beta)}$.

46. Given two ordinates β, β', and the difference of the corresponding abscissæ δ, of the logarithmic curve, to construct it.

The equation to the logarithmic curve being

$$y = a^x,$$

if we can determine a in terms of known quantities β, β', δ, the construction will be effected by assuming any abscissa at pleasure, and finding the corresponding ordinate from the equation.

Let u be the abscissa corresponding to β; then $u+\delta$ is the abscissa due to β', and by the equation we have

$$\beta = a^u,\ \beta' = a^{u+\delta};$$

$$\therefore a^\delta = \frac{\beta'}{\beta}$$

$$\therefore a = \left(\frac{\beta'}{\beta}\right)^{\frac{1}{\delta}}.$$

Hence $y = \left(\dfrac{\beta'}{\beta}\right)^{\frac{x}{\delta}}$, which gives the construction required.

47. Required the curve whose perpendicular, drawn from a given point upon the tangent, is constant.

Let the line of abscissæ AM pass through the given point S. Let also AM $= x$, PM $= y$,

$AS = a$, and $Sy = b = \perp$ upon the tangent PT.

Then from similar Δ we have

$$b : ST :: y : PT$$

$$:: \dot{y} : \sqrt{y^2 + \frac{y^2\,dx^2}{dy^2}}$$

$$:: 1 : \sqrt{1 + \frac{dx^2}{dy^2}}.$$

$$\therefore b\sqrt{1 + \frac{dx^2}{dy^2}} = ST = MT - SM.$$

$$= \frac{ydx}{dy} - x + a.$$

Hence $y = b.\sqrt{1 + \frac{dy^2}{dx^2}} + (x - a)\frac{dy}{dx}$,

or $y = b.\sqrt{1+p^2} + (x - a)\,p$....($a$) which being reducible to *Clairaut's Formula*, may be integrated as follows:

$$dy = \frac{bpdp}{\sqrt{1 + p^2}} + p.dx + (x - a)\,dp.$$

But $p.dx = y$.

$$\therefore \left(\frac{bp}{\sqrt{1 + p^2}} + x - a\right) dp = 0,$$

$$\left.\begin{array}{l}\therefore dp = 0 \\ \text{or } \dfrac{bp}{\sqrt{1+p^2}} + x - a = 0\end{array}\right\} \quad \left.\begin{array}{l}\therefore p = c \\ \text{or } p = -\dfrac{x-a}{\sqrt{b^2-(x-a)^2}}\end{array}\right\}$$

The first value gives, by substituting in ($eq.a$), the *general solution*

$$y = b\sqrt{1 + c^2} + c\;(x - a),....(b)$$

The other gives, after proper reductions,

$$y = \frac{b^2 - (x-a)^2}{\sqrt{b^2 - (x-a)^2}} = \sqrt{b^2 - (x-a)^2}$$

or $y^2 = b^2 - a^2 + 2ax - x^2$...(c) an equation involving no arbitrary constant and not deducible from (b), and therefore, a particular solution. This latter result evidently shews the curve to be a *circle*. It is, in fact, the *locus* of the intersections of all the

straight lines made by giving every possible value to the constant c in the *general solution* (*b*). See No. 44, Vol. II.

48. Given two distances r, r' from the focus of a parabola and the angle between them α, to construct it.

Let the angular distance of r from the axis be θ; then the $\angle$ between r' and the axis is $\theta + \alpha$. Let also $p =$ the parameter of the parabola, to be determined.

Now the polar equation to the parabola, deduced very readily from $y^2 = px$, being

$$\rho = \frac{p}{2} \cdot \frac{1}{1 + \cos. \phi} = \frac{p}{4} \cdot \frac{1}{\cos.^2 \frac{\phi}{2}}, \text{ we have}$$

$$r = \frac{p}{4 \cos.^2 \frac{\theta}{2}}$$

$$\text{and } r' = \frac{p}{4 \cos.^2 \frac{\theta + \alpha}{2}}$$

$$\therefore \sqrt{\frac{r}{r'}} = \frac{\cos. \frac{\theta + \alpha}{2}}{\cos. \frac{\theta}{2}}$$

$$= \frac{\cos. \frac{\theta}{2} . \cos. \frac{\alpha}{2} - \sin. \frac{\theta}{2} . \sin. \frac{\alpha}{2}}{\cos. \frac{\theta}{2}}$$

$$= \cos. \frac{\alpha}{2} - \sin. \frac{\alpha}{2} \tan. \frac{\theta}{2}.$$

$$\therefore \tan. \frac{\theta}{2} = \left(\cos. \frac{\alpha}{2} - \sqrt{\frac{r}{r'}}\right) \frac{1}{\sin. \frac{\alpha}{2}} = m$$

$$\text{Hence } \cos.^2 \frac{\theta}{2} = \frac{1}{1 + m^2} = \frac{1 - 2 \cos. \frac{\alpha}{2} . \sqrt{\frac{r}{r'}} + \frac{r}{r'}}{\sin.^2 \frac{\alpha}{2}}$$

$$\therefore \frac{p}{4} = \frac{r}{\sin.^2 \frac{\alpha}{2}} \cdot \left(1 - 2\cos. \frac{\alpha}{2} \cdot \sqrt{\frac{r}{r'}} + \frac{r}{r'}\right)$$

Hence the equation between the radius-vector ρ and its $\angle$ of inclination to the axis φ, which affords the required construction, is

$$\rho = \frac{r}{\sin.^2 \frac{\alpha}{2}} \cdot \left(1 - 2\cos. \frac{\alpha}{2} \cdot \sqrt{\frac{r}{r'}} + \frac{r}{r'}\right) \frac{1}{\cos.^2 \frac{\varphi}{2}}.$$

49. A and B being two given points (Fig. 36), and C another such that AC : CB :: n : 1; then the *locus* of C is a circle.

For, take AD : DB :: n : 1

and DN $= x$, CN $= y$, AB $= a$.

Then $AC^2 + CB^2 = (n^2 + 1)\,CB^2$

$= (n^2 + 1)(y^2 + BN^2) = AN^2 + BN^2 + 2y^2$

$\therefore n^2 - 1)\,y^2 + n^2.(BD - x)^2 = (AD + x)^2.$

But $BD = \frac{a}{n+1}$, and $AD = \frac{na}{n+1}$.

Hence, by substitution and reduction, we get

$$y^2 = \frac{2na}{n^2 - 1}\,x - x^2$$

which is the equation to a circle whose radius is $\frac{na}{n^2 - 1}$.

50. To find the polar equation to an ellipse, the pole being in the centre.

The equation between the rectangular co-ordinates x, y, measured from the centre, is

$$y^2 = \frac{b^2}{a^2}(a^2 - x^2) \;....\; (a)$$

a and b being the principal semi-axes.

Let ρ be the radius-vector measured from the centre, and θ its

inclination to the $\frac{1}{2}$ axis a; then $x = \rho \cos.\theta, y = \rho \sin.\theta$, and by substitution in equation (a) we have

$$\rho^2 \sin.^2 \theta = \frac{b^2}{a^2}(a^2 - \rho^2 \cos.^2 \theta)$$

Hence $\rho^2 = \dfrac{a^2 b^2}{a^2 - (a^2 - b^2)\cos.^2\theta}$

Now it is well known that

$a^2 - b^2 =$ (dist. between focus and centre)^2

$=$ (eccentricity)^2 $= (ae)^2$ by supposition.

$\therefore \rho^2 = \dfrac{a^2 b^2}{a^2 - a^2 e^2 \cos.^2 \theta} = \dfrac{b^2}{1 - e^2 \cos.^2\theta}$ the equation required.

51. To trace the curve whose equation is

$$y^3 = ax^2 + x^3.$$

Let $x = 0$, and $- a$. Then the corresponding values of y are each $= 0$. Hence, putting AB (Fig. 37), $= - a$, the curve will pass through A and B.

Again, let $x = \pm \infty$. Then $y = \pm \infty$, or we have two infinite branches AQ, Bq lying on different sides of the axis Cc.

To find the greatest ordinate of the branch AMB, we have

$$\frac{dy}{dx} = \frac{2a + 3x}{3x^{\frac{1}{3}}(a + x)^{\frac{2}{3}}} = 0$$

which gives $x = -\dfrac{2a}{3}$, and $\therefore y = (ax^2 + x^3)^{\frac{1}{3}} = \dfrac{a}{3} . 4^{\frac{1}{3}}$ Hence, taking Am $= -\dfrac{2}{3}$ AB and drawing mM $= \dfrac{a}{3} . 4^{\frac{1}{3}}$ at right angles to Cc, we obtain the position and magnitude of the greatest ordinate required.

To find the asymptotes, we have

$$y = x.(1 + ax^{-1})^{\frac{1}{3}} = x(1 + \frac{a}{3}x^{-1} - \frac{1}{9}a^2 x^{-2} + \&c.)$$

$$= x + \frac{a}{3} - \frac{a^2}{9}x^{-1}, \&c.$$

Hence, the equation to the *rectilinear-asymptote*, (see *Francœur, Mathemat. Pures*, p. 323, or *Stirling, Lineæ Tertii Ordinis Newtonianæ*, p. 48,) is

$$y' = x' + \frac{a}{3}, \ x' \text{ being measured from A.}$$

Let $y' = 0$. Then $x = -\frac{a}{3} = \text{AD}$ (Fig. 37).

Let $x' = 0$. Then $y' = \frac{a}{3} = \text{AR}$.

Therefore DR being joined and produced indefinitely will give the asymptotes DE, D*e* to the respective branches AQ, B*q*.

For the common method of finding the asymptotes to this curve, see *Vince's Fluxions*, p. 52.

Since $\tan.\theta = \frac{dy}{dx} = \frac{2a+3x}{3x^{\frac{1}{3}}(a+x)^{\frac{2}{3}}}$, θ being the inclination of the tangent to the axis. By substituting the *singular* values of x, *viz.*, $0, -a$, and $-\frac{2a}{3}$, the corresponding values of $\tan.\theta$, are ∞, ∞ and 0. Consequently the tangents at A and B are $\perp$ to the axis, and that at M is parallel to it.

52. To trace the curve whose equation is

$$y = \pm x\sqrt{\frac{a^2 - x^2}{a^2 + x^2}}.$$

Let $y = 0$; then $x = 0$ or $= \pm a$. Take therefore $\text{AB} = +a$, $\text{A}b = -a$ (Fig. 1), in the same straight line, and the curve passes through the points A, B, b.

Again, B and b are the limits of x, since y is imaginary for every value of $x > a$.

Again, putting $x = 0$ and $\pm a$, in

$$\tan.\theta = \frac{dy}{dx} = \frac{a^4 - 2a^2x^2 - x^4}{(a^2+x^2)^{\frac{3}{2}}\sqrt{a^2 - x^2}} \text{ we get}$$

$$\tan.\theta = \pm 1 \text{ and } \frac{1}{0}, = \begin{Bmatrix}\tan.\ 45^\circ \\ \tan.\ 135^\circ\end{Bmatrix} \text{and } \tan.\ 90^\circ.$$

$\therefore$ The point A is double, its two tangents being inclined to the axis at the angles 45° and 135° respectively, and the tangents at B,b are each $\perp$ axis.

The greatest ordinates PM, pm are given by putting $a^4 - 2a^2x^2 - x^4 = 0$, and thence obtaining $x = \pm a\sqrt{\sqrt{2}-1} =$ AP or Ap; which being substituted in the given equation afford $y = \pm a\sqrt{\dfrac{\sqrt{2}-1}{\sqrt{2}}}$ for the pairs of maximum ordinates at P and p, viz, PM, PM$'$ and pm, pm' respectively.

Again, to find the area, we have

$\int ydx = \int xdx\sqrt{\dfrac{a^2-x^2}{a^2+x^2}}$, and putting $a^2 + x^2 = u^2$, and substituting, we get

$$\int ydx = \int du\sqrt{2a^2 - u^2}. \text{ Again let}$$

$$P = u.\sqrt{2a^2 - u^2}, \text{ and we finally get}$$

$$\int ydx = \frac{P}{2a^2+1} + \frac{2a^2}{2a^2+1}\int\frac{du}{\sqrt{2a^2-u^2}}$$

$$= \frac{\sqrt{a^4-x^4}}{2a^2+1} + \frac{2a^2}{2a^2+1}.\int\frac{\frac{du}{\sqrt{2}}}{\sqrt{a^2-\frac{u^2}{2}}}$$

$$= \frac{\sqrt{a^4-x^4}}{2a^2+1} + \frac{2a^2}{2a^2+1}.\sin.^{-1}\frac{\sqrt{a^2+x^2}}{\sqrt{2}} + C$$

$$\text{But } C = -\frac{a^2}{2a^2+1} - \frac{2a^2}{2a^2+1}.45^\circ$$

$$\therefore \int ydx = \frac{\sqrt{a^4-x^4}-a^2}{2a^2+1} + \frac{2a^2}{2a^2+1}.\left(\sin.^{-1}\frac{\sqrt{a^2+x^2}}{\sqrt{2}} - 45^\circ\right)$$

the general value of the area A M P.

Let $x = a$; then the area

$$AMB = -\frac{a^2}{2a^2+1} + \frac{2a^2}{2a^2+1}(90^\circ - 45^\circ)$$

$$= -\frac{a^2}{2a^2+1} + \frac{a}{2(2a^2+1)}.180^\circ \times a$$

$$= -\frac{a^2}{2a^2+1} + \frac{a}{2(2a^2+1)} \times (\text{area of } \odot, \text{rad.} = a.)$$

Hence the whole area AMBM'A$m'bm$

$$= \frac{2a}{2a^2+1} \times (\odot \text{ rad.} = a) - \frac{4a^2}{2a^2+1} = \frac{2}{2a^2+1} \times (a.\odot - a^2).$$

53. To find the equation to the *curve of pursuit.*

Let T, moving uniformly along the straight line TM with a velocity $= v$, be pursued by P moving also uniformly with a velocity $= \frac{av}{n}$; and let PT, be that path of P which is $\perp$ TM.

Let also P'T' be any other contemporaneous positions of P,T.

Put PT $= a$, TN $= x$, PN $= y$, and PP' $= s$.

Then, since P'T' is evidently a tangent at P' and PN is decreasing, we have

$$\text{TT}' = x + \text{NT}' = x - \frac{ydx}{dy},$$

and by the question

$$s = \text{PP}' = \frac{\text{TT}'}{n}, \text{ since they move uniformly.}$$

$$\therefore\ ns = x - \frac{ydx}{dy}$$

Hence, making dy constant, we have

$$n. \sqrt{dx^2+dy^2} = dx - dx - y\frac{d^2x}{dy}$$

$$= -y.\frac{d^2x}{dy}.$$

Let $\frac{dx}{dy} = p$.

Then $-ny\frac{dp}{dy} = .\sqrt{1+p^2}$

$$\therefore -n\frac{dy}{y} = \frac{dp}{\sqrt{1+p^2}}$$

and $l.c - l.y^n = l.(p + \sqrt{1+p^2})$.

$$\therefore \frac{c}{y^n} = p + \sqrt{1+p^2}$$

Let $x = 0$; then $y = a$

and $p = \dfrac{dx}{dy} = \cot.\ \theta = \cot.\ 90^\circ = 0$,

$\therefore c = a^n$.

Hence $\left(\dfrac{a}{y}\right)^n - p = \sqrt{1+p^2}$

and squaring both sides, we get

$$2\,a^n p = \frac{a^{2n}}{y^n} - y^n;$$

$$\therefore 2a^n dx = a^{2n} y^{-n} dy - y^n dy,$$

and integrating, on the supposition that dy is negative, we obtain

$$2a^n x = -\frac{a^{2n}}{1-n}.y^{1-n} + \frac{y^{n+1}}{n+1} + \text{C}.$$

$$= -\frac{a^{2n}}{1-n}\,y^{1-n} + \frac{y^{n+1}}{1+n} + \frac{2a^{n+1}\,n}{1-n^2},$$ the general equation of the *Curve of Pursuit.*

Let $n = \dfrac{1}{3}$ and $y = 0$.

Then $\text{TM} = \dfrac{a.\frac{1}{3}}{1-\frac{1}{9}} = \dfrac{3a}{8}$, the distance described by T before P overtakes it.

The problem admits of being further generalized. T might be made to move in a curve, and with a velocity varying according to a given law, P following it likewise with a variable velocity. The determination of the problem under these circumstances we recommend, as an useful exercise.

54. To transform the equation to the Lemniscata,

$$(x^2 + y^2)^2 = a^2.(x^2 - y^2).$$

from rectangular to polar co-ordinates,

Make A, (Fig. 1) the origin of the rectangular, the pole of the polar co-ordinates, and put $\text{AM} = \rho$, and $\angle\text{MAB} = \theta$. Then, since $y = \text{AM. sin.}\ \theta = \rho.\ \sin.\ \theta$, and $x = \text{AP} = \rho \cos.\ \theta$, by substituting in the given equation, we get

$$(\rho^2 \sin.^2 \theta + \rho^2 \cos.^2 \theta)^2 = a^2 (\rho^2 \cos.^2 \theta - \rho^2 \sin.^2 \theta)$$

$$\text{or } \rho^2 = a^2 (\cos.^2 \theta - \sin.^2 \theta) = a^2. \cos. 2\theta.$$

$$\therefore \rho = a. \sqrt{\cos. 2\theta}, \text{ the equation required.}$$

55. The loci of the corresponding values of x and y, in an homogeneous equation, are straight lines.

Let the sum of the indices in each term of the equation determining the degree of its homogeneity be m. Then if we divide each term of the equation by x^m, the dimension of each term will obviously be zero, i.e., each term will be of the form $P \times \left(\frac{y}{x}\right)^p$. Hence, after dividing by the coefficient of $\left(\frac{y}{x}\right)^m$ and ordering the terms according to the powers of $\frac{y}{x}$, the equation will be reduced to the form,

$$\left(\frac{y}{x}\right)^m + A_1\left(\frac{y}{x}\right)^{m-1} + A_2\left(\frac{y}{x}\right)^{m-2} + \ldots A_{m-1}\left(\frac{y}{x}\right) + A_m = 0.$$

Let the roots of this equation be denoted by $a_1, a_2, a_3 \ldots a_m$; then, by the general theory of equations, we have

$$\left(\frac{y}{x} - a_1\right). \left(\frac{y}{x} - a_2\right). \left(\frac{y}{x} - a_3\right) \ldots \left(\frac{y}{x} - a_m\right) = 0$$

$$\text{or } \frac{y}{x} - a_1 = 0, \frac{y}{x} - a_2 = 0, \ldots \frac{y}{x} - a_m = 0,$$

or $y = a_1 x$, $y = a_2 x$, ... $y = a_m x$, which are equations to straight lines. Consequently the corresponding loci of x and y are straight lines. Q.E.D.

Ex. To find what lines are defined by

$y^3 - 2xy^2 + x^3 = 0$, we have

$\left(\frac{y}{x}\right)^3 - 2\left(\frac{y}{x}\right)^2 + 1 = 0$, whose roots being $1, \frac{1+\sqrt{5}}{2}$ and $\frac{1-\sqrt{5}}{2}$, we have

$y = x, y = \frac{1+\sqrt{5}}{2}x$, and $y = \frac{1-\sqrt{5}}{2}x$, which define three straight lines issuing from the origin of abscissæ, and inclined to the line of abscissæ at the angles $45°$, $\tan.^{-1}\frac{1+\sqrt{5}}{2}$, and $\tan.^{-1}\frac{1-\sqrt{5}}{2}$ respectively.

56. **The rectangle under the two segments of the tangents at the extremities of the axis major of an ellipse, cut off by any other tangent, is constant.**

Let APM (Fig. 39) be an ellipse whose axis-major $AV = 2a$. Vt, AT and Tt being tangents, we have $AT \times Vt = \text{const.}$

For $Vt = VQ.\tan.Q = (MQ - MV)\tan.Q.$

$$= y - x\tan.Q,$$

and $AT = (AV + VQ)\tan.Q.$

$$= y + (2a - x)\tan.Q.$$

Hence $AT \times Vt = y^2 + 2(a-x)y\tan.Q - (2ax - x^2)\tan.^2 Q.$

But $y^2 = \frac{b^2}{a^2}.(2ax - x^2)$ b being the axis-minor,

$$\text{and } \tan.Q = \frac{dy}{dx} = \frac{b}{a}.\frac{a-x}{\sqrt{2ax - x^2}}$$

$\therefore$ by substitution, we get

$$AT \times Vt = \frac{b^2}{a^2}\{2ax - x^2 + 2(a-x)^2 - (a-x)^2\}$$

$$= b^2 \text{ which is constant.}$$

57. **Conversely, if $AT \times Vt$ be constant, and $= b^2$, AT, Vt being $\perp$ AV, then the curve to which Tt is perpetually a tangent, is an ellipse.**

By the preceding problem, we have

$$AT \times Vt = y^2 + 2.(a-x)y\tan.Q - (2ax - x^2)\tan.^2Q$$

And $\therefore$ by supposition,

$$y^2 + 2.(a-x)y\tan.Q - (2ax - x^2)\tan.^2Q = b^2 \ldots\ldots (a)$$

Now, supposing Tt to receive a minute change of position, y and x will remain the same, whilst tan. Q will undergo a variation. Differentiating, therefore, on that hypothesis, we get

$$2.(a-x)y \times d\,\text{tan. Q} - 2.(2ax - x^2)\,\text{tan. Q.}\,d\,\text{tan. Q} = 0.$$

$$\left.\begin{array}{l} \therefore\ d\,\text{tan. Q} = 0 \\ \text{Or } (a-x)\,y = (2ax - x^2)\,\text{tan. Q.} \end{array}\right\}$$

$$\begin{array}{ll} \therefore\ \text{tan. Q} = \text{const.} = c & (1) \\ \text{And tan. Q} = \dfrac{(a-x)\,y}{2ax - x^2} & (2) \end{array}$$

Substituting the value of (1) in equation (a) we get, after solving the resulting quadratic,

$$y = (a-x)\,c \pm \sqrt{b^2 + c^2a^2}$$

for the *general solution*, which defines all the lines Tt, T$'t'$, &c., which can be made by giving to c all possible values, and by substituting for tan. Q in (a) its value in equation (2), we get, after proper reduction,

$$y^2 = \frac{b^2}{a^2}.\sqrt{2ax - x^2}$$

the *particular solution*, defining the *locus* of the intersections of Tt, T$'t'$,&c., which is, therefore, an ellipse whose semi-axes are a and b.

If for tan. Q we put its equivalent $\frac{dy}{dx}$, equation (a) will become

$$y^2 + 2\,(a-x)y\frac{dy}{dx} - (2ax - x^2)\frac{dy^2}{dx^2} = b^2,$$

And differentiating, all the terms will be destroyed by opposition of signs, except

$$2\,(a-x)\,y\,\frac{d^2y}{dx} - 2\,\frac{dyd^2y}{dx^2}\,(2ax - x^2) = 0$$

$$\left.\begin{array}{l} \therefore\ \dfrac{d^2y}{dx} = 0 \text{ or } \dfrac{dy}{dx} = c \\ \text{And } \dfrac{dy}{dx} = \dfrac{a-x}{2ax - x^2} \end{array}\right\} \text{which give the same results as before.}$$

58. To find the curve which cuts off a third part from all circular arcs described upon the same chord.

Let SPT (Fig. 40) be any one of these arcs described upon the chord ST.

Suppose $SP = PP' = PT$

and $SA = AA' = AT$.

Join SP, PP', PT, and from C the middle of ST draw $CG \perp ST$. Then it is easily proved that chord $SP =$ chord $PP' =$ chord PT, $PG = GP'$ and $\therefore SP = 2PG$, and that $SA = 2AC$.

Let $AC = c$, $PN = y$, and (since the curve evidently passes through A) $AN = x$.

Then $SP^2 = (2PG)^2 = (2NC)^2 =$ also $SN^2 + PN^2$.

But $4NC^2 = 4(c+x)^2 = 4c^2 + 8cx + 4x^2$

And $SN^2 = (2c - x)^2 = 4c^2 - 4cx + x^2$

$\therefore y^2 = PN^2 = 12cx + 3x^2 = 3(4cx + x^2)$.

Put $a = 2c$

and $\frac{b^2}{a^2} = 3, \quad \therefore b^2 = 12c^2$

Then $y^2 = \frac{b^2}{a^2} \cdot (2ax + x^2) = \frac{12c^2}{2c} \cdot (4cx + x^2)$, the equation to an *hyperbola*, whose semi-axes are $2c$ and $2c\sqrt{3}$.

That the *Hyperbola* is the curve required, is also evident from the property of that curve with regard to its directrix. If S be the focus, and CG the directrix; then SP : PG in a constant ratio, viz. 2 : 1. Consequently AP is an hyperbola.

59. Find the equation to a straight line any how disposed in space, and thence deduce the equation to another passing through a given point, at right angles to the former.

Let the straight line B, (Fig. 41) whose equation is required, be referred to the rectangular co-ordinates AX, AY, AZ, and suppose it to meet the plane YAX in B. Draw from P, any point in the line BL, $Pp \perp$ plane YAX, and join Bp. Draw, also pm, $pn \perp$ AY, AX, and through B draw bC, aD parallel to pm, pn, respectively.

Then, putting $Pp = z$, $pm = An = x$, $pn = Am = y$, $\angle PBp$ = inclination of BL to the plane YAX $= \gamma$, and $\angle pBC =$ its

inclination to the plane ZAX $= \alpha$, and the co-ordinates Bb, Ba, of the given point B, a and b, we have
$x = \text{A}n = \text{A}a + an = a + \text{BC} = a + \text{B}p \cos.\alpha = a + \cos.\alpha.\cot.\gamma \times z$, and $y = \text{A}m = \text{A}b + bm = b + \text{BD} = b + \text{B}p, \sin.\alpha = b + \sin.\alpha.\cot.\gamma \times z$; or the two equations, which evidently determine the position of BL in space (since they do any point of it, P) are

$$\left.\begin{array}{l} x = a + z.\cos.\alpha.\cot.\gamma \\ y = b + z.\sin.\alpha.\cot.\gamma \end{array}\right\} \ldots\ldots (a)$$

Again, let another straight line B′Q, passing through the given point P′. meet BP at right-angles in Q, and the plane YAX in B′. Then, calling its co-ordinates x', y', z', by the preceding determination, we get

$$\left.\begin{array}{l} x' = a' + z'.\cos.\alpha'.\cot.\gamma' \\ y' = b' + z'.\sin.\alpha'.\cot.\gamma' \end{array}\right\} \ldots\ldots (b)$$

in which it remains for us to obtain the values of a', b', $\cos.\alpha'$, $\sin.\alpha'$, $\cot.\gamma'$, in terms of the known constants, a, b, $\cos.\alpha$, $\sin.\alpha$. $\cot.\gamma'$.

Now, since B′Q is $\perp$ BP, if we make Qq $\perp$ plane YAX, and join B′q, B′q will be $\perp$ Bp. Hence the inclination of BQ to the plane ZAX $=$ B′sC $= \alpha + 90° = \alpha'$. Therefore, $\sin.\alpha' = \sin.(\alpha + 90°) = \cos.\alpha$, and $\cos.\alpha' = \cos.(\alpha + 90)° = -\sin.\alpha$; and by substitution, equations (b) becomes

$$\left.\begin{array}{l} x' = a' - z'.\sin.\alpha.\cot.\gamma' \\ y' = b' + z'.\cos.\alpha.\cot.\gamma' \end{array}\right\} \ldots\ldots (c)$$

Again, supposing the co-ordinates of the given point P to be m, n, r, we have, by equations (c)

$$\left.\begin{array}{l} m = a' - r\sin.\alpha.\cot.\gamma' \\ n = b' + r\cos.\alpha.\cot.\gamma' \end{array}\right\}$$

Hence, deducing a' and b', and substituting in (c) we get

$$\left.\begin{array}{l} x' = m - (z' - r)\sin.\alpha.\cot.\gamma' \\ \text{and } y' = n + (z' - r)\cos.\alpha.\cot.\gamma' \end{array}\right\} \ldots\ldots (d).$$

Lastly, since the point Q is common to both lines, at that point we have, $x' = x$, $y' = y$, and $z' = z$. Hence, by (a) and (d),

$$\left.\begin{array}{l} a + z'\cos.\alpha.\cot.\gamma = m - (z' - r)\sin.\alpha.\cot.\gamma' \\ \text{and } b + z'\sin.\alpha.\cot.\gamma = n + (z' - r)\cos.\alpha.\cot.\gamma' \end{array}\right\}$$

And eliminating z and c we find

$$\cot. \gamma' = \frac{(m-a)\sin.\alpha - (n-b)\cos.\alpha}{(m-a)\cos.\alpha + (n-b)\sin\alpha - r\cot.\gamma} \times \cot.\gamma.$$

$\therefore$ substituting this value of $\cot. \gamma'$ in (d), we ultimately obtain two equations between x', y' and z' and known constants, determining the exact position of the straight line B′Q, which is $\perp$ BP, and passes through the given point P′. Q. E. I.

Eliminating $z \cot. \gamma$ from equation (a), we get

$$y = b - a\tan.\alpha + x.\tan.\alpha \;\ldots\ldots\; a'$$

which, being independent of z, determines the straight line BL when $z = 0$, or BL lies in the plane YAX.

Eliminating $(z'-r)\cot.\gamma'$ from equation (d) there results, after proper reduction,

$$y' = n + m\cot.\alpha - x'\cot.\alpha \;\ldots\ldots\; (d')$$

which defines the straight line B′Q when $z = 0$, and $z' = 0$, or both BP, B′Q lie in the plane YAX.

60. Required the curve whose nature is such that its abscissa, ordinate, subtangent, and distance between the origin of abscissæ, and the point of intersection of the tangent and axis, are in continued proportion.

Let $x' =$ abscissa, $y =$ ordinate.

Then, by the question, we have

$$x : y :: \frac{ydx}{dy} : \frac{ydx}{dy} - x$$

$$\therefore x : x - y :: \frac{ydx}{dy} : x$$

$\therefore \dfrac{dy}{dx} = \dfrac{yx-y^2}{x^2}$, which being homogeneous, put $y = ux$,

and we get

$$\frac{udx + \dot{x}du}{dx} = u - u$$

$$\text{or } \frac{dx}{x} = -\frac{du}{u^2}$$

$$\therefore lx = \frac{1}{u} + C = \frac{x}{y} + C$$

$$\therefore y = \frac{x}{c + l.x} \text{ ..... } (a) \text{ the equation to the curve.}$$

Let $x = 0$; then $y = \frac{0}{-\infty} = 0$, or the curve meets the origin of abscissæ A (Fig. 42). Let $x = -x'$; then $y = l(-1) + lx'$.

But $l(-1) = l.(\cos.\pi + \sqrt{-1}\sin.\pi) = l.e^{\pi\sqrt{-1}} = \pi\sqrt{-1}$, an imaginary quantity. $\therefore$ the curve does not go beyond A. Again, let $lx = -C$, or $x = e^{-c} = AB$; then $y = \frac{e^{-c}}{0} = \infty =$ BC, which is $\therefore$ an asymptote; also for every value of $x > e^{-c}$ we have a real value of y, and when $x = \infty$, $y = 0$, $\therefore$ the curve has another branch D′A′ to which BC, BD are asymptotes.

Since $\tan.\theta = \frac{dy}{dx} = \frac{C - 1 + lx}{(C + lx)^2} = \frac{-\infty}{\infty^2} = -\frac{1}{\infty} = 0$, the curve touches the line of abscissæ at A.

61. Let C (Fig. 43) be a right angle, and from A, B given in position, let two points move with equal velocities; and let A′, B′ be any contemporaneous position of those points; required the curve to which A′B′, or it produced, is perpetually a tangent?

Let CA $>$ CB $= a$, and CB $= b$. Then it is evident, that CA′ will touch the curve in V, (AV being taken $=$ BC,) and it will therefore be convex to CA′.

Let $\therefore$ B′A′ touch the curve in P, and draw PM $\perp$ CA′ and make PM $= y$, MV $= x$.

Now, by similar $\triangle$, we have

$$y : \text{MA}' :: \text{CB}' : \text{CA}'$$
$$:: b + \text{BB}' : a + \text{AA}'.$$

But, by the question, BB′ $=$ AA′ $=$ AV′ $- b =$ VM $-$ MA′$-b =$

$x - b - MA'$, and $MA' =$ subtangent $= \frac{ydx}{dy}$; hence, by substitution, and proper reduction, we get

$$\left(\frac{dx}{dy} - 1\right) . \left(x - \frac{ydx}{dy}\right) = a - b \text{ ..... (1)}$$

and differentiating, there results

$$\frac{d^2x}{dy} . \left(x + y - 2y . \frac{dx}{dy}\right) = 0$$

$$\therefore \frac{d^2x}{dy} = 0, \text{ and } x + y - 2y . \frac{dx}{dy} = 0$$

$$\therefore \frac{dx}{dy} = c \text{ ..... (2)}$$

$$\text{and } \frac{dx}{dy} = \frac{x+y}{2y} \text{ ..... (3)}$$

Hence, by substituting these values of $\frac{dx}{dy}$ in equation (1), the corresponding results are,

$$\left.\begin{array}{l} x = cy + \dfrac{a-b}{c-1} \\ \text{And } (x-y)^2 = 4(a-b)\, y \end{array}\right\} \text{the former the } \textit{general integral} \text{ of (1),}$$

and the latter its *particular solution.*

Now, the latter being of the second degree, shews the curve to be a conic section. Also, since

$$x = y \pm 2\sqrt{a-b} . \sqrt{y} \text{ ..... (2)}$$

there are but two infinite values of y for one of x, or the curve is a parabola.

To find the vertex, and latus-rectum, we proceed as follows:

Let $BV' = AC$; then the point A' will be in C, and CV' will touch the curve in V'. Also $CV = CV'$. Hence, by a well known property of the parabola, C is the intersection of the axis and the directrix. Making $\therefore$ CR the axis, and drawing $PM' \perp CR$, and putting $CM' = x'$, $PM' = y'$, we shall easily obtain $y = \frac{x' - y'}{\sqrt{2}}$, and $x = b - a + \frac{x' + y'}{\sqrt{2}}$, and substituting in (2), we get

$$y'^2 = \sqrt{2} . (a - b)\, x' - \frac{(a-b)^2}{2} \text{ ..... (3)}$$

Hence, making $y' = 0$, we get $Cv = \frac{a-b}{2\sqrt{2}}$, giving the vertex v, and the latus-rectum $= 4Cv = \sqrt{2}.(a-b)$.

Hence, the equation to the curve, reckoning the abscissæ from v, is

$$y'^2 = \sqrt{2}.(a-b)\,x''.$$

The problem may be generalized, without increasing, materially, the difficulty of its solution, by supposing the angle C any whatever, and the points A′, B′ to move according to any given law. For problems involving other *particular solutions*, see pp. 47, 48, and the next problem.

62. Let A (Fig. p. 205 of the Problems) be the origin of abscissæ, PT any tangent to the curve, and AD be drawn $\perp$ line of abscissæ meeting the tangent in D. Required the nature of the curve, when AT $\propto$ AD^m; or the enunciation may be stated, *If from* A, *a given point in the given straight line* AT, AD *be drawn at right-angles to* AT, *and* AT *be always taken* $= a \times$ AD^m (a *being a constant*) *required the curve touched by the several lines passing through* T, D.

Making A the origin of x, we have

$$\text{AT} = \text{subtangent} - x = \frac{ydx}{dy} - x,$$

$$\text{and AD} = \text{AT}.\frac{dy}{dx} = y - \frac{xdy}{dx}.$$

$$\therefore \frac{ydx}{dy} - x = a.\left(y - \frac{xdy}{dx}\right)^m$$

$\therefore$ dividing by $ydx - xdy$

$$\frac{dx}{dy} = a.\left(y - \frac{xdy}{dx}\right)^{m-1}$$

$$\text{and } y = \frac{xdy}{dx} + a^{\frac{1}{1-m}}.\left(\frac{dy}{dx}\right)^{\frac{1}{1-m}} \quad \ldots\ldots (a)$$

which, coming under *Clairaut's Formula*, is integrable by differentiation. By that process, and proper reduction, we get

$$\frac{d^2y}{dx} \cdot \left\{ \frac{a^{\frac{1}{1-m}}}{1-m} \cdot \left(\frac{dy}{dx}\right)^{\frac{m}{1-m}} + x \right\} = 0$$

$$\left. \begin{array}{l} \therefore \ \dfrac{d^2y}{dx} = 0 \\ \text{or } \dfrac{a^{\frac{1}{1-m}}}{1-m} \left(\dfrac{dy}{dx}\right)^{\frac{m}{1-m}} + x \end{array} \right\}$$

$$\left. \begin{array}{ll} \text{Hence } \dfrac{dy}{dx} = \text{const.} = c & \ldots\ldots (1) \\ \text{or } \dfrac{dy}{dx} = \dfrac{(m-1)^{\frac{1-m}{m}}}{a^{\frac{1}{m}}} \cdot x^{\frac{1-m}{m}} & \ldots\ldots (2) \end{array} \right\}$$

which being substituted in equation (a) give respectively,

$$y = cx + a^{\frac{1}{1-m}} \cdot c^{\frac{1}{1-m}} \ \ldots\ldots (b)$$

for the *general solution*, and

$$y = \frac{(m-1)^{\frac{1-m}{m}}}{a^{\frac{1}{m}}} \ x^{\frac{1}{m}} + a^{\frac{1}{m}} \cdot (m-1)^{\frac{1}{m}} \ x^{\frac{1}{m}}$$

$$\text{or } y = \frac{m}{a^{\frac{1}{m}} \cdot (m-1)^{\frac{m-1}{m}}} \cdot x^{\frac{1}{m}} \ \ldots\ldots (c)$$

for the *particular solution*.

The equation (c) expresses the nature of the curve required. See pp. 47, 48, 67.

63. *Required the nature of a curve* G'P, (Fig. 44,) *such, that if,* PN *being the ordinate, and* PG *the normal, we bring the* △ PNG *into the position* P'N'G' *determinable from that of* PNG *by making* NN' *some given function of the abscissa, denoted by* NN'$= f.x$, *then* G' *shall be a point in the curve, and* G'P' *the normal at that point.*

Let PN $= y$, G'N' $= y'$
AN $= x$, AN' $- x'$.
Then, $x' = $ AN' $=$ AN $-$ NN' $= x - f.x$
and $y' = $ G'N' $=$ subnormal $= \dfrac{ydy}{dx}$

$\therefore$ making dx constant, $d.x' = dx - dx\,f'x$

and $dy' = \frac{dy^2}{dx} + \frac{yd^2y}{dx}$

$$\therefore \frac{dy'}{dx'} = \frac{dy^2 + yd^2y}{dx^2(1 - f'x)}.$$

But $\frac{dy'}{dx'}$ = tan. N'G'P' = cot. N'P'G'

$$= \frac{1}{\text{tan. N'P'G'}} = \frac{1}{\left(\frac{dy}{dx}\right)}$$

$\therefore$ substituting &c., and ordering the terms, we finally get a differential equation

$$y\frac{dy}{dx}.\frac{d^2y}{dx^2} + \frac{dy^3}{dx^3} = 1 - f'x, \ldots (a)$$

whose integral, which can be found only in certain particular cases, will give an equation between x, y, expressing the nature of the curve required.

To prepare the equation (a) for these particular cases, let $\frac{dy}{dx} = p$; then $\frac{d^2y}{dx^2} = \frac{dp}{dx} = \frac{pdp}{dy}$. Hence by substituting, equat. (a) becomes

$$\frac{y}{dy}.p^2dp + p^3 = 1 - f'x \ldots.(b)$$

which is integrable whenever $f(x)$ is such that its derived function $f'x$ is a function of p, and of $yp\frac{dp}{dy}$.

Ex. 1. Let P' and G coincide, or fx

$= \text{NN}' = \text{PN}' - \text{GN} = y - \frac{ydy}{dx}$.

$$\text{Here } f'x = \frac{dy}{dx} - \left(\frac{dy}{dx}\right)^2 - \frac{yd^2y}{dx^2}$$

$$= p - p^2 - \frac{y}{dy}.pdp.$$

And substituting equation (b) becomes

$$\frac{y}{dy}.pdp.(p-1) = 1 - p + p^2 - p^3 \ldots.(e)$$

$$\therefore \frac{dy}{y} = -\frac{pdp\,(1-p)}{1-p+p^2-p^3} = -\frac{pdp}{1+p^2}$$

$$\therefore ly = lc - l.\sqrt{1+p^2} = l.\frac{c}{\sqrt{1+p^2}}$$

$$\therefore 1+p^2 = \frac{c^2}{y^2}, \text{ and } p = \frac{\sqrt{c^2-y^2}}{y} = \frac{dy}{dx}$$

$$\therefore dx = \frac{ydy}{\sqrt{c^2-y^2}}$$

$\therefore x - c' = -\sqrt{c^2-y^2}$, which reduces to

$y^2 = c^2 - c'^2 + 2c'x - x^2$ the equation to a circle.

Let the origin of abscissæ be that of the ordinates; then putting $x = 0$, we have also $y = 0$, and $c' = c$.

$\therefore y^2 = 2cx - x^2 \ldots\ldots (1)$

the equation to a circle, whose radius is c, which evidently satisfies the conditions of the problem.

Again, since $p - 1$ is also a factor of the equation (e), we have

$$p - 1 = 0, \text{ or } p = \frac{dy}{dx} = 1.$$

$\therefore y = x + c \ldots\ldots (2)$

the equation to a straight line inclined to the line of abscissæ at an angle of 45°, which also affords a solution to the problem.

Ex. 2. Let N,N′ be separated by a constant interval, or let $fx = a$.

Then $f'x = 0$, and equation (b) becomes

$\frac{y}{ay}p^2dp + p^3 = 1$. Hence

$$\frac{dy}{y} = \frac{p^2dp}{1-p^3} \text{ which gives}$$

$$ly = lc - \frac{1}{3}.\,l.(1-p^3) = l.\frac{c}{(1-p^3)^{\frac{1}{3}}}$$

$\therefore 1 - p^3 = \frac{c^3}{y^3}$ whence substituting $\frac{dy}{dx}$ for p, &c., we easily get

$$dx = \frac{ydy}{(y^3-c^3)^{\frac{1}{3}}}$$

which being integrated (if possible) will give the relation between x and y expressing the nature of the curve required.

For a solution of this problem, and of some others of the like nature, by the method of Finite Differences, the reader may consult a very excellent work entitled *A Collection of Examples of the Applications of the Calculus of Finite Differences*, by J. F. W. Herschel, A. M., &c. p. 127.

In p. 129 of that work, the author appears to have made an oversight. By the assumption of $a=b=1$, he makes the subnormal $\frac{ydy}{dx}$ and consequently y' constant, which is contrary to the question. Hence the *common parabola does not satisfy the conditions of the problem.*

Mr. Herschel's method, however holds good in other cases; for instance, (we suppose the reader possessed of the book) when

$$f(y.\ \phi y) = \sqrt{a^2+2y\phi y}, \text{ we have}$$

$$a^2 + 2y\phi y = (y + \phi y)^2 = y^2 + 2y\phi y + \phi^2 y,$$

and $\therefore\ \phi y = \sqrt{a^2 - y^2}$. Hence $x = \int \frac{ydy}{\phi y} = \int \frac{ydy}{\sqrt{a^2-y^2}}$

which, as we have already shewn, gives the equation to a circle.

64. Let A'P, (Fig. 45), a parabola whose vertex is A, roll upon another AP, equal and similar to the former; then supposing the vertices A, A' to have coincided at the beginning of the motion, required the curve described (1) by the focus of A'P – and (2) that described by the vertex A'. (1). Let AN $= x'$, PN $= y'$. Then, the curves having the same tangent at P, viz., PT, and equal ordinates and abscissæ, AA' is $\perp$ PT. Hence S, S' being the foci S'M $= y$ and AM $= x$, we have

SB $=$ BS', ST $=$ TS', $\angle$ STB $= \angle$ BTS', &c., and putting SA' $=$ A'S' $= a$, we get

$$y = \text{SM cot. MS'S} = (x+a).\text{cot. PTN}$$

$$= (x + a).\ \frac{dx'}{dy'} = (x + a).\ \frac{dx'}{d.\ \sqrt{4ax'}}$$

$$= (x+a)\sqrt{\frac{x'}{a}}$$

$$\therefore x' = \frac{ay^2}{(a+x)^2} \ldots\ldots (1)$$

Again, $y = \text{TM. tan. } 2\text{ PTM}$

$$= 2\text{TM. } \frac{\text{tan. PTM}}{1 - \text{tan.}^2\text{PTM}}.$$

$$\text{Hence } x' = \frac{y^2 + x^2 - a^2}{2(x+a)} \ldots. (2)$$

and equating values of x', we get after proper reductions

$$y^2(x-a) = x^3 + ax^2 - a^2x - a^3,$$

$$\therefore y^2 = x^2 + 2ax + a^2$$

$$\therefore y = x + a \ldots (b)$$

which is the equation to a straight line inclined at an angle of 45° to the line of abscissæ, &c.

Secondly, let $AM' = x$, $A'M' = y$, &c., as before; then

$$y = x.\text{ cot. M'A'A} = x\sqrt{\frac{x'}{a}}$$

$$\therefore x' = \frac{ay^2}{x^2} \ldots. (1)$$

Again, $y = \text{TM}'.\text{ tan. } 2\text{QTM}$. Whence as before,

$$x' = \frac{y^2 + x^2}{2x} \ldots. (2)$$

$$\therefore \frac{ay^2}{x^2} = \frac{x^2 + y^2}{2x}$$

$$\text{Hence } y^2 = \frac{x^3}{2a - x} \ldots. (c)$$

the equation to a cissoid, the diameter of whose generating circle is $2a$ or $\frac{1}{2}$ the *latus rectum* of the parabola.

The above process will apply to all equal and similar curves, the contact being always at the same points in both of them. The général theory however, of *roulettes*, as they are termed, is best explained in *Lacroix*, Vol. I. p. 430.

PROPERTIES OF CURVES.

65. *Let r_1, r_2, ... r_n be the values of y, which satisfy the equation*
$y^n - (a + bx)\, y^{n-1} + (c + dx + ex^2)\, y^{n-2} - \ldots\ldots = 0$, *and* S_1, S_2 .... S_n *the subtangents of the curve expressed by its corresponding to these values of* y ; *then*

$$\frac{r_1}{S_1} + \frac{r_2}{S_2} + \ldots \frac{r_n}{S_n} = \textit{a constant quantity.}$$

For, by the theory of equations, $r_1 + r_2 + \ldots\, r_n = a + bx$

$$\therefore\ \frac{dr_1}{dx} + \frac{dr_2}{dx} + \ldots \frac{dr_n}{dx} = b,$$

and $S_1 = \dfrac{r_1 dx}{dr_1}$, $S_2 = \dfrac{r_2 dx}{dr_2}$, ... $S_n = \dfrac{r_n dx}{dr_n}$ by the common differential expression for the subtangent. Hence, by substitution, $\dfrac{r_1}{S_1} + \dfrac{r_2}{S_2} + \ldots \dfrac{r_n}{S_n} = b$, a constant quantity.

66. The axis minor of an Ellipse is a mean proportional between the axis major and the latus rectum.

For, supposing x to originate at the centre, we have by the equation to the ellipse

$$y^2 = \frac{b^2}{a^2} \cdot (a^2 - x^2)$$

$$\text{where } a = \frac{\text{axis major}}{2}, \text{ and } b = \frac{\text{axis minor}}{2}.$$

Hence, at the focus where $y = \dfrac{1}{2}$ latus rectum, and $x = \sqrt{a^2 - b^2}$, we have

$$\frac{l^2}{4} = \frac{b^2}{a^2} \cdot (a^2 - a^2 + b^2) = \frac{b^4}{a^2}$$

$$\therefore b^2 = \frac{al}{2}, \text{ or}$$

$$(2 \cdot b)^2 = 2a \cdot l$$

$$\therefore 2a : 2b :: 2b : l. \text{ Q.E.D.}$$

A *geometrical* demonstration may be found in most books of Conics.

67. The equation to the Logarithmic Curve is $y = a^x$, and the polar equation between the radius vector ρ and angle described θ of the Reciprocal or Hyperbolic Spiral is $\theta = \frac{m}{\rho}$. Hence the Subtangent of the former $= \frac{ydx}{dy} = \frac{1}{1a}$, and of the latter $= \frac{\rho^2 d\theta}{d\rho} = m$. $\therefore$ by the question, $\frac{1}{1 \cdot a} = m$. Again the length of the arc of the Logarithmic Curve $= \int dy \sqrt{1 + \frac{dx^2}{dy^2}} = \int dy \times \sqrt{1 + \frac{1}{y^2 1 a^2}} = \int dy \sqrt{1 + \frac{m^2}{y^2}} = \int \frac{dy}{y} \sqrt{m^2 + y^2} = \int \frac{m^2 dy}{y \sqrt{m^2+y^2}} + \int \frac{ydy}{\sqrt{m^2+y^2}} = \frac{m}{2} \cdot l. \frac{\sqrt{m^2+y^2} - m}{\sqrt{m^2+y^2} + m} + \sqrt{m^2+y^2} + C$. Also, that of the spiral $= \int d\rho \sqrt{1 + \rho^2 \frac{d\theta^2}{d\rho^2}}$
$= \int \frac{d\rho}{\rho} \sqrt{m^2 + \rho^2} = \frac{m}{2} l. \frac{\sqrt{m^2 + \rho^2} - m}{\sqrt{m^2 + \rho^2} + m} + \sqrt{m^2 + \rho^2} + C'$.

Let $\rho = y$, then $C = C'$, and the two arcs are equal. Q. E. D.

68. Two curves are similar when *any* rectilinear figure whatever being inscribed in the one, it is possible to inscribe a similar one in the other.

Hence, describing upon any two corresponding sides $(a, a',)$ of

these figures, which shall remain fixed, innumerable others similar, each to each, so as to meet the curves in every point of them, and from those points of concourse letting fall pairs of perpendiculars upon the respective fixed lines, these perpendiculars, (by the properties of similar $\triangle$), as also the parts of (a, a') cut off by them, will be to each other in the ratio of $a : a'$. If we, therefore, make a, a' lines of abscissæ, the perpendiculars (called y, y') will represent the ordinates, and the parts of a, a' cut off by them, the abscissæ, (x, x') of the respective curves; and the *condition of their similarity* is $\therefore$

$$x : x' :: a : a' :: y : y' \text{ .... } (1)$$

Referred to polar co-ordinates $\rho, \rho', \theta, \theta'$, it may be shewn in like manner, that the *condition of similarity* is properly expressed by (the poles being similarly situated with respect to a, a', and $\therefore$ $\theta = \theta'$)

$$\rho : \rho' :: a : a' \text{ .... } (2)$$

The definition, &c., for any number of curves is the same in principle.

Ex. Hence ellipses are similar when their axes majores are proportional to their axes minores. See the Commentators on Newton's 5th Lemma.

By 'the chords of curvature of conterminous arcs of similar curves at their point of contact,' is merely meant those chords similarly situated in the curves with respect to a, a'.

Hence, since the chord of curvature RE : its Radius $= \dfrac{dz^3}{-dxd^2y} :: dx : dz$. (See *Simpson's Fluxions*, p. 73).

$\text{RE} = \dfrac{dz^2}{-d^2y} = \dfrac{dx^2+dy^2}{-d^2y}$, and for another similar curve, by proport. (1) we have

$$\text{R}'\text{E}' = \frac{dx'^2+dy'^2}{-d^2y'} = \frac{\left(\frac{a'}{a}\right)^2 dx^2 + \left(\frac{a'}{a}\right)^2 dy^2}{-\frac{a'}{a}d^2y}$$

$$= \frac{a'}{a}.\text{RE}$$

$$\therefore \text{RE} : \text{R}'\text{E}' :: a : a',$$

and the same mode of proof may be extended to any number of similar curves.

69. *There cannot be more than five regular solids, supposing the equal and regular polygonal faces of them to be of finite magnitude.*

If S = surface of a sphere whose radius is unity, and A_1, A_2, &c., A_n the angles of a spherical polygon of n . angles described on its surface by arcs of great circles, then (*Woodhouse's Trig.*) the surface of this polygon

$$= \frac{A_1 + A_2 + \ldots. A_n - (n-2)\pi}{\pi} \cdot \frac{S}{4}.$$ Let $A_1 = A_2$

$= A_3 =$ &c. $= A_n$, and this surface becomes

$$\frac{nA_1 - (n-2)\pi}{\pi} \cdot \frac{S}{4}.$$

Supposing now that p such polygons will *exactly* cover the whole sphere, we have

$$\frac{nA_1 - (n-2)\pi}{\pi} \cdot \frac{S}{4} = \frac{S}{p}$$

$\therefore \dfrac{nA_1 - (n-2)\pi}{\pi} = \dfrac{1}{p}$ ..... (a) and since at the points of concourse of the angles of these pologons, a certain number (a) of angles each $= A_1$ together fill the spherical space $= 2\pi$, we have $A_1 = \dfrac{2\pi}{a}$. $\therefore$ substituting, we get, after proper reduction, the equation

$$p = \frac{4a}{2n-(n-2)a} \; \ldots\ldots \; (1)$$

where n, p, and a, are finite and positive whole numbers, and a, n, (by the nature of the question) each > 2.

From equation (1) we find, as follows, what number and what kind of equilateral and equiangular spherical polygons will exactly cover the surface of a sphere, and thence the number and kind of regular solids, whose faces will evidently be the rectilinear polygons formed by the chords of the arcs of the respective spherical polygons.

First. Let $n = 3$ or the faces of the solids be equal equilateral Δ, then

$$p = \frac{4a}{6-a} \;\ldots\ldots\; (2),$$

which being satisfied only by $a = 3, 4, 5$, and the corresponding values of p being 4, 8, 20, shew that the only regular solids whose faces are triangular are the *Tetraedron, having three plane angles at each solid angle, the Octaedron with four plane angles at each solid angle*, and the *Icosaedron with five plane angles to each solid angle.*

Secondly. Let $n = 4$, or the faces of the solids be square. Then

$$p = \frac{4a}{8-2a} = \frac{2a}{4-a} \;\ldots\ldots\; (3)$$

which is satisfied only by $a = 3$, giving $p = 6$. Hence the *Cube, having three plane right angles to each solid angle*, is the only regular solid whose faces are equal squares.

Thirdly. Let $n = 5$, or the faces of the solids be pentagonal. Then

$$p = \frac{4a}{10-3a} \;\ldots\ldots\; (4)$$

which is satisfied only by $a = 3$, giving $p = 12$. Hence the *Dodecaedron, having three angles to each solid angle*, is the only regular solid whose faces are equal and regular pentagons.

Fourthly. Let $n > 5 = 5 + w$. Then

$$p = \frac{4a}{10 + 2w - (3+w)\,a}.$$

and supposing $a > 2 = 2 + w'$

$$p = \frac{8+4w'}{4 - 3.(w+w') - w\,w'},$$ which being negative for every value of w and $w' =$ or > 1, shews that there are no more regular solids than the five above named, scil. the *Tetraedron, Octaedron, Icosaedron, Cube*, and *Dodecaedron.* Q.E.D.

This problem, as above enunciated, might have been solved with much less trouble by the consideration of the limits of solid angles, (*Euc.* B. XI). It appeared, however, desirable to discuss the problem analytically, not only on account of the further use

which we are about to exemplify, of the results thus obtainable, but also as an exercise to the student in such processes.

To find those regular polyedrons whose faces are infinitely small.

Here $p = \infty$. $\therefore$ by eq. (1) we have

$$2n = (n - 2) \cdot a$$

$$\text{or } a = \frac{2n}{n-2} \text{ ...... } (5)$$

where a and n are positive integers.

Let $n = n' + 2$. Then $a = \frac{2n'+4}{n'} = 2 + \frac{4}{n'}$, and $\frac{4}{n'} =$ a whole number, which can only be when $n' = 1, 2, 4$, or when $n = 3, 4, 6$, or when $a = 6, 4, 3$, which respective values of n, a, $\therefore$ satisfy eq. (5) and shew that the *only regular polyedrons having faces infinitely small, viz., spheres, are those whose innumerable faces are equilateral* $\triangle$, *with solid angles composed of six plane angles, or those whose faces are squares with solid angles composed of four right angles, or those whose faces are hexagons with solid angles composed of three plane angles.*

This conclusion is also apparent from the consideration that in this case the solid angles become plane, and that equilateral $\triangle$, squares and hexagons are the only figures whose angles measure 360°.

By the form

$$A_1 = \frac{\pi}{np}(1 + \overline{n-2} \cdot p) \text{ ...... } (b)$$

deduced from (a), we get in the tetraedron

$A_1 = \frac{\pi}{3 \cdot 4}(1 + 4) = \frac{5\pi}{12}$. Hence the difference the spherical angle A_1 and the corresponding plane angle $= \frac{5\pi}{12} - \frac{\pi}{3} = \frac{\pi}{12}$, and similarly for the other solids. Many other results are deducible from the above equations, which we leave to be effected by the student.

70. $SP = r$ and HP being the distances from the foci to the point P, $a \times e$ the eccentricity SC, (a being $\frac{1}{2}$ axis major), then it is evident that

$$HP^2 = SP^2 . \sin.^2 A + (2ae + SP \cos. A)^2$$
$$= \text{also } (2a - SP)^2 \text{ by the property of the ellipse.}$$

$\therefore r^2 \sin.^2 u + 4a^2 e^2 + 4aer \cos. u + r^2 \cos.^2 u = 4a^2 - 4ar + r^2$. But $r^2 . (\sin.^2 u + \cos.^2 u) = r^2$,

$$\therefore r . (a + ae \cos. u) = a^2 . (1 - e^2)$$
$$\therefore r = \frac{a^2 . (1 - e^2)}{a . (1 + e \cos. u)} = \frac{a . (1 - e^2)}{1 + e \cos. u}.$$

71. To find the difference of the latera-recta of an ellipse and parabola whose least distances are equal.

The equations to the respective curves are $y^2 = \frac{b^2}{a^2} . (a^2 - x^2)$, $y'^2 = L' x'$, where b, a, are the axes-minor and major of ellipse, and L' the latus rectum of the parabola. Now at the foci the least distances being the same quantity (suppose $= d$) we have corresponding to those points $y = \frac{L}{2}$ (L = latus rectum of ellipse) $y' = \frac{L'}{2}$.

$$\text{Hence } \frac{L^2}{4} = \frac{b^2}{a^2} . (a^2 - d^2), \text{ and } \frac{L'^2}{4} = L'd.$$

But by a common property of the ellipse $4b^2 = aL$. Consequently

$$L = \frac{1}{a} . (a^2 - d^2), \text{ and } L' = 4d,$$

$$\therefore L - L' = \frac{1}{a} . (a^2 - d^2) - 4d \text{ the difference required.}$$

If $a = 300$, and $d = 1$, according to the problem, we get

$$L - L' = 300 - \frac{1}{300} - 4 = \frac{90000 - 1201}{300}$$
$$= \frac{88799}{300}. \quad \text{Q.E.D.}$$

72. To find the side of a regular decagon inscribed in a circle in terms of the radius r.

The side being a chord of $\frac{360^\circ}{10} = 36^\circ$, and the chord of an arc $= 2 \sin. \left(\frac{1}{2} \text{ that arc}\right)$, this side $= 2r \sin \frac{36^\circ}{2} = 2r \sin. 18^\circ$. But

$$\text{Sin } (2 \times 18^\circ) = \sin. 36^\circ = \cos. 54^\circ = \cos. 3 \times 18^\circ$$

$$\therefore 2 \sin. 18^\circ . \cos. 18^\circ = \cos. 2 \times 18^\circ . \cos. 18^\circ - \sin. 2\, 18^\circ . \cos. 18^\circ$$

$$= \cos.^3 18^\circ - \cos. 18^\circ . \sin.^2 18 - 2 \sin^2. 18^\circ \cos. 18^\circ$$

$$= \cos.^3 18^\circ - 3 \cos. 18^\circ . \sin.^2 18^\circ$$

$$\therefore 2 \sin. 18^\circ = 1 - \sin.^2 18^\circ - 3 \sin.^2 18^\circ$$

$$\therefore 4 \sin.^2 18^\circ + 2 \sin. 18^\circ = 1$$

and solving the equation according to $2 \sin. 18^\circ$ we get

$$2 \sin. 18^\circ = -\frac{1}{2} \pm \sqrt{\frac{1}{4} + 1} = \frac{\sqrt{5} - 1}{2}.$$

Hence the side of a regular decagon inscribed in a ⊙ whose rad. $= r$ is

$$\frac{r}{2} . (\sqrt{5} - 1).$$

73. Let the ordinates y, y', of two curves whose abscissæ x, x', are measured on the same axis, be always in the constant ratio of $m : n$; they have the same tangents at corresponding points.

For by problem (68) it is evident that $x : x' :: m : n$.

Hence $y : y' :: x : x'$

and $dy : dy' :: dx : dx'$

$\therefore \frac{dy}{dx} = \frac{dy'}{dx'}$, i. e., the tangents of the inclinations of the respective tangents to the axis are equal, or the tangents coincide, &c.

This proposition relates to Newton's Lemmas in the Principia.

74. Let the polar co-ordinates of P, R, T, (Fig. Cam. Prob.

p. 194) be (ρ, θ), (ρ', θ'), (ρ'', θ''), and put $\angle$ RSQ $= \phi$. Then by equation to the curve, we have

$$\left.\begin{aligned} \theta &= c \,.\, l.\ \frac{\rho}{a} \\ \theta' &= c \,.l.\ \frac{\rho'}{a} \\ \theta'' &= c.\ l.\ \frac{\rho''}{a} \end{aligned}\right\} \cdots\cdots\cdots (1)$$

Hence $\theta'' - \theta = c\,l.\,\frac{\rho''}{\rho}$, and $\theta' - \theta = c\,l.\,\frac{\rho'}{\rho}$

But $\theta'' - \theta = n \times (\theta' - \theta)$ by the question,

$$\therefore l.\frac{\rho''}{\rho} = n\,l.\,\frac{\rho'}{\rho} = l.\ \frac{\rho'^n}{\rho^n}.$$

$$\therefore \rho'' = \frac{\rho'^n}{\rho^{n-1}}.$$

Now DT $=$ ST $-$ SD $= \rho'' - 2\rho\,.\cos.\,n\,\frac{\text{QR}}{2}$, and $\rho' =$ SR $= 2\rho \cos.\,\frac{\text{QR}}{2} = 2\rho\cos.\,\phi$,

$$\begin{aligned} \therefore \text{DT} &= 2^n\rho\cos.^n\phi - 2\rho\cos.\,n\phi \\ &= 2\rho\,(2^{n-1}\cos.^n\phi - \cos.\,n\phi) \end{aligned}$$

But (*Woodhouse's Trig.*)

$2^{n-1}\cos.^n\phi = \cos.\,n\phi + n\,.\cos.\,(n-2)\phi + n\,.\,\frac{n-1}{2}\,.$ $\cos.(n-4)\,\phi + \&c.$ to $\frac{n+1}{2}$ or $\frac{n}{2}$ terms, according as n is odd or even. Hence DT $= 2\,(n\cos..\,\frac{n-2}{2}\,\text{QR} + n\,.\,\frac{n-1}{2}\cos.\frac{n-4}{2}\,\text{QR} + \&c.)$ Q.E.D.

75. For the definition of similar curves, see No. (68).

Let ρ, ρ' be the radii vectores,

$\left.\begin{matrix}\text{R, R}' \\ r,\ r'\end{matrix}\right\}$ the radii of the pairs of generating circles.

Then, R, R′ being those radii vectores from which the arcs (θ)

are measured (the centres of the circles are poles) the equations to the Epicycloids give

$$\left.\begin{array}{l}\rho^2 = (R + r)^2 - 2r.(R + r)\cos.\theta \\ \rho'^2 = (R' + r')^2 - 2r'(R' + r')\cos.\theta\end{array}\right\}$$

But by the conditions of similarity (No. 68.)

$$\rho : \rho' :: R : R', \text{ and } \theta = \theta'$$

$$\therefore (R + r)^2 - 2r.(R + r).\cos.\theta = \frac{R^2}{R'^2}\{(R' + r')^2 -$$

$$2r'(R' + r')\cos.\theta\}$$

Let $\theta = 0$. Then

$$R^2 - r^2 = \frac{R^2}{R'^2}.(R'^2 - r'^2) = R^2 - \frac{R^2r'^2}{R'^2}$$

$$\therefore R : R' :: r : r',$$

so that all Epicycloids are similar, the radii of whose pairs of generating circles are respectively proportional.

76. The equation to the catenary being $z = \sqrt{2ax + x^2}$ and $8a$ the latus rectum of the parabola whose abscissæ are the same, let y' be the ordinate of the latter, and y of the former; then

$$\left.\begin{array}{l}z + y = \sqrt{2ax + x^2} + y \\ \text{and } y' = 2\sqrt{2a}\,.\sqrt{x}\end{array}\right\}$$

$$\text{Now } dz' = \sqrt{dx^2 + dy'^2} = \frac{dx}{\sqrt{x}}\sqrt{x+2a}$$

$$\therefore = \frac{(x+a)\,dx}{\sqrt{x^2 + 2ax}} + \frac{adx}{\sqrt{x^2 + 2ax}}$$

$$\therefore z' = \sqrt{x^2 + 2ax} + a\,l.\,(x + a + \sqrt{x^2 + 2ax})$$

$$\text{Again, } dz = \frac{(a+x)\,dx}{\sqrt{2ax + x^2}} = \sqrt{dx^2 + dy^2}$$

$$\therefore dy^2 = \frac{(a + x)^2\,dx^2}{2ax + x^2} - dx^2 = \frac{a^2dx^2}{2ax + x^2}$$

$$\therefore dy = \frac{adx}{\sqrt{2ax + x^2}}$$

$$\text{and } y = a\,l.\,(a + x + \sqrt{2ax + x^2})$$

Hence $z + y = z'$. Q.E.D.

77. Let ACV (Fig. 43,) be a *semitrochoid* generated by the point C given in position in the revolving wheel $v'd'$, during half the revolution, and let V′D′ be the position of the wheel when V is highest, or at the vertex V; then drawing CF $\perp$ axis VD, and $\therefore \perp vd$ and cutting $\frac{1}{2} \odot$ VED in E, &c., required to prove that CE is to the arc EV in a constant ratio.

Let O, o be the centres of the circles, join OE, oC and produce them to e and c respectively. Then, by the nature of the rotation, we have

$$\left.\begin{aligned} AD &= A'D' = D'eV' \\ Ad &= A'd' = cd' = eD' \end{aligned}\right\}.$$

$$\therefore Ff = Dd = AD - Ad = \text{arc } V'e = \frac{R}{r} \times \text{arc VE},$$ R, r being the radii of VeD' VED respectively.

But Ff also $= fE + EF = fE + Cf = CE$.

$$\therefore CE = \frac{R}{r} VE$$

or CE : VE :: R : r. Q.E.D.

Hence we easily get an equation to the trochoid.

Let $CO = \rho$, and $\angle EOV = \theta$; then

$$\rho^2 = CF^2 + FO^2 = (CE + EF)^2 + r^2 \cos.^2 \theta = \frac{R^2}{r^2}. EV^2 + 2\frac{R}{r} EV \times r \sin. \theta + r^2.$$

But $EV = r \times \angle \theta$. Hence

$\rho^2 = R^2.\theta^2 + 2 Rr\theta \sin. \theta + r^2$ ...... (a) the equation between ρ and θ.

78. Let Q be any point in the direction Mm of the parabola AP (Fig. 47), of which also S is the focus, A the vertex. Again, join QS and draw PSp $\perp$ QS, and join QP, Qp; QP, Qp touch the parabola, and the $\angle$ PQp = a right angle.

For, drawing PM and $pm \perp$ Mm, by property of the parabola, SP = PM, Sp = pm, and the angles at M and m are right,

and PQ is common to the two Δ; $\therefore$ the angles at P are equal, and likewise those at p, or QP, Qp, touch the parabola.

Again $\angle$ PQS $=$ $\angle$ PQM, and $\angle$ pQS $=$ $\angle$ pQm,

$\therefore$ 2 $\angle$ PQS $+$ 2 $\angle$ pQS $=$ 2 right angles, or $\angle$ PQp $=$ right $\angle$.

79. The equation to the reciprocal spiral being

$$\varrho = \frac{a}{\theta}$$

where ϱ is the radius vector, and θ the traced angle, if we make ϕ the angle between the radius vector and that tangent, at the point of contact, (which is the inclination stated in the problem,) we have

$$\text{Subtangent} = \varrho \,.\, \tan.\ \phi,$$

$$\text{But, subtangent} = \frac{\varrho^2 d\theta}{d\varrho} = a$$

$$\therefore \tan.\ \phi = \frac{a}{\varrho} \propto \frac{1}{\varrho}. \quad \text{Q.E.D.}$$

80. The angle of a Δ included by two sides a, b, being $= \frac{4}{3}$ right-angle, the (base)^2 $= \dfrac{a^3 \backsim b^3}{a \backsim b}$.

For, by *Euclid*, B. II.; and *Trig.*

$$(\text{Base})^2 = a^2 + b^2 - 2ab\cos.\left(\frac{4}{3}\text{ right}\,.\,\angle\right)$$

$$= a^2 + b^2 - 2ab\cos.\,(90^\circ + 30^\circ)$$

$$= a^2 + b^2 + 2ab\cos.\ 30^\circ$$

$= a^2 + b^2 + ab$, which is the required result when divided by its denominator.

81. Let VP (Fig. 48) be the $\perp$ let fall from the vertex V of the pyramid, upon its base ACB; join AP, CP, PB, and produce AP to Q. Then, since the faces are equal equilateral Δ (each $= a$), we have .

$$AP^2 = VA^2 - PV^2 = CV^2 - PV^2 = PC^2.$$

Hence AP = CP = PB, and the Δ APC, APB having two sides in each =, and one common, we have $\angle$ CAQ = $\angle$ BAQ. Hence AQ is $\perp$ CB.

And $\therefore$ CQ = QB, &c., and, as is well known,

$$AP = 2PQ, \therefore AQ = 3\,PQ.$$

Now $VP^2 = VA^2 - PA^2 = 4CQ^2 - 4PQ^2$

$$= 4AQ^2.\ \tan.^2 30 - \frac{4}{9} AQ^2$$

$$= \frac{4}{3} AQ^2 - \frac{4}{9} AQ^2 = \frac{8}{9} AQ^2$$

$$\therefore VP = \frac{2}{3} \sqrt{2} \times AQ$$

$$\text{But } AQ = \frac{\text{area of ACB}}{CQ} = \frac{n}{AQ.\ \tan.\ 30^\circ} = \frac{\sqrt{3} \times a}{AQ}$$

$$\therefore AQ = \sqrt{\sqrt{3} \times a}. \text{ Hence}$$

$$VP = 2\sqrt{2} \times \frac{3^{\frac{1}{4}}}{3}.\sqrt{a} = \frac{2\sqrt{2}}{3^{\frac{3}{4}}} \times \sqrt{a}. \quad \text{Q. E. I.}$$

82. Let Bb, Aa (Fig. 49) be any two conjugate diameters of the ellipse $ABab$, and let a tangent $A'a'$ at any point P cut off from the tangents at A, a the segments AA', aa'; then $AA' \times aa' = BC^2$.

Let the axes of co-ordinates be taken parallel to the conjugate diameters, i.e., let PN (parallel to BC) = y, and AN = x, and put BC = b, AC = a. Then, by a well known property of the ellipse, we have

$$PN^2 = \frac{BC^2}{AC^2} \times AN \times Na$$

$$\text{Or } y^2 = \frac{b^2}{a^2} \times (2ax - x^2) \ldots\ldots (1)$$

Now, taking $PN' = y'$ and $AN' = x'$ for the co-ordinates of the tangent Aa' we get

$$y' = y - PQ, \therefore y' - y = -PQ$$
$$x' = x + NN' = x + P'Q, \therefore x' - x = P'Q$$

$\therefore \frac{y'-y}{x'-x} = \frac{-PQ}{P'Q}$, which being invariable for the same point P,

$$\frac{y'-y}{x'-x} = \frac{dy}{dx}$$

$$\therefore y' = x'\frac{dy}{dx} + y - \frac{xdy}{dx} \;\ldots\ldots (2)$$

$$\text{But } \frac{dy}{dx} = \frac{b}{a}\cdot\frac{a-x}{\sqrt{2ax-x^2}}$$

$$\therefore y' = \frac{b}{a} \times \frac{a-x}{\sqrt{2ax-x^2}} \times x' + \frac{bx}{\sqrt{2ax-x^2}} \;\ldots\ldots (3)$$

$$\text{Let } x' = 0, \text{ then } AA' = y' = \frac{bx}{\sqrt{2ax-x^2}}$$

$$\text{Let } x' = 2a, \text{ then } aa' = y' = b.\frac{2a-x}{\sqrt{2ax-x^2}};$$

$$\text{Hence } AA' \times aa' = \frac{b^2}{2ax-x^2}.(2ax-x^2) = b^2. \qquad \text{Q. E. D.}$$

The same process will apply to the *hyperbola*, the only difference in the equations being in the sign of x^2, which in the hyperbola is positive.

The proposition is also true for the parabola and circle, which are ellipses in extreme cases.

It may be useful to the Student, in his future inquiries, to bear in mind, that all equations of conic sections, not involving the angle made by their conjugate diameters, are the same in form, and admit of similar applications, whatever conjugate diameters may be taken. Hence, as above, the equations referred to the conjugate, are the same as those referred to the principal or rectangular diameters.

83. This problem will be more intelligible when enunciated; *If the abscissa of a curve (the origin of co-ordinates being at the vertex) be to the ordinate ultimately*, i. e., *at the vertex, in a finite ratio, the abscissa will cut the curve in an angle whose tangent is finite.*

For it is evident that the line joining the vertex and top of the ordinate is ultimately a tangent to the curve at the vertex, and therefore makes the same angle with the abscissa as the curve itself. Now, calling the abscissa x, ordinate y, and the $\angle$ between x and the curve θ, we have ultimately

$$\frac{y}{x} = \tan.\,\theta$$

which is finite, infinitesimal, or infinite, according as $\frac{y}{x}$ is finite, and of the form $\frac{0}{0}$, infinitesimal, and $\therefore$ of the forms $\frac{a}{\infty}, \frac{0}{a}$, or infinite, and of the forms $\frac{\infty}{a}$, $\frac{a}{0}$, i. e., according as θ is neither $p \times \pi$ nor $p.\ \frac{\pi}{2}, p \times \pi$, or $p \times \frac{\pi}{2}$, p being any of 0, 1, 2, 3, &c.

The considerations to which this problem conducts us are useful in illustration of some of the Lemmas in the first section of the *Principia*.

84. Let Ll (Fig. 36,) cut the diameter Dd of the circle at any angle A. Draw from the centre O, OR $\perp$ Dd, Ol' parallel to Ll, and from l', $l'm$ parallel to Dd. Take also $DA' = Ad$, and $LM = Al$, join A'M, and draw $Om \perp Ll$. Then

$$DA - Ad : LA - Al :: Dd : \text{chord of } 2 \times Rl'.$$

For by property of the $\odot$, $Lm = ml$, and we have also $LM = Al$. $\therefore$ $Mm = mA$ and $A'O = OD - DA' = Od - Ad = OA$. Hence $A'A : MA :: OA : mA :: Ol' : l'm :: Dd : 2l'm$ from similar $\triangle$ $ml'O$, OmA. But $2l'm =$ chord of $2\,Rl'$, $A'A = DA - Ad$, and $MA = LA - Al$, $\therefore$

$$DA - Ad : LA - Al :: Dd : \text{ch. } 2Rl'.$$

85. Let aPA (Fig. 50) be any section of the cone generated by the revolution of the asymptotes AC, Cr, about the axis CV, made by a plane touching the hyperboloid QV at the point

Q. Also suppose Aa the intersection of the generating plane, in a position perpendicular to that of the section APa, with the plane APa, and let PM be the intersection of a plane $\perp$ axis CV with the plane APa, $R'r'$ being its intersection with the plane ACa.

Then, since both the planes APa, $R'Pr'$ are $\perp$ plane ACa, their intersection PM is $\perp$ $R'r'$, Aa. Now drawing through Q, Rr parallel to $R'r'$, since $R'Pr'$ is evidently a circle, we have

$$PM^2 = R'M \times Mr' \quad . \quad . \quad . \quad . \quad (1)$$

But, by similar $\triangle$,

$$R'M : MA :: RQ : QA$$

$$\text{and } r'M : Ma :: rQ : Qa$$

$$\text{Hence } R'M \times r'M = \frac{MA \ . \ Ma \times RQ \ . \ Qr}{QA \ . \ aQ}$$

Now $RQ \times Qr = b^2$, b being the axis-minor of the hyperbola, and, by a well known property of the hyperbola, $QA \times aQ = QA^2$. Hence, putting $QA = a$, $AM = x$, $PM = y$, we get

$$y^2 = \frac{b^2}{a^2} . (2ax - x^2)$$

the equation to an ellipse having the same axis-minor as the generating hyperbola.

86. Let $y^2 = \frac{b^2}{a^2} . \ (2ax - x^2)$ the equation to the ellipse. Then at the focus the equation to the circle described on the axis-major (a) is $y'^2 = (2ax - x^2) = \frac{a^2}{b^2} \times$ (latus rectum)^2.

But latus rectum $= \frac{b^2}{a}$ by a known property of the ellipse.

$$\therefore y'^2 = \frac{a^2}{b^2} . \ \frac{b^4}{a^2} = b^2$$

$$\therefore y' = b. \qquad \text{Q. E. D.}$$

87. Let x, y be the co-ordinates of the cissoid originating in the extremity of the diameter (a) of the generating circle.

Then the equation to the curve being

$$y^2 = \frac{x^3}{a-x} \quad . \quad . \quad . \quad . \quad (1)$$

And by the question, the subtangent $= \frac{ydx}{dy} = \frac{a}{4}$, we have

$$\frac{2x(a-x)}{3a-2x} = \frac{a}{4}$$

$$\therefore x^2 - \frac{5}{2}ax = -\frac{3}{4}a^2$$

$$\therefore x = \frac{5}{4}a \pm a\frac{\sqrt{13}}{4}$$

$$= \frac{5 \pm \sqrt{13}}{4} \times a$$

which determines the abscissa of the point required, and substituting in equation (1) we get the corresponding ordinate, which will determine the actual position of the point. There are two solutions giving points on different sides of the axis of y.

88. Let S, H, (Fig. 51,) be the foci of the ellipse APa, and PT a tangent at any point P. Then Hy being $\perp$ PT and Cy, SP joined, Cy is parallel to SP.

For, producing SP, Hy to meet in h, since PT (by property of the ellipse) bisects the angle hPH, we easily prove that $yh = y$H. Hence, since SC = CH, we have CHy similar to SHh. $\therefore \angle y$Ca $= \angle h$SH, or Cy is parallel SP.

Similarly, Sy' being drawn $\perp$ PT, and Cy' joined, it may be shewn that Cy' is parallel to PH.

89. Let C, A (Fig. 52,) be the circle of curvature to the parabola AP_2P_3, &c. at its vertex A, and suppose a series of circles whose centres are C_2, C_3, &c., to be described each touching the one preceding it and the two branches of the parabola, then r_1, r_2, r_3, &c., denoting the radii of the circles, required to prove that r_1, r_2, r_3, &c., r_n, are in the proportion of 1, 3, 5 ..... $2n-1$.

Joining C_2P_2, C_3P_3, &c., and making $AN_2 = x_2$, $P_2N_2 = y_2$, &c. we have $r_2 = C_2a = aN_2 + C_2N_2 = x_2 - 2r_1 + \frac{p}{2}$

$$= \frac{y_2^2}{p} + \frac{p}{2} - 2r_1 = \frac{r_2^2}{p} - \frac{p}{4} + \frac{p}{2} - 2r_1$$

$\therefore$ after proper reductions, we get

$$r_2^2 - pr_2 = 2pr_1 - \frac{p^2}{4}$$

$$\therefore r_2 = \frac{p}{2} \pm \sqrt{2pr_1}$$

By the same process with respect to the next circle, we get

$$r_3 = \frac{p}{2} \pm \sqrt{2p \,.\, (r_1 + r_2)}$$

And generally,

$$r_n = \frac{p}{2} \pm \sqrt{2p\,(r_1 + r_2 + \ldots\ldots r_{n-1})} \ \ldots\ldots (a)$$

Now it is well known that the diameter of curvature $2r_1$ at the vertex of a parabola = latus rectum p.

$$\therefore r_1 = \frac{p}{2} \ \ldots\ldots (1)$$

Hence $r_2 = \frac{p}{2} \pm p = \frac{3p}{2}$ or $- \frac{p}{2}$

$$r_3 = \frac{p}{2} \pm 2p = \frac{5p}{2} \text{ or } - \frac{3p}{2}$$

$$r_4 = \frac{p}{2} \pm 3p = \frac{7p}{2} \text{ or } - \frac{5p}{2}$$

&c. = &c.

$$r_n = \frac{p}{2} + (n-1)p = \frac{(2n-1)p}{2} \text{ or } - \frac{(2n-3)p}{2}$$

which give the relations required, the negative values merely indicating those circles which touch both the preceding one, and the branches *internally*, and evidently coincide *in toto* with it.

If $x = \varphi\,(y)$ be the equation of the curve, then we easily find that

$$r_n = \frac{ydy}{dx} + \varphi. \sqrt{r_n^2 - \frac{ydy}{dx} - 2(r_1 + r_2 + \ldots\ldots r_{n-1})}$$

whence it appears possible to find a series of circles such as are represented (r_1 however being any *given* ⊙ touching the branches, which somewhat generalizes the problem,) in the figure, only for such curves as give $\frac{ydy}{dx}$ constant, i. e., for the common parabola only.

90. This enunciation ought to have specified that the radius vector in arriving at SP had just completed a revolution.

Let SP $= \varrho$, and the corresponding angle $= \theta$, then the equation to the spiral (which is that of Archimedes), is

$$\theta' = a\varrho'$$

Hence the subtangent $= \frac{\varrho'^2 d\theta'}{d\varrho'} = a\varrho'^2 = a\,\varrho^2$ at the point P.

Let n be the number of revolutions,

Then $\theta = n \times 4.$ R. $\measuredangle = a\varrho$.

Hence $n \propto \varrho \propto \frac{\sqrt{\text{subtan.}}}{a} \propto \sqrt{\text{subtan.}}$ Q. E. D.

91. This enunciation is very ambiguously expressed. By the " acute angle" is meant that of the angles which is the most acute, and the tangent is considered " the base."

Let R_1, r_1 be the radii of the respective circles, C, c their circumferences, L the straight line $=$ C, and l the one to be found; then from similar $\triangle$ whose sides are L, C; l, c, we have

$$L : C :: l : c$$

But $L = C$, $\therefore l = c$. Q. E. D.

92. Let R $=$ the radius of the circle, r that of the circular base of the cone, and $s =$ side of the cone; then $\pi R^2 =$ area of the circle, $2\pi \times r =$ circumference of the base.

Hence the surface of the cone $= \pi r \times s$, and by the question

$$\pi R^2 = \pi r \times s$$

$$\therefore\ R^2 = rs, \text{ or}$$

$$r : R :: R : s.$$ Q. E. D.

93. Let R be the radius of the sphere. Then the solidity of the cylinder whose base $= R^2\pi$ and altitude $2R$, is $2R^3\pi$, and that of the cone with same base and altitude as the cylinder, is $\frac{2R^3\pi}{3}$. Also the sphere $= \frac{4R^3\pi}{3}$. Hence the cone $= \frac{2}{3}R^3\pi =$

$2R^3\pi - \frac{4}{3}R^3\pi =$ Cylinder − Sphere. Q. E. D.

94. Let PL (Fig. 53) $=$ CD $= a'$, CP $= b'$, CK $=$ KL $=$ KM $=$ KN, then PM $= a$ the axis-major of the ellipse, and PN $= b$, the axis-minor.

Suppose PM $= m$, PN $= n$. Then

$$\begin{aligned} m^2 + n^2 &= (MK + PK)^2 + (MK - PK)^2 \\ &= 2MK^2 + 2PK^2 = 2KL^2 + 2PK^2 \\ &= PL^2 + PC^2 = a'^2 + b'^2 \\ &= a^2 + b^2 \text{ by property of the ellipse.} \end{aligned}$$

Again, $(m + n)^2 = CL^2 = CP^2 + PL^2 \pm 2PL \times PE$

$$\therefore m^2 + n^2 + 2mn = a'^2 + b'^2 + 2a'b' \sin. PCE.$$

Hence $mn = a'b' \sin. \angle C$

$= ab$ by property of the ellipse.

$$\therefore \left.\begin{aligned} m^2 + n^2 &= a^2 + b^2 \\ mn &= ab \end{aligned}\right\} \text{ whence } \left.\begin{aligned} m &= a \\ n &= b \end{aligned}\right\}$$

95. Given the figure of an ellipse, to find its centre and foci *practically*.

In the ellipse draw any two parallel lines GH, gh, (Fig. 53) bisect them in I, i, and Ii being joined, let it be produced to meet the curve in P,p. Then, since every diameter bisects its ordinates and reciprocally, Pp is a diameter, of which the middle point C is the required centre of the ellipse.

Again, through C draw Dd parallel HG, and we get Dd a diameter conjugate to Pp. Produce EP, (which is $\perp$ CD,) to L, making PL $=$ CD, join CL and through its middle point K and P draw MKN, making MK $=$ KN $=$ CK or KL. Then, by the preceding problem, PM $= a$ the axis-major, and PN $= b$ the

axis-minor of the ellipse. Hence with C as centre and radii $= a$, b describing circular arcs, the points A, B, where they will meet the ellipse, will determine the position of its axes AC, BC; and since the distance of the focus from B $=$ AC, if with centre B and radius $=$ AC we describe arcs cutting AC in S, S′, the foci S, S′ of the ellipse will also be determined.

96. Let Ioi, Qoq (Fig. 54), intersecting in o, be parallel to *any two* conjugate diameters BCD, ACM; then $Io \times oi : Qo \times oq :: BC^2 : AC^2$.

For drawing Com through o and C and mn parallel to BD, we have

$$\left.\begin{array}{l} Iv^2 : mn^2 :: AC^2 - Cv^2 : AC^2 - Cn^2 \\ \text{And } mn^2 : ov^2 :: Cn^2 : Cv^2 \end{array}\right\}$$

$$\therefore Iv^2 : ov^2 :: AC^2 . Cn^2 - Cv^2 . Cn^2 : AC^2 . Cv^2 - Cn^2 . Cv^2$$

$$\left.\begin{array}{l} \therefore Iv^2 - Ov^2 : Iv^2 :: AC^2 (Cn^2 - Cv^2) : Cn \; (AC^2 - Cv^2) \\ \text{Also } Iv^2 : BC^2 :: AC^2 - Cv^2 : AC^2 \end{array}\right\}$$

$$\therefore Iv^2 - Ov^2 : BC^2 :: Cn^2 - Cv^2 : Cn^2$$

$$:: Cm^2 - Co^2 : Cm^2$$

$$\left.\begin{array}{l} \text{Or } Io \times oi : mo \times om' :: BC^2 : Cm^2 \\ \text{Similarly } mo \times om' : Qo \times oq :: Cm^2 : AC^2 \end{array}\right\}$$

$$\therefore Io \times oi : Qo \times oq :: BC^2 : AC^2$$

BC, AC being any conjugate diameters whatever, which is more general, although solved with the same facility as the problem proposed.

97. Let the cycloid acb (Fig. 55), equal and similar to ACB, have its vertex c in the base AB, and its base ab parallel to AB, and let them intersect in P; they cut at right angles.

For drawing PEn $\perp$ axes CD, cd, and intersecting the generating circles in E, e, and the axes in N, n, and joining EC, ed, we have the tangents at P to the curves AC, ac, parallel to CB, ee respectively. But it is easily shewn that ed is parallel to EC, and dee is a right angle; $\therefore$ the tangents at P are at right angles, and consequently the curves themselves.

98. Let $\left.\begin{aligned} y^2 &= \frac{b^2}{a^2}.\ (2ax \mp x^2) \\ y'^2 &= \frac{b'^2}{a'^2}\ (2a'x' \mp x'^2) \end{aligned}\right\}$

be the equations of two conic sections of the same kind, the sign $-$ indicating the ellipse, circle, or parabola, and $+$ the hyperbola; then, by the question, we have

$$a' = a,\ x' = x, \text{ and } \therefore$$
$$y^2 : y'^2 :: b^2 : b'^2.$$

$$\text{But } b^2 = \frac{\text{latus rect.}}{2} \times a = \frac{\text{L} \times a}{2}$$

$$\text{and } b'^2 = \frac{\text{latus rect.}}{2} \times a' = \frac{\text{L}' \times a'}{2}$$

$$\therefore\ y^2 : y'^2 :: \text{L} : \text{L}'$$

And $y : y :: \sqrt{\text{L}} : \sqrt{\text{L}'}$. Q. E. D.

99. The side of an equilateral polygon of n sides inscribed in a circle (rad. $=1$) is $\pm\sqrt{2 \pm \sqrt{2 \mp \sqrt{2 \mp \sqrt{2 \mp \&c.}}}}$, to $(n-1)$ extractions.

Since the side of the polygon $=$ the chord of $\frac{2\pi}{2^n} = 2 \sin. \frac{\pi}{2^n}$, we have merely to shew that

$2 \sin. \frac{\pi}{2^n} = \pm\sqrt{2 \pm \sqrt{2 \mp \sqrt{2 \mp \sqrt{2 \mp \ldots\ldots}}}}$ to $(n-1)$ terms.

Now, suppose the proposition true in one case, i. e., let

$2 \sin. \frac{\pi}{2^m} = \pm\sqrt{2 \pm \sqrt{2 \mp \sqrt{2 \mp \ldots}}}$ to $m-1$ terms; then,

since $2 \sin. \frac{\pi}{2^m} = 2^2.\ \sin. \frac{\pi}{2^{m+1}}.\ \cos. \frac{\pi}{2^{m+1}}$

$= \sqrt{2 \pm \sqrt{2 \mp \sqrt{2 \mp \ldots}}}$ to $m-1$ terms, squaring both sides, and changing signs, &c., we get

$$2^4 \sin.^4 \frac{\pi}{2^{m+1}} - 2^2.\ 2^2 \sin.^2 \frac{\pi}{2^{m+1}} = -2 \mp \sqrt{2 \mp \sqrt{2 \mp \&c.}}$$

Hence resolving the equation in $2^2 \sin.^2 \frac{\pi}{2^{m+1}}$ we have

$$2^2 \sin.^2 \frac{\pi}{2^{m+1}} = 2 \pm \sqrt{2 \mp \sqrt{2 \mp \sqrt{2 \mp \&c.}}}, \text{ and}$$

$$2 \sin. \frac{\pi}{2^{m+1}} = \pm \sqrt{2 \pm \sqrt{2 \mp \sqrt{2 \mp \&c.}}}, \text{ to } m \text{ terms, which}$$

shews that if the proposition be true for any one case, it is likewise so for that immediately succeeding; but it is true in the case of $m = 1$ (since $2 \sin. \frac{\pi}{2^2} = 2 \times \frac{1}{\sqrt{2}} = \pm \sqrt{2}$), $\therefore$ it is for $m = 2$, and $\therefore$ for $m = 3$, &c., up to $m = n$, i. e., the side of an equilateral polygon inscribed in a circle whose rad. $= 1$, is

$$\pm \sqrt{2 \pm \sqrt{2 \mp \sqrt{2 \mp \sqrt{2 \mp \&c.}}}}, \text{ to } n - 1 \text{ terms.}$$

100. Let AB, AC, (Fig. 56), the sides of a spherical Δ ABC be produced to B′, C′, so that BB′, CC′, may equal half the supplements BA′, CA′ of AB, AC, respectively; then B′C′ being joined, and the chords α, β, γ, being drawn, the angle between γ, β = the angle at the centre of the sphere subtended by B′C′.

For the sides opposite A, B, C, being put $= a, b, c$, and those opposite A′, B′, C′, $= a', b', c'$, and taking radius of the sphere $= 1$, we have by Spherical Trig.

$$\text{Cos. A} = \frac{\cos. a - \cos. b \,.\, \cos. c}{\sin. b \,.\, \sin. c}$$

$$\text{and cos. A}' = \frac{\cos. a' - \cos. b' \,.\, \cos. c'}{\sin. b' \,.\, \sin. c'}$$

$$\text{But A}' = \text{A}, \; b' = \frac{\pi - b}{2}, \; c' = \frac{\pi - c}{2}$$

$$\text{and } \therefore \cos. b' \,.\, \cos. c' = \sin. \frac{b}{2} \,.\, \sin. \frac{c}{2}$$

$$\text{and } \sin. b' \,.\, \sin. c' = \cos. \frac{b}{2} \,.\, \cos. \frac{c}{2}$$

Hence, equating cos. A, cos. A′, and substituting, we get

$$\frac{\cos. a' - \sin. \frac{b}{2} . \sin. \frac{c}{2}}{\cos. b . \cos. c} = \frac{\cos. a - \cos. b \cos. c}{\sin. b . \sin. c}$$

But $\sin. b . \sin. c = 4 \sin. \frac{b}{2} \sin. \frac{c}{2} . \cos. \frac{b}{2} . \cos. \frac{c}{2}$

$$\therefore \cos. a' = \frac{4 \sin.^2 \frac{b}{2} . \sin.^2 \frac{c}{2} - \cos. b . \cos. c + \cos. a}{4 \sin. \frac{b}{2} . \sin. \frac{c}{2}}$$

which easily reduces (since $\cos. b = 1 - 2 \sin.^2 \frac{b}{2}$, &c.) to

$$\text{Cos } a' = \frac{\sin.^2 \frac{b}{2} + \sin.^2 \frac{c}{2} - \sin.^2 \frac{a}{2}}{2 \sin. \frac{b}{2} . \sin. \frac{c}{2}} \text{ ...... (1).}$$

Again, making the angle between $\beta, \gamma = \theta$, by Plane Trig. we have

$$\text{Cos. } \theta = \frac{\beta^2 + \gamma^2 - \alpha^2}{2\beta\gamma}.$$

But $\beta = 2 \sin. \frac{b}{2}$, $\gamma = 2 \sin. \frac{c}{2}$, $\alpha = 2 \sin. \frac{a}{2}$

$$\text{Hence } \cos. \theta = \frac{\sin.^2 \frac{b}{2} + \sin.^2 \frac{c}{2} - \sin.^2 \frac{a}{2}}{2 \sin. \frac{b}{2} . \sin. \frac{c}{2}} \text{ ...... (2)}$$

$$\therefore \cos. \theta = \cos. a' = \cos. \angle \text{B}'\text{OC}',$$

and since the radius of sphere B′O was supposed $= 1$; $\therefore \theta = \angle \text{B}'\text{OC}'$. Q. E. D.

101. Let a, the diameter of the circle, be the first term, and r the common ratio, then the other sides of the Δ described in the $\frac{1}{2} \odot$, will be $\frac{a}{r}$, and $\frac{a}{r^2}$.

But by property of the $\frac{1}{2}\odot$, we have

$$a^2 = \left(\frac{a}{r}\right)^2 + \left(\frac{a}{r^2}\right)^2 = \frac{a^2}{r^2} + \frac{a^2}{r^4}$$

$$\therefore\ r^4 - r^2 = 1$$

$$\therefore\ r^2 = \frac{1 \pm \sqrt{5}}{2}$$

$$\text{And } r = \sqrt{\frac{1 \pm \sqrt{5}}{2}}$$

Whence the side $\frac{a}{r^2} = \frac{2a}{1 \pm \sqrt{5}}$ which being inflected in the $\frac{1}{2}\odot$ will give the point required.

102. Let A, A′, be two opposite angles of the quadrilateral, and let the sides which contain them be a, b; a', b', respectively, and B,B′, be the other two angles; then putting the diagonals AA′ $=$ D, BB′ $=$ E, by trig. we have

$$\text{Cos. A} = r \,.\, \frac{a^2 + b^2 - E^2}{2ab}$$

$$\text{And cos. A}' = r \,.\, \frac{a'^2 + b'^2 - E^2}{2a'b'}$$

$$\text{But A}' = \pi - \text{A},\ \therefore \text{cos. A}' = -\text{cos. A}.$$

$$\therefore\ \frac{a^2 + b^2 - E^2}{ab} = -\frac{a'^2 + b'^2 - E^2}{a'b'}$$

$$\therefore\ E^2 = \frac{(a^2 + b^2)\,a'b' + (a'^2 + b'^2)\,ab}{ab + a'b'}$$

$$\text{and } E = \sqrt{\frac{(a^2 + b^2)\,a'b' + (a'^2 + b'^2)\,ab}{ab + a'b'}}$$

In the same manner it may be shewn that

$$D = \sqrt{\frac{(a^2 + a'^2)\,bb' + (b^2 + b'^2)aa'}{aa' + bb'}}$$

103. Let c, C (Fig. 57,) be the centres of the revolving and fixed circles, and let their radii be denoted by r, R, respec-

tively; also suppose A the point in which the generating point P meets the fixed $\odot$, and join Cc (passing through the point of contact m,) PC, CA, and draw PN, $Cn \perp$ CA produced, and $Pn' \perp Cn$.

Then, putting $\angle\, c\mathrm{CN} = \theta$, $\mathrm{PN} = y$, and $\mathrm{CN} = x$, we have

$$x = \mathrm{C}n + \mathrm{N}n = (\mathrm{R} + r)\cos.\ \theta + r\ .\ \sin.\ \mathrm{P}cn' \text{ and}$$
$$y = cn - cn' = (\mathrm{R} + r)\sin.\ \theta - r\ .\ \cos.\ \mathrm{P}cn'.$$

But since by the nature of the revolution $\mathrm{A}m = \mathrm{P}m$, $\mathrm{A}m = \mathrm{R}\ .\ \theta$ and $\mathrm{P}m = r\ .\ \angle\ \mathrm{P}cm$, we get

$$\mathrm{R}\ .\ \theta = r\ .\ \mathrm{P}cm = r\ .\ \mathrm{P}cn' + r\ .\ \left(\frac{\pi}{2} - \theta\right)$$

$$\therefore\ \angle\ \mathrm{P}cn' = \frac{\mathrm{R} + r}{r}\ .\ \theta - \frac{\pi}{2}$$

$$\therefore\ \sin.\ \mathrm{P}cn' = -\cos.\ \frac{\mathrm{R} + r}{r}\ .\ \theta, \text{ and} \cos.\ \mathrm{P}cn' = \sin.\ \frac{\mathrm{R} + r}{r}\ \theta$$

Hence

$$\left.\begin{array}{l} x = (\mathrm{R} + r)\cos.\ \theta - r\ .\ \cos.\dfrac{\mathrm{R} + r}{r}\theta \\ y = (\mathrm{R} + r)\sin.\ \theta - r\ .\ \sin.\dfrac{\mathrm{R} + r}{r}\theta \end{array}\right\} \ldots\ldots (1) \text{ the equa-}$$

tions to the curve between x, y, and θ.

Now supposing, according to the question, that $\mathrm{R} = mr$, where m is an assignable whole number, then equation (1) becomes

$$\left.\begin{array}{l} x = (m + 1)\ .\ r\cos.\ \theta - r\ .\ \cos.\ (m + 1)\ \theta \\ y = (m + 1)\ .\ r\sin.\ \theta - r\ .\ \sin.\ (m + 1)\ \theta \end{array}\right\} (2)$$

and the latter of these equations being reducible to

$$4\,(m + 1)^2\ .\ (1 - \cos.^2\theta)\,(1 - \cos.^2\overline{m + 1}\ .\ \theta) = \left\{(m+1)^2 \times (1 - \cos.^2\theta) + 1 - \cos.^2\overline{m + 1}\ .\ \theta - \frac{y^2}{r^2}\right\}^2,$$ and substituting

$$(2\cos.\ \theta)^{m+1} - (m + 1)\,(2\cos.\ \theta)^{m-1} + \frac{(m + 1)\ .\ (m - 2)}{2} \times (2\cos.\ \theta)^{m-3} - \&c.,$$ for its equivalent $\cos.\ (m + 1)\ \theta$ in the equations, the results will contain integral powers of $\cos.\ \theta$. Hence resolving those results, i. e., finding any two values of $\cos.\ \theta$ in terms of x, y, and constants, we may equate them, and thereby obtain an Algebraic equation between x and y. Q. E. D.

Ex. Let $m = 1$, or $\mathrm{R} = r$;

$$\left.\begin{array}{l}\text{Then } x = 2r \cdot \cos.\ \theta - r \cos.\ 2\,\theta \\ \text{and } y = 2r \ \sin.\ \theta - r \sin.\ 2\,\theta\end{array}\right\}$$

From the former we easily find

$$\text{Cos.}\ \theta = \frac{1 \pm \sqrt{3r-2x}}{2} \text{ which gives}$$

$$\text{Sin.}\ \theta = \frac{\sqrt{3 - 3r + 2x \mp 2\sqrt{3r-2x}}}{2}, \text{ which being sub-}$$

stituted in the latter we get

$$\frac{y}{2r} = \sin.\ \theta\ .\ (1 - \cos.\ \theta) = \frac{\sqrt{3 - 3r + 2x \mp 2\sqrt{3r-2x}}}{2}$$

$$\times \frac{1 \mp \sqrt{3r-2x}}{2}, \text{ which, after proper reductions, will give}$$

$$(x^2 + y^2 - rx)^2 = r^2 \{(x - r)^2 + y^2\}.$$

104. Let $x =$ the part produced of the given line a; then by the question

$$x^2 = (a + x)\ .\ a = a^2 + ax$$

$$\therefore x^2 - ax = a^2$$

$$\text{and } x = \frac{a}{2} \pm \sqrt{a^2 + \frac{a^2}{4}} = \frac{a \pm a\sqrt{5}}{2}$$

$$= \frac{a}{2} \cdot (1 \pm \sqrt{5})$$

which is the production required.

105. Let AB (Fig. 58,) $= a$, be the given line, required to construct a Tetraedron upon it.

In the plane ABC construct the equilateral $\triangle$ ABC, and let O be the point in it equally distant from A, B, C, and make OD $\perp$ plane ABC $= a\sqrt{\frac{2}{3}}$; then AD, BD, CD, being joined, the figure ABCD will be the Tetraedron required.

$$\text{For AO : BA :: sin. } 30^\circ : \text{sin. } 60^\circ$$

$$:: \frac{1}{2} : \frac{\sqrt{3}}{2} :: 1 : \sqrt{3}$$

$$\therefore\ AO = \frac{a}{\sqrt{3}}.$$

Hence $AD = \sqrt{AO^2 + DO^2} = \sqrt{\frac{a^2}{3} + \frac{2a^2}{3}} = a$, and similarly DC, BD, may be shewn to be $= a$.

$\therefore$ the four faces are equal equilateral $\triangle$ &c.

Again, r being taken in OD so that $rA = rD$, we have $AO^2 + Or^2 = Ar^2 = rD^2 = AO^2 + (OD - rD)^2 = AO^2 + OD^2 + rD^2 - 2OD \times rD$.

$\therefore\ 2\,OD \times rD = AO^2 + OD^2 = AD^2 = a^2$ or $rD = \frac{a^2}{2OD} = \frac{a^2}{2a\sqrt{\frac{2}{3}}} = \frac{a}{2}\sqrt{\frac{3}{2}}$, which is the radius of the circumscribing sphere.

106. To find the focus of a given parabola, either use the method adopted in Prob. 95, or the following one; drawing two parallel lines in the parabola QV, Q′V′, and let them be bisected in V, V′, join VV′ and produce it to meet the curve in P, thus determining an abscissa PV, corresponding to the ordinate QV. In like manner drawing any two parallel lines not parallel to the former, we get another pair of oblique co-ordinates pv, qv.

Now by property of the curve, we have (S being the focus)

$$\left.\begin{array}{l} QV^2 = 4SP \times PV \\ \text{and } qv^2 = 4\,Sp \times pv \end{array}\right\} \quad \therefore \left.\begin{array}{l} SP = \dfrac{QV^2}{4PV} \\[2ex] Sp = \dfrac{qv^2}{4pv} \end{array}\right\} \text{being known, if}$$

with P, p, as centres, and radii $=$ SP, Sp, we describe two circular arcs, their intersection will be the focus required.

107. From similar $\triangle$ OPM, ONQ, and the property of the parabola, we have

$$AN : AM :: QN^2 : PM^2 :: ON^2 : OM^2$$
$$:: (AO - AN)^2 : (AM - AO)^2$$

$\therefore$ AN $\times$ (AM^2 $-$ 2AO $\times$ AM $+$ AO^2) $=$ AM $\times$ (AN^2 $-$ 2AO $\times$ AN $+$ AO^2), and by reduction

$$AO^2 \times (AM - AN) = AN \,.\, AM\,(AM - AN)$$
$$\therefore AO^2 = AN \times AM, \text{ or}$$
$$AN : AO :: AO : AM. \qquad \text{Q. E. D.}$$

108. Let AP (Fig. 47,) be any conic section whose focus is S, and vertex A; also let La, Aa, be the tangents at the extremity of the latus rectum SL, and vertex A intersecting in a; then Aa $=$ AS.

For supposing $y = \frac{b}{a}\sqrt{2ax \mp x^2}$ to be the equation to the conic section, (which will be an ellipse, parabola, or circle, when the negative sign is taken, and an hyperbola when the positive,) a and b being axes, and x, y, originating in A, we have.

$y' - y = \frac{dy}{dx} \,.\, (x' - x)$ ..... (1) the equation to the tangent at any point. Hence, since $\frac{dy}{dx} = \frac{b}{a} \,.\, \frac{a \mp x}{\sqrt{2ax \mp x^2}}$

$$y' = \frac{b}{a} \,.\, \frac{a \mp x}{\sqrt{2ax \mp x^2}} \,.\, x' + \frac{bx}{\sqrt{2ax \mp x^2}} \text{ ..... (2)}$$

Let $x = AS = m$, and $x' = 0$;

Then $Aa = y' = \frac{b \,.\, m}{a\sqrt{2am \mp m^2}}$;

But $a^2 = b^2 + (a \mp m)^2 = b^2 + a^2 + m^2 \mp 2am$ from a well-known property of the curve,

$$\therefore 2am \mp m^2 = b^2$$
and $Aa = m = AS$. Q. E. D.

109. If R, r, be the radii of the circles, circumscribing and inscribing a $\triangle$ whose sides are a, b, c, then

$$R\,r = \frac{abc}{2\,(a+b+c)}.$$

For, supposing A, B, C, the angles to which a, b, c, are respectively opposite, since those angles are each bisected by the lines joining them and the centre of the *inscribed* ⊙, we have

$$a = r \cot. \frac{B}{2} + r \cot. \frac{C}{2}$$

$$b = r . \cot. \frac{C}{2} + r \cot. \frac{A}{2}$$

$$c = r . \cot. \frac{A}{2} + r \cot. \frac{B}{2}$$

$$\therefore \frac{a+b+c}{2r} = \cot. \frac{A}{2} + \cot. \frac{B}{2} + \cot. \frac{C}{2}$$

$$\therefore \frac{a+b-c}{2r} = \cot. \frac{C}{2}$$

$$\text{Hence } \frac{4r^2}{(a+b-c)^2 + 4r^2} = \sin.^2 \frac{C}{2} = \frac{1-\cos. C}{2}$$

$$= \frac{2ab - a^2 - b^2 + c^2}{4ab}$$

which gives by reduction

$$4r^2 = \frac{a+b-c}{a+b+c} \{c^2 - (a-b)^2\} \text{ ..... 1.}$$

Again, since the chord of an arc = 2 sin. of half that arc, we have

$$C = 2R \sin. \frac{2C}{2} = 2R \sin. C$$

$$\therefore R^2 = \frac{c^2}{4 \sin.^2 C}$$

$$= \frac{a^2 b^2 c^2}{(a+b+c).(a+b-c)(c+b-a).(c+a-b)} \text{ .... (2)}$$

$$\therefore 4R^2 r^2 = \frac{a^2 b^2 c^2}{(a+b+c)^2}$$

$$\text{and } Rr = \frac{abc}{2(a+b+c)}. \qquad \text{Q. E. D.}$$

110. The equation to an equilateral hyperbola whose axes are unity, and co-ordinates at the centre being

$$y = \sqrt{1+x^2}$$

we have $\int y dx = \int dx \sqrt{1 + x^2}$

$$= \frac{x\sqrt{1+x^2}}{2} - \frac{1}{2} . l. (x + \sqrt{1+x^2})$$

$\therefore$ Sector ACP $= \frac{1}{2} l. (x + \sqrt{1+x^2})$

and similarly ACp $= \frac{1}{2} . l. (x' + \sqrt{1+x'^2})$

$\therefore$ by the question

$$l. (x + \sqrt{1+x^2}) = n . l. (x' + \sqrt{1+x'^2})$$

$$\therefore x + \sqrt{1+x^2} = (x' + \sqrt{1+x'^2})^n$$

Now AC : AT :: CN : NP

Or 1 : T :: x : $\sqrt{1+x^2}$

$$\therefore \frac{1-T}{1+T} = \frac{x - \sqrt{1+x^2}}{x + \sqrt{1+x^2}} = \frac{1}{(x + \sqrt{1+x^2})^2}$$

Similarly $\frac{1-t}{1+t} = \frac{1}{(x' + \sqrt{1+x'^2})^2}$

$$\therefore \frac{1-T}{1+T} = \frac{1}{(x' + \sqrt{1+x'^2})^{2n}} = \left(\frac{1-t}{1+t}\right)^n,$$

111. Let LP′ (Fig. 47,) be a tangent at L the extremity of the latus rectum of the ellipse, meeting any ordinate PN produced in P′; then S being the focus, required to shew that SP = NP′.

a and b being the semi-axes of the ellipse, y, x, its co-ordinates at the vertex, and y', x', those of the tangent at any point, originating also at the vertex, we have

$$y = \frac{b}{a}\sqrt{2ax - x^2}$$

and $y' = \frac{b}{a} . \frac{a-x}{\sqrt{2ax - x^2}} \times x' + \frac{bx}{\sqrt{2ax - x^2}}$. (See 108.)

Let $x = AS = m = a \pm \sqrt{a^2 - b^2}$

Then, after the proper reductions, we get

$$y' = \frac{\pm\sqrt{a^2 - b^2}}{a} x' + a \pm \sqrt{a^2 - b^2}$$

and $\therefore y'^2 = \frac{a^2 - b^2}{a^2} x'^2 - \frac{2\sqrt{a^2 - b^2}}{a} (a \pm \sqrt{a^2 - b^2})\, x' +$

$2a^2 - b^2 + 2a\sqrt{a^2 - b^2}$ .... (1)

But $SP^2 = y^2 + SN^2 = \frac{b^2}{a^2} \cdot (2ax - x^2) + (a \pm \sqrt{a^2 - b^2})^2$

which reduces to

$$SP^2 = \frac{a^2 - b^2}{a^2} x^2 - \frac{2\sqrt{a^2 - b^2}}{a} \cdot (a \pm \sqrt{a^2 - b^2})\, x' + 2a^2$$

$$- b^2 + 2a\sqrt{a^2 - b^2} \text{ . . . . } (2)$$

which being identical with (1) when $x = x'$, we have

$SP^2 = NP'^2$, and $\therefore$

$SP = NP'$. Q. E. D.

The geometrical proof, although easier, is not grounded on such obvious and general principles as this.

112. Euler, who invented this theorem, conceived it to be universally applicable; but we shall presently see that it is true only in a very particular class of polyedrons.

Case 1. Let the polyedron be without perforation, and consist of one surface only.

Let F, S, E, be the number of faces, solid angles and edges of the polyedron, respectively.

Then supposing any one face projected upon a given plane in such a manner that the interior angles of the projection may be each less than two right angles, and all the others upon the same plane, so that their projections may be wholly within the former, (which is evidently possible), the projection thus formed of the polyedron will consist of F plane polygons, united by the S points of concourse of the E different sides, one of them circumscribing the rest. Now, supposing $n, n_1, n_2 \ldots\ldots n_F$ to represent the number of edges in the respective faces or polygons; then, since it can evidently make no difference in the sum of the inner angles of the interior polygons, whether or no any of them be greater than two right angles, (for they equally fill space about the points of concourse), we have

$$\left.\begin{array}{c} 2n_1 R - 4R \\ 2n_1 R - 4R \\ \&c. \\ 2n_F R - 4R \end{array}\right\} = 4(S - n) R + 2n R - 4R$$

$$\text{Or } n_1 + n_2 + \ldots\ldots - 2(F - 1) = 2S - n - 2.$$

But, since each of the sides, except those of the circumscribing polygon, is common to two of the polygons, we have

$$n_1 + n_2 + \ldots\ldots = 2(\mathrm{E} - n)$$

$$\therefore 2\mathrm{E} - 2\mathrm{F} = 2\mathrm{S} - 4$$

$$\therefore \mathrm{S} + \mathrm{F} = \mathrm{E} + 2 \ldots\ldots (1)$$

which shews the Theorem to be true in this case.

Case 2. Let the polyedron be supposed to be hollow, *i. e.*, to have two surfaces consisting of plain faces, one interior with respect to the other.

Then, supposing s, f, e; s', f', e', to be the number of solid angles, faces and edges of these surfaces, we get, by case 1,

$$\left.\begin{array}{l} s + f = e + 2 \\ s' + f' = e' + 2 \end{array}\right\}$$

and $s + s' = \mathrm{S}'$, $f + f' = \mathrm{F}$, and $e + e' = \mathrm{E}$

$\therefore$ in this case

$$\mathrm{S} + \mathrm{F} = \mathrm{E} + 4;$$

and, generally, if there be h interior boundaries, we get

$$\mathrm{S} + \mathrm{F} = \mathrm{E} + 2(h + 1) \ldots\ldots\ldots (2).$$

Case 3. Let the polyedron be annular, or have a perforation.

Then, supposing a plane to cut it into two polyedrons of the first class, making (m) new edges, and $\therefore$ (m) new solid angles, and *two* new faces in each, we have (using the above notation)

$$\left.\begin{array}{l} s + f = e + 2 \\ s' + f' = e' + 2 \end{array}\right\}$$

$$\therefore s + s' + f + f' = e + e' + 4$$

$$\text{Or } \mathrm{S} + m + \mathrm{F} + 4 = \mathrm{E} + m + 4$$

$$\therefore \mathrm{S} + \mathrm{F} = \mathrm{E};$$

and, generally, if there be p such perforations, we have

$$\mathrm{S} + \mathrm{F} = \mathrm{E} + 2 - 2p \ldots\ldots\ldots (3).$$

Case 4. Suppose the polyedron formed by the union of two others of the first class, in making the planes of two of their unequal faces coincide, so that the face of the one may be wholly within that of the other.

Let S, F, E, be the number of solid angles, faces, and edges of

the compound polyedron: s, f, e; s', f', e', those of the components, then by case (1)

$$\left.\begin{array}{l} s + f = e + 2 \\ s' + f' = e' + 2 \end{array}\right\} \therefore s + s' + f + f' = e + e' + 4$$

But $s + s' = S, f + f' = F + 1, e + e' = E$

$$\therefore S + F = E + 3;$$

and similarly, if the polyedron be formed by the union of n others, upon any face of one of them, we have

$$S + F = E + 2 + n.$$

Hence, generally, if $n_1, n_2, n_3, \ldots n$ be the number of such formations upon each of m faces, we have

$$S + F = E + 2 + n_1 + n_2 + n_3 + \ldots n_m \ldots (4)$$

Hence, then, Euler's Theorem may be generalized, by stating it, *If a Polyedron have* (h) *interior surfaces,* (p) *perforations,* and $n_1, n_2, n_3, \ldots n_m$ *augments upon* m *of its faces severally, then*

$$S + F = E + 2 + 2(h - p) + n_1 + n_2 + \ldots n_m.$$

113. In the axis AB of the cycloid AP (Fig. 59,) of which C is the centre of its generating ⊙, let CN = AM, then if MP be ⊥ AB, PN be joined, &c., as in the fig. required to shew that the sector ANP = △ MQB.

Let $AC = r$, $AM = x$, $PM = y$, then the equation to the cycloid is

$$y = AQ + QM = \text{vers.}^{-1} x + \sqrt{2rx - x^2}$$

$$\therefore \int x dy = \int \frac{x \sqrt{2r - x}}{\sqrt{x}} dx = \int \sqrt{2rx - x^2} \times dx$$

$$= \int QM \times d \,.\, AM = \text{area } AMQ, \, i.e., \, AmP = AMQ.$$

Hence sector $ANP = AMP - PNM$

$$= PM \times AM - \frac{PQ \times AC}{2} + \frac{CM \times MQ}{2} - \frac{MN \times PM}{2}$$

$$= \frac{PM \times (2AM - MN) - PQ \times AC}{2} + \frac{CM \times MQ}{2}$$

$$= \frac{MQ \times AC}{2} + \frac{CM \times MQ}{2} = \frac{MB \times QM}{2}$$

$= \triangle$ BMQ. Q. E. D.

TANGENTS AND ASYMPTOTES.

114. Let $x =$ AE, then $y = r \cdot \sin. x$

$$\therefore \text{ Subtangent } = \frac{y dx}{dy} = \tan. x.$$

And the area $= \int y dx = r \int dx \cdot \sin. x = c - r \cos. x$.

Let $x = 0$. Then $c = r^2$, and the area $= r^2 - r \cos. x$; let $x =$ AC, then the whole area $= r^2$.

115. Let AP (Fig. 60,) be any curve whatever, and suppose ordinates PP′ ⊥ AN, to be erected on AP as a line of abscissæ, tracing out the curve AP′; then T being the intersection of the tangents at P, P′, and $p'p$ the next position of the ordinate, if P′m' be drawn parallel to TPp we have, by similar △

$$\text{PT} : \text{PP}' :: \text{P}'m : mp'$$

$$\text{or PT} : y' :: d \cdot \text{AP} : dy'$$

and calling AP $= s$, we get

$$\text{PT} = \frac{y' ds}{dy'} \ldots\ldots\ldots\ldots\ldots (1)$$

which will always give the value of PT, when y' is a function of s, and the nature of AP is given.

In the problem, $y' = as^n$

And AP is a circle,

$$\therefore \text{PT} = \frac{s}{n}; \text{ and } s \text{ is known,}$$

∴ PT is known, and joining PT we get the tangent PT.

In the case of the cycloid AP is the generating circle, and $n = 1$, and ∴ PT = PP′.

Hence ∠ PPT $= \frac{1}{2}$ supplement of TPP′ $= \frac{1}{2}$ TPN = ∠ TPA by property of the circle. See Prob. 26, p. 22.

116. Let the base AB of the $\triangle$ ABC $= a$, and the angle A $= n$B, then AC being the radius vector ρ, and $\angle$ A $= \theta$ the angle traced by it, we have

$$\rho : a :: \sin. B : \sin. C = \sin. (A + B)$$
$$:: \sin. n\theta : \sin. (n + 1)\theta$$
$$\therefore \rho = \frac{a \sin. n\theta}{\sin. (n + 1)\theta} \ldots\ldots\ldots\ldots\ldots (1)$$

the polar equation to the locus of C, which will be a straight line, hyperbola, &c., according as $n = 1, 2,$ &c.

Now Subtangent (See Appendix to new edit. of *Simpson*, or *Lacroix*).

$$\text{ST} = \frac{\rho^2 d\theta}{d\rho} = \frac{a \sin.^2 n\theta}{n \cos. n\theta . \sin. \overline{n+1} . \theta - (n+1) \sin. n\theta \cos. \overline{n+1\theta}}$$

$= \dfrac{a \sin.^2 n\theta}{n \sin. \theta - \sin. n\theta . \cos. \overline{n+1}\,\theta}$, the subtangent at any point (ρ, θ).

Let $\rho = \infty$, then C $\doteq \pi - (n + 1)\theta = 0$, $\therefore \theta = \dfrac{\pi}{n+1}$,

$\therefore$ the subtangent to the asymptote is

$$\frac{a \sin.^2 \dfrac{n}{n+1} . \pi}{\overline{n + 1} . \sin. \dfrac{\pi}{n+1}}.$$

which is also given in position by its inclination to AB, which is $\dfrac{\pi}{2} - \theta = \dfrac{\pi}{2} - \dfrac{\pi}{n+1} = \dfrac{n-1}{n+1} . \dfrac{\pi}{2}$.

Hence drawing a $\perp$ at its extremity, we shall have the asymptote required.

CURVATURE AND EVOLUTES.

117. To find the radius R of curvature to the common cycloid, whose equation is

$$y = \sin.^{-r} x + \sqrt{2rx - x^2}$$

we have

$$R = \frac{dx^2}{-d^2y} \times \left(1 + \frac{dy^2}{dx^2}\right)^{\frac{3}{2}}$$

$$= \frac{x\sqrt{2ax - x^2}}{a}\left(1 + \frac{2a - x}{x}\right)^{\frac{3}{2}}$$

which reduces to

$$R = 2\sqrt{2a} \times \sqrt{2a - x}.$$

118. Required the chord of curvature parallel to the axis, of the common parabola, whose equation is $y^2 = px$.

$$R = \frac{dx^2}{-d^2y} \times \left(1 + \frac{dy^2}{dx^2}\right)^{\frac{3}{2}}$$

$$\text{But } \frac{dy}{dx} = \frac{1}{2}\sqrt{\frac{p}{x}}$$

$$-\frac{d^2y}{dx^2} = \frac{\sqrt{p}}{4} \cdot \frac{1}{x^{\frac{3}{2}}}$$

$$\therefore R = \frac{1}{2\sqrt{p}} \times (4x + p)^{\frac{3}{2}}.$$

Now the chord of curvature parallel to the axis $= 2R \therefore \frac{dy}{dz} =$

$$\frac{2R}{\sqrt{1 + \frac{dx^2}{dy^2}}} = \frac{4x + p}{2}.$$

119. By the preceding problem we learn that

$$R = \frac{1}{2\sqrt{p}}(4x + p)^{\frac{3}{2}}.$$

Again, in the expressions

$$\left.\begin{aligned} x - \alpha + (y - \beta)\frac{dy}{dx} = 0, \\ 1 + \left(\frac{dy}{dx}\right)^2 + (y - \beta)\frac{d^2y}{dx^2} = 0 \end{aligned}\right\}$$ (See Appendix to *Simpson's Fluxions* or *Lacroix*).

α, β are the co-ordinates of the centre of curvature, and $\therefore$ of the evolute of a curve; and if we can eliminate from them y and x, the co-ordinates of the curve, the result will be the equation to the evolute.

But since $y^2 = px$,

by substituting for $\frac{dy}{dx}, \frac{d^2y}{dx^2}$ hence obtained, we easily get

$$y - \beta = \frac{4x + p}{\sqrt{p}} \cdot \sqrt{x} = \frac{\frac{4y^2}{p} + p}{p} \cdot y, \text{ and } \therefore x - \alpha$$

$$= -2x - \frac{p}{2}.$$

$$\text{Hence } x = \frac{1}{3}\left(\alpha - \frac{p}{2}\right)$$

$$\text{and } -\beta = \frac{4x^{\frac{3}{2}}}{p^{\frac{1}{2}}} = \frac{4}{p^{\frac{1}{2}}} \cdot \frac{1}{\sqrt{27}}\left(\alpha - \frac{p}{2}\right)^{\frac{3}{2}}$$

$$\therefore \beta^2 = \frac{16}{27p}\left(\alpha - \frac{p}{2}\right)^3,$$

the equation to the evolute.

120. By the question, the equation to the curve is

$$x = r \cdot \text{vers.}\, y = r - r\cos. y,$$

$$\therefore \frac{dx}{dy} = r \sin. y$$

$$\text{and } \frac{d^2x}{dy^2} = r\cos. y.$$

$$\text{Hence } R = \frac{dz^3}{dy\ d^2x} = \frac{dy^2}{d^2x}\left(1 + \frac{dx^2}{dy^2}\right)^{\frac{3}{2}}$$
$$= r\cos. y\,(1 + r^2 \sin.^2 y)^{\frac{3}{2}}.$$

121. By problem 118 the radius of curvature of the parabola is

$R = \frac{1}{2\sqrt{p}}(4x + p)^{\frac{3}{2}}$, and the equation to the parabola being $y^2 = px$ the normal (N) $= \sqrt{y^2 + \frac{y^2dy^2}{dx^2}}$

$$= y\sqrt{1 + \frac{dy^2}{dx^2}} = y\sqrt{1 + \frac{p}{4x}} = \frac{\sqrt{p}}{2}\sqrt{4x+p}.$$

$$\therefore R = \frac{4N^3}{p^2}, \text{ or}$$

$$\therefore R : N :: N^2 : \frac{p^2}{4}. \qquad \text{Q. E. D.}$$

122. By problem 118 we have generally

$$R = \frac{1}{2\sqrt{p}}(4x + p)^{\frac{3}{2}}$$

$\therefore$ at the vertex of the parabola, where $x = 0$,

$$R = \frac{1}{2\sqrt{p}} \cdot p^{\frac{3}{2}} = \frac{p}{2}.$$

123. To find the evolute of the common cycloid, whose equation is

$$y = \text{vers.}^{-1} x + \sqrt{2rx - x^2}$$

Here

$$\frac{dy}{dx} = \sqrt{\frac{2r - x}{x}}$$

$$\frac{d^2y}{dx} = -\frac{r}{x\sqrt{2rx - x^2}}$$

which being substituted in the equations

$$\left.\begin{array}{r} x - \alpha + (y - \beta)\dfrac{dy}{dx} = 0 \\ 1 + \left(\dfrac{dy}{dx}\right)^2 + (y - \beta)\dfrac{d^2y}{dy^2} = 0 \end{array}\right\}$$

give

$$\left.\begin{array}{l} x - \alpha = -4r + 2x \\ \text{and } y - \beta = 2\sqrt{2rx - x^2} \end{array}\right\}$$

$$\therefore x = \alpha - 4r$$

$$\beta + 2\sqrt{2rx - x^2} = \text{vers.}^{-r}x + \sqrt{2rx - x^2}$$

$\therefore \beta = \text{vers.}^{-r}(\alpha - 4r) - \sqrt{2r(\alpha - 4r) - (\alpha - 4r)^2}$ the equation to the evolute, which is therefore a cycloid equal to the curve itself, but having its base $\perp$ base of the given cycloid at either extremity.

CONTRARY FLEXURE.

124. To find the points of contrary flexure of a curve, whose equation is

$$x = (ly)^3, \text{ we have}$$

$$\frac{d^2x}{dy^2} = \frac{3l y}{y^3}(2 - ly) = 0 \text{ or } \infty, \text{ by the rule, which gives}$$

$$ly = 2, \text{ or } x = (ly)^3 = 8,$$

and $y = e^2$, which are the co-ordinates of the point required.

125. If R be the radius of the wheel, r the distance of the generating point from its centre, x the abscissa of the trochoid, measured from the vertex or highest point of it; then the equation to the curve is

$$y = \frac{R}{r} \text{vers.}^{-r}\ x + \sqrt{2rx - x^2} \ \ldots\ldots (1)$$

$$\therefore \frac{dy}{dx} = \frac{R + r - x}{\sqrt{2rx - x^2}} \ \ldots\ldots (2)$$

$$\therefore \frac{d^2y}{dx^2} = \frac{Rx - (R+r)r}{(2rx - x^2)^{\frac{3}{2}}}$$

which being put $= 0$ or ∞, will give

$$\left.\begin{aligned} Rx - (R + r)\, r &= 0 \\ \text{or } 2rx - x^2 &= 0 \end{aligned}\right\}$$

and we $\therefore$ have two points of contrary flexure, whose abscissæ are

$$(R + r)\frac{r}{R}, \text{ and } 2r,$$

and ordinates

$$\frac{R}{r}. \text{vers.}^{-r} (R + r)\frac{r}{R} + \frac{r}{R}\sqrt{R^2 - r^2}, \text{ and } \frac{R}{r} \text{ vers.}^{-r}\ 2r \text{ re-}$$

spectively, the latter point being a ceratoid, as we learn by substituting in equation 2.

THE QUADRATURE OF CURVES.

126. The equation to the curve being

$$y = \frac{4x^2}{\sqrt{a^2 - x^2}}$$

we have $\int y dx = \int \frac{4x^2 dx}{\sqrt{a^2 - x^2}}$

$$= 4a \int \frac{a dx}{\sqrt{a^2 - x^2}} - 4 \int dx \sqrt{a^2 - x^2}$$

$$= 4a \sin.^{-1} x - 4A$$

where A = that part of the area of a ⊙ whose rad. is a, which is comprised between the ordinates at the centre and extremity of x.

Let $x = a$, which it cannot exceed, then the whole area

$$4 \int y dx = 16a^2 \frac{\pi}{2} - 4a^2\pi$$

$$= 4a^2\pi = 4 \odot \text{ (rad.} = a).$$

127. Here $y = a.l. \frac{a - \sqrt{a^2 - x^2}}{x + \sqrt{a^2 - x^2}} = 2a.\, l\, (a - \sqrt{a^2 - x^2})$
$- 2al.\, x.$

$\therefore \int y dx = 2a \int dx\, l\, (a - \sqrt{a^2 - x^2}) - 2a \int dx l x$

$$= 2ax\, l(a - \sqrt{a^2 - x^2}) - 2a \int \frac{x^2 dx}{\sqrt{a^2 - x^2}\, (a - \sqrt{a^2 - x^2})} - 2ax \times$$

$$lx + 2ax = 2ax \times \left\{ 1 + l. \frac{a - \sqrt{a^2 - x^2}}{x} \right\} -$$

$$2a \int \frac{x^2 dx}{\sqrt{a^2 - x^2}\, (a - \sqrt{a^2 - x^2})},$$

Let $a - \sqrt{a^2 - x^2} = u$,

Then $\int \frac{x^2 dx}{\sqrt{a^2-x^2}(a-\sqrt{a^2-x^2})} = \int(2a-u)du$

$= 2au - \frac{u^2}{2} = a^2 + \frac{x^2}{2} - a\sqrt{a^2-x^2}.$

Hence

$\int ydx = 2ax\left\{1 + l.\frac{a-\sqrt{a^2-x^2}}{x}\right\} - 2a^3 - ax^2$

$+ 2a^2\sqrt{a^2-x^2} + C.$

Let $y = 0$, or $x = a$. Then $\int ydx = 0$ and $C = 3a^3 - 2a^3$.

$\therefore \int ydx = 2ax\left\{1+l.\frac{a-\sqrt{a^2-x^2}}{x}\right\} + a^3 - 2a^3 - ax^2 + 2a^2\sqrt{a^2-x^2}$

and putting $x = b$, we shall have the area contained between the abscissæ a and b.

128. Here $y = \frac{a^x}{x}$

$\therefore \int ydx = \int \frac{a^x}{x} dx$, which cannot be expressed in a finite number of terms.

We know that

$a^x = 1 + la + \frac{(la)^2}{2}.x + \frac{(la)^3}{2.3}.x^2 + \&c.$

$\therefore \int ydx = lx + xla + \frac{x^2(la)^2}{2.2} + \frac{x^3(la)^3}{2.3.3} + \&c. + C.$ which will converge for some particular values of a and x.

For the whole theory of the integration of $\frac{a^x dx}{x}$, the reader may consult *Lacroix*, Vol. III., pp. 475....

129. Here $y = \frac{a^3}{a^2 - x^2}$.

$\therefore \int ydx = \int \frac{a^3 dx}{a^2 - x^2} = \frac{a^2}{2} l.\frac{a+x}{a-x} + C$

Let $x = 0$, then $y = a$, and area $= 0$ $\therefore C = 0$

and $\int y dx = \frac{a^2}{2} l. \frac{a+x}{a-x}$.

130. The regular octagon may be divided into eight $=$ $\triangle$ whose $=$ bases shall be equally distant from the concourse of their vertices, and the area of each $\triangle = \frac{\text{base} \times \perp}{2}$

$$= \frac{a}{2} \times \frac{a}{2} \cot. \frac{\pi}{8} = \frac{a^2}{4} \cot. \frac{\pi}{8}.$$

$\therefore$ the whole area $= 2a^2 \cot. \frac{\pi}{8}$.

131. Since $y = ax + \frac{b}{\sqrt{1-x^2}}$, we have

$\int y dx = \frac{ax^2}{2} + b \sin.^{-1} x + \text{C}$.

Let $x = 0$, then $y = b$, and area $= 0$
and $\text{C} = - b \sin.^{-1} 0 = 0$.

$\therefore \int y dx = \frac{ax^2}{2} + b \sin.^{-1} x$.

132. The area of a $\triangle = \frac{\text{base} \times \perp}{2}$.

$\therefore$ if a be its base, since $\perp$ will

$= b. \sin. (\angle \text{ opposite to C}) = b \times \frac{2}{ab} \times$

$\sqrt{\frac{\text{S}}{2} \times \left(\frac{\text{S}}{2} - a\right).\left(\frac{\text{S}}{2} - b\right).\left(\frac{\text{S}}{2} - c\right)}$ by trig.

$\therefore$ area $= \sqrt{\frac{\text{S}}{2} \cdot \left(\frac{\text{S}}{2} - a\right)\left(\frac{\text{S}}{2} - b\right)\left(\frac{\text{S}}{2} - c\right)}$.

133. Let the faces PAB, PAC, ABC, (Fig. 61) of the

pyramid, $\perp$ to one another, be represented by F, F′, F″, then $(PCB)^2 = F^2 + F'^2 + F''^2$.

For passing through AB a plane $\perp$ BC, the intersections PN, AN will each be $\perp$ BC;

$$\text{Hence } (BCP)^2 = \frac{BC^2 \times PN^2}{4}$$

$$= \frac{AN^2 + AP^2}{4} \times BC^2 = \frac{AN^2 \times BC^2}{4}$$

$$+ \frac{AP^2 \times AB^2}{4} + \frac{AP^2 \times AC^2}{4}$$

$$= F^2 + F'^2 + F''^2.$$

134. Let the radius of the $\odot = r$, AM (Fig. in Enunciation) $= y$, PM $= x$; then, by the question,

$$y = AC - x = \sqrt{x^2 + 2rx - x^2} - x$$

$$= \sqrt{2rx} - x$$

the equation to curve, which is a common parabola.

Hence the maximum value of y is obtained by putting

$$\frac{dy}{dx} = \frac{\sqrt{2r}}{2\sqrt{x}} - 1 = 0$$

which gives $x = \frac{r}{2}$ and $\therefore y = r - \frac{r}{2} = \frac{r}{2}$

Again the area $= \int y dx$

$$= \sqrt{2r} \int \sqrt{x} dx - \int x dx = \frac{2\sqrt{2r}}{3} x^{\frac{3}{2}} - \frac{x^2}{2}.$$

135. Since $xy^2 = a^3$ or $y = a^{\frac{3}{2}} \frac{1}{\sqrt{x}}$, we have

$$\int y dx = 2a^{\frac{3}{2}} \sqrt{x} + C = \frac{2a^3}{y} + C$$

Let $x = c$, then $y = \infty$ and $C = 0$.

$\therefore \int y dx = \frac{2a^3}{y}$, and $\therefore$ the area between the ordinates b and c is

$$2a^3 \left(\frac{1}{c} - \frac{1}{b}\right) = \frac{2a^3}{bc}(b - c).$$

136. For the investigation, see *Vince*, *Simpson*, or *Lacroix*. Since $a^m x^n = y^{n+m}$, we have

$$\int y dx = a^{\frac{m}{n+m}} \int x^{\frac{n}{n+m}} dx$$
$$= \frac{n+m}{2n+m} \cdot a^{\frac{m}{n+m}} x^{\frac{2n+m}{n+m}}$$
$$= \frac{n+m}{2n+m} a^{\frac{m-2nm-m^2}{n+m}} \times y^{\frac{2n+m}{n}} = a' y^{\frac{2n+m}{n}}$$

$\therefore$ between $y = b$ and c, the area is

$$a' \left(c^{\frac{2n+m}{n}} - b^{\frac{2n+m}{n}}\right).$$

137. Here $y \sqrt{2ax - x^2} = ax$,

$$\therefore \int y dx = \int \frac{ax dx}{\sqrt{2ax - x^2}} = \int \frac{a^2 dx}{\sqrt{2ax - x^2}} -$$
$$\int \frac{a(adx - xdx)}{\sqrt{2ax - x^2}} = a \text{ vers.}^{-1} x - a \sqrt{2ax - x^2}$$

Let $x = a$. Then $\text{vers.}^{-1} a = a \times AC$.

$$\text{And } \int y dx = a \times AC - a^2$$
$$= 2ABC - a^2$$
$$= ABC - (a^2 - ABC)$$
$$= ABC - AED \text{ (Fig. in enun.)}$$

138. Let r be the radius of the given $\odot$, then, by the question

$$x = \sin. y \text{ (to rad. } r)$$
$$\therefore \int x dy = \int dy \sin. y = C - r \cos. y$$
$$\therefore \int y dx = xy - \int x dy = y \sin. y + r \cos. y - r^2.$$

139. Since the equation to the parabola is $y^2 = px$, we have

$$\int y dx = \sqrt{p} \int \sqrt{x}\, dx = \frac{2\sqrt{p}}{3} x^{\frac{3}{2}} = \frac{2}{3} \frac{y^3}{p}.$$

$\therefore$ the area between $y = b$ and c, is

$$\frac{2}{3p} \cdot (c^3 - b^3).$$

140. By the question

$$\frac{ydx}{dy} = \frac{b}{1 + y^2} \dots (1)$$

$$\therefore \int ydx = b \int \frac{dy}{1 + y^2} = b \tan.^{-1} y + C$$

where C is perfectly arbitrary, because if the equation be integrated it will contain an arbitrary constant.

141. Let PN, P′Q (Fig. 62) be any two of the rectangles inscribed in the interior and exterior of the parabola AP, and put $PN = y$, $P'N' = y'$, $AN = x$, $AN' = x'$, and let p be the parameter of the parabola, then

$$PN : P'Q :: y'(x - x') : x'(y - y')$$

$$:: \frac{y'}{p}(y^2 - y'^2) : \frac{y'^2}{p} \cdot (y - y')$$

$$:: y + y' : y'$$

Let the number of rectangles be indefinitely increased, and $\therefore$ NN′ diminished, then $y = y'$ and by the fourth lemma of the Principia,

$$\begin{aligned} APN : APQ &:: PN : P'Q \text{ ultimately} \\ &:: 2y : y \\ &:: 2 \ : 1 \end{aligned}$$

$$\therefore APN : ANP :: 2 : 2 + 1 :: 2 : 3$$

$$\therefore APN = \frac{2}{3} ANP \text{ the circumscribing rectangle.}$$

142. For the investigation, see *Vince*, *Simpson*, or *Lacroix*. The equation to the hyperbola, referred to its centre, being $y = \frac{b}{a}\sqrt{x^2 - a^2}$, we have

$$\int yz = \frac{b}{a} \int dx \sqrt{x^2 - a^2}$$

$$= \frac{b}{2a} x \sqrt{x^2 - a^2} - \frac{ab}{2} \, l. \, (x + \sqrt{x^2 - a^2}) + C$$

as we learn, by putting $x\sqrt{x^2-a^2} = u$ &c.

Let $x = a$. Then $C = \frac{ab}{2}\, la$.

And $\int y dx = \frac{b}{2a}\, x\sqrt{x^2-a^2} - \frac{ab}{2} . l . \frac{x+\sqrt{x^2-a^2}}{a}$

Now, since the asymptotes originate in the centre, and, by a known property of the curve, the ordinate of an asymptote corresponding to its abscissa a, $= b$, its co-ordinates are always in the ratio of $b : a$. $\therefore$ the Δ formed by the asymptote and its co-ordinates x and $x \times \frac{b}{a}$, $= \frac{bx^2}{2a}$. Hence the area between the asymptotes and curve

$$= \frac{x^2 b}{a} - \frac{b}{a} . x\sqrt{x^2-a^2} + ab\ l . \frac{x+\sqrt{x^2-a^2}}{a}.$$

143. Since the diagonals (d,d') of a rhombus bisect each other at right angles, its area will evidently be

$$\frac{d \times d'}{2}.$$

But if θ be the $\angle$ subtended by d', we have (from the nature of the rhombus)

$$d' = d \times \tan. \frac{\theta}{2}$$

$$\therefore \text{ the area} = \frac{1}{2}\, d^2 \tan. \frac{\theta}{2}.$$

In the problem $d = a$, and $\theta = 60^\circ$.

$$\therefore \text{ area} = \frac{1}{2}\, a^2 \times \tan.\ 30^\circ = \frac{a^2}{2\sqrt{3}}.$$

144. Let $AM = x$, $PV = y$, $AB = 2$, the radius being supposed unity; then by the question

$$y = \tan. \left(\frac{1}{2} \text{ vers.}^{-1}\, x\right).$$

Let $\text{vers.}^{-1}\, x = z$; then $x = \text{vers.}\ z$

$= 1 - \cos. z = 2 \sin.^2 \frac{z}{2}$; hence $\tan. \frac{z}{2}$

$= \sqrt{\frac{x}{2-x}}$, $\therefore$

$$y = \sqrt{\frac{x}{2-x}} \text{ ..... } (1)$$

the equation to the curve.

Now $\int y dx = \int \frac{x dx}{\sqrt{2x - x^2}}$

$\therefore$ AM $= \text{vers.}^{-1} x - \sqrt{2x-x^2}$ (137)

$$\left.\begin{array}{l} \text{But AMP} = \int \sqrt{2x-x^2}\, dx \\ \text{and } \triangle \text{ AMP} = \dfrac{x\sqrt{2x-x^2}}{2} \end{array}\right\}$$

$\therefore$ segment AP $= \int \sqrt{2x-x^2}.\, dx - \frac{x\sqrt{2x-x^2}}{2}$

Hence $d.$AP $= \frac{x dx}{2\sqrt{2x-x^2}} = \frac{1}{2}.\, d.$AMV;

and they begin together,

$\therefore$ AMV $=$ 2AP.

Again $\frac{dy}{dx} = \frac{x}{(2x-x^2)^{\frac{3}{2}}}$

and $\frac{d^2y}{dx^2} = \frac{2}{(2x-x^2)^{\frac{3}{2}}}$

which $= \infty$, when $x = 2$; whence the position of the point of contrary flexure.

145. The equation to an ellipse referred to the centre being

$$y^2 = \frac{b^2}{a^2}(a^2 - x^2)$$

the rectangle inscribed in a quadrant of it which $= y \times x$, is

$$\frac{b}{a}.\, x\sqrt{a^2 - x^2}$$

Hence $d.\frac{y\times x}{dx}=\frac{b}{a}.\sqrt{a^2-x^2}-\frac{b}{a}\frac{x^2}{\sqrt{a^2-x^2}}$ being assumed $=0$, we get

$$a^2-2x^2=0,$$

which gives $x=\frac{a}{\sqrt{2}}$, and $y=\frac{b}{a}\sqrt{a^2-\frac{a^2}{2}}=\frac{a}{\sqrt{2}}$. $\therefore$ the greatest rectangle inscriptible in the quadrant of the ellipse is (Fig. 63).

$$NM=\frac{ab}{2}.$$

Again, $\int ydx=\frac{b}{a}\int\sqrt{a^2-x^2}\times dx$

$=\frac{b}{2a}x\sqrt{a^2-x^2}+\frac{b}{2}\sin.^{-1}x$, as we learn by assuming $x\sqrt{a^2-x^2}=u$, substituting, &c.

Let $x=\frac{a}{\sqrt{2}}$

Then $\int ydx=\frac{ab}{4}+\frac{b}{2}\sin.^{-1}\left(\frac{a}{\sqrt{2}}\right)=$ BCNP.

Let $x=a$. Then CBPA $=\frac{b}{2}\sin.^{-1}a$.

Hence PMB $=$ BCNP $-$ NM $=\frac{ab}{4}+\frac{b}{2}.(45^\circ$ of $\odot$ rad. $=a.)$

$-\frac{ab}{2}=\frac{b}{2}45^\circ-\frac{ab}{4}$.

And ANP $=$ ACB $-$ CBPN $=\frac{b}{2}.\,90^\circ-\frac{ab}{4}$

$-\frac{b}{2}45^\circ=\frac{b}{2}45^\circ-\frac{ab}{4}$.

$\therefore$ PMB $=$ ANP. Q. E. D.

146. Let a, b, p; a', b', p' be the axes and parameters of the ellipses E, E', respectively; then, by the preceding problem, we have, $E=4\times\frac{b}{2}\sin.^{-1}a=\frac{b}{a}.\frac{a}{2}\times$ circumference of a

circle (rad. $= a$) $= \frac{b}{a} \times$ its area $= \frac{b}{a} a^2\pi = ab\,\pi$. Similarly $E' = a'b'\pi$.

$$\therefore E : E' :: ab : a'b'.$$

But by the property of the ellipse

$$b = \sqrt{ap} \text{ and } b = \sqrt{a'p'}$$

$$\therefore E : E' :: a^{\frac{3}{2}}\sqrt{p} : a'^{\frac{3}{2}}\sqrt{p'}. \quad \text{Q.E.D.}$$

147. Let A,A′ be the arcs of the circles whose radii are R, R′; then by the question (area of sector $= \frac{R \times A}{2}$)

$$\frac{R \times A}{2} = \frac{R' \times A'}{2}$$

$$\therefore R : R' :: A' : A :: \frac{1}{A} : \frac{1}{A'}.$$

148. Let Pap (Fig. 64.) be the section required, aN being the intersection of the generating and cutting planes, and BC the diameter of the base $\perp$ Pp the intersection of its plane with Pap; then putting

$ab =$ BN $= a$, aN $= x$, PN $= p$N $= y$, we have, by the property of $\odot$,

$$y^2 = BN \times NC = ab \times NC = a \times x \times \frac{\sin.A}{\sin.C}$$

$$= ax \times \frac{\sin.A}{\cos.\frac{A}{2}} = 2\sin.\frac{A}{2} \times ax$$

the equation of any parabola of the cone.

But by the question,

$$y^2 = BN \times NC = \text{area of parabola} = \frac{2}{3}xy$$

$$\therefore 2\sin.\frac{A}{2} \times ax = \frac{2}{3}xy$$

$$\therefore y = 3\sin.\frac{A}{2} \times a$$

and $x = \dfrac{y^2}{2a \sin. \dfrac{A}{2}} = \dfrac{9a \sin. \dfrac{A}{2}}{2}$

which determine the position of the parabola.

149. Let SP $= \rho$, and the $\angle$ traced by it $= \theta$, then by the question

$$\theta \propto \rho^m = a\rho^m$$

and we hence obtain the area

$$\int \frac{\rho^2 d\theta}{2} = \frac{ma}{2} \int \rho^{m+1}\, d\rho = \frac{ma}{2\,(m+2)}\,\rho$$

$$= \frac{ma}{2\,(m+2)} \left(\frac{\theta}{a}\right)^{\frac{m+2}{m}} \propto \theta^{\frac{m+2}{m}}$$

Let $\theta = 2\pi,\ 4\pi,\ 6\pi \ldots\ldots 2n\pi$ successively, then the corresponding areas, described by $1,\ 2,\ 3, \ldots\ldots n$ revolutions of ρ are evidently as

$$1, 2^{\frac{m+2}{m}}, 3^{\frac{m+2}{m}}, 4^{\frac{m+2}{m}} \ldots\ldots n^{\frac{m+2}{m}}.$$

150. The equation to the conchoid referred to the centre C (Fig. 65,) of revolution of its generating line ρ, is

$$\rho = \frac{b}{\cos. \theta} + a$$

$\therefore$ Area ACP traced by ρ, is

$$\int \frac{\rho^2 d\theta}{2} = \int \left(\frac{b}{\cos. \theta} + a\right)^2 \frac{d\theta}{2}$$

$$= \frac{b^2}{2} \cdot \int \frac{d\theta}{\cos.^2 \theta} + ab \int \frac{d\theta}{\cos. \theta} + \frac{a^2}{2}\theta + C$$

$$= \frac{b^2}{2} \tan. \theta + ab \,.\, l. \tan. \left(\frac{\pi}{4} + \frac{\theta}{2}\right) + \frac{a^2}{2}\theta + C$$

See Appendix to new edition of *Simpson*, p. 335, or *Lacroix*.

Let $\theta = 0$. Then

$$C = -ab\, l. \tan. \frac{\pi}{4} = -ab \,.\, l.\, 1 = 0.$$

$$\therefore \int \frac{\rho^2 d\theta}{2} = \frac{b^2}{2} \tan. \theta + ab\, l. \tan. \left(\frac{\pi}{4} + \frac{\theta}{2}\right) + \frac{a^2}{2} \theta.$$

If the area ABNP be required, we have

$$\triangle\ BCQ = \frac{b}{2} \times BQ = \frac{b}{2} \sqrt{(\rho - a)^2 - b^2} = \frac{b^2}{2} \tan. \theta,$$

and since $\triangle$ BCQ, PQN are similar

$$\triangle\ PQN = \triangle\ BCQ \times \frac{a^2}{(\rho - a)^2}$$

$$= \frac{a^2 b^2}{2} \cdot \frac{\tan. \theta}{b^2} \times \cos.^2 \theta = \frac{a^2}{2} . \sin. \theta \cos. \theta$$

Hence

$$ABNP = ACP - BCQ + PQN$$

$$= ab\, l. \tan. \left(\frac{\pi}{4} + \frac{\theta}{2}\right) + \frac{a^2}{2}\theta + \frac{a^2}{4} \sin. 2\,\theta.$$

This may also be expressed in terms of $PN = y$, or $BN = x$.

For $y = a . \cos. \theta$

$$\therefore \sin. \theta = \frac{\sqrt{a^2 - y^2}}{a}$$

$$\text{Hence } \theta = \cos.^{-1} \frac{y}{a} = \frac{1}{a} \cos.^{-a} y$$

$$\text{and } \tan. \left(\frac{\pi}{4} + \frac{\theta}{2}\right) = \frac{1 + \sin. \theta}{\cos. \theta} = \frac{a + \sqrt{a^2 - y^2}}{y}$$

$$\text{and } \frac{a^2}{4} \sin. 2\,\theta = \frac{a^2}{2} \sin. \theta . \cos. \theta = \frac{y\sqrt{a^2 - y^2}}{2}$$

$$\therefore ABNP = ab\, l. \frac{a + \sqrt{a^2 - y^2}}{y} + \frac{a}{2} \cos.^{-a} y + \frac{y . \sqrt{a^2 - y^2}}{2}$$

a result which is the same as that deduced by *Simpson* in his Fluxions, p. 147, from the equation

$$x^2 y^2 = (b + y)^2 \times (a^2 - y^2.)$$

151. Let d, d', be the diagonals of the trapezium, θ their inclination, m, m', the segments of d'. Then, since the trapezium

= sum of two $\triangle$, whose common base is $d = \frac{md \sin. \theta}{2} + \frac{m'd \sin. \theta}{2} = \frac{m+m'}{2} . d \sin. \theta = \frac{dd'}{2} \sin. \theta.$

152. Let x, y, be the co-ordinates of the hyperbola Vv, (Fig. 33,) originating in the centre A, and measured along the asymptotes, then by property of the curve

$$xy = \text{const.} = m, \text{ the equation.}$$

Let BC touch the curve in P, then the subtangent MC (as it may easily be proved,) $= \frac{ydx}{dy} = -\frac{m^2}{x} \times \frac{m}{y^2}$ $= x = \text{AM}.$

Hence PB = PC, and $\triangle$ ACB = 2 AMPN = $2xy \times \sin. \text{A}$ $= 2m \sin. \text{A} =$ a const. Q. E. D.

A geometrical demonstration may be seen in most books on conic sections.

153. Let the radius of the circle CA (Fig. 66,) $= r$, AN $= x$, PN $= y$; then by similar $\triangle$

$$y = (1 - x) \times \frac{\text{D}n}{\text{C}n} = (1 - x) \tan. \left(\frac{1}{2} \text{vers.}^{-1} x\right).$$

Let $\text{vers.}^{-1} x = 2z$,

then $x = \text{vers. } 2z = 1 - \cos. 2z = 2 \sin.^2 z$,

$$\therefore \sin. z = \sqrt{\frac{x}{2}}$$

and $\cos. z = \sqrt{\frac{2-x}{2}}$ $\therefore \tan. z = \sqrt{\frac{x}{2-x}}$

$$\therefore y = (1 - x) \sqrt{\frac{x}{2-x}}$$

the equation to the curve.

Hence $\int ydx = \int dx \sqrt{2x - x^2} + \int \frac{(1-x)\, dx}{\sqrt{2x - x^2}} - \int \frac{dx}{\sqrt{2x - x^2}}$

$$= \text{area CDA} + \sqrt{2x - x^2} - \text{vers.}^{-1} x.$$

Let $x = \text{CA} = 1$. Then

$$\int ydx = \text{ACQ} + 1 - \text{ABQ} \times \text{CA}$$
$$= \text{CA}^2 - \text{ACQ}.$$

154. Let A, B, C, be angles of the $\triangle$, of which make $B = \theta$, $A = \pi - 3\theta$, the side $AB = a$, and $BC = \rho$; then

$$\rho : a :: \sin. 3\theta : \sin. (\pi - 3\theta + \theta)$$

$$\therefore \rho = \frac{a \sin. 3\theta}{\sin. 2\theta} \ldots\ldots\ldots (1)$$

the polar equation of the locus of C.

$$\therefore \text{Area} = \int \frac{\rho^2 d\theta}{2} = \frac{a^2}{2}\int \frac{d\theta \sin.^2 3\theta}{\sin.^2 2\theta}$$

$$= \frac{a^2}{2}\int \left(2 \cos. \theta - \frac{1}{2 \cos. \theta}\right)^2 d\theta$$

$$= 2a^2 \int d\theta \cos.^2 \theta - a^2 \int d\theta + \frac{a^2}{8}\int \frac{d\theta}{\cos.^2\theta}$$

$$= a^2 \sin. \theta . \cos. \theta + a^2 \theta - a^2 \theta + \frac{a^2}{8} \tan. \theta$$

$$= \frac{a^2}{8} . \tan. \theta \times (8 \cos.^2 \theta + 1)$$

the area required.

155. Since $a^2y^2 - a^2x^2 + x^4 = 0$

or $y = \pm \frac{x}{a}\sqrt{a^2 - x^2}$, we have

$$\int ydx = C - \frac{2}{3a}(a^2 - x^2)^{\frac{3}{2}}$$

$$= \frac{2}{3}a^2 - \frac{2}{3a}(a^2 - x^2)^{\frac{3}{2}}.$$

Let $x = \pm a$ (which it cannot exceed,)

Then the whole area is

$$4 \times \frac{2}{3}a^2 = \frac{8}{3}a^2.$$

156. Let a, b, C, denote the two sides of the $\triangle$ plane or spherical, and C the included $\angle$; then in the former case, the area

$$\triangle = \frac{\text{base} \times \perp}{2} = \frac{ab}{2} . \sin. C.$$

Now in the latter, A, B, being the other two $\angle$, it is well known that the area

$$= A + B + C - \pi.$$

But by Napier's Analogies

$$\tan. \frac{A+B}{2} = \cot. \frac{C}{2} \times \frac{\cos. \frac{a-b}{2}}{\cos. \frac{a+b}{2}}$$

$$\therefore A + B = 2 \tan.^{-1} \left\{\cot. \frac{C}{2} . \frac{\cos. \frac{a-b}{2}}{\cos. \frac{a+b}{2}}\right\}$$

$\therefore$ the spherical Δ, T is given by the equation

$$T = C + 2 \tan.^{-1} \left\{\cot. \frac{C}{2} . \frac{\cos. \frac{a-b}{2}}{\cos. \frac{a+b}{2}}\right\}$$

Another expression for T is

$$\text{Cot.} \frac{T}{2} = \frac{\cot. \frac{a}{2} . \cot. \frac{b}{2} + \cos. C}{\sin. A}$$

which the reader will have no difficulty in investigating.

157. Let ABC (Fig. 67,) be either end of the oblique triangular prism A′C, and (supposing the prism to be produced,) let Cbc be a section made by a plane $\perp$ its axis, and $\therefore$ to its sides; then NCD being the intersection of the planes ACB, bcC, and D that of the lines bc, ND, if through Ab, Bc we make planes to pass $\perp$ ND, the $\angle$ bNA, cMB between the intersections of these planes with those of ACB, bCc, will each be the inclination of the plane ABC to the plane bCc, *i. e.*, the complement of the inclination of ABC to the axis, (since it is $\perp$, by construction, to bCc). Let this $\angle = \frac{\pi}{2} - \theta$.

$$\left.\begin{aligned} \text{Now } \Delta\ ACD &= \frac{CD \times AN}{2} \\ \text{and } \Delta\ BCD &= \frac{CD \times BM}{2} \end{aligned}\right\}$$

$$\therefore \triangle\, ACB = CD \times \frac{AN - BN}{2}$$

$$= \frac{CD}{2} \cdot \left\{ \frac{bN}{\cos.\left(\frac{\pi}{2} - \theta\right)} - \frac{cM}{\cos.\left(\frac{\pi}{2} - \theta\right)} \right\}$$

$$= \frac{CD}{2 \sin. \theta} \cdot (bN - cM) \ldots\ldots 1$$

Again $\triangle\, bCc = bCD - CcD = \frac{CD}{2} \cdot (bN - cM)$

$$\therefore \triangle\, ACB = \frac{1}{\sin. \theta} \times \triangle\, bCc.$$

But for the same prism, the section $\perp$ axis bCc is evidently constant.

$$\therefore \text{end } ACB \propto \frac{1}{\sin. \theta}$$

i. e., $ACB : A'C'B' :: \frac{1}{\sin. \theta} : \frac{1}{\sin. \theta'}$.

This proof which only applies to triangular prisms, may be extended to those of whose ends are any polygons whatever, (and ∴ to cylinders), by dividing them into their component triangular prisms.

158. Let the radius of the circle $= 1$.

Then, by the question

$$y = \text{vers. } x = 1 - \cos x,$$

the equation to the curve.

$\therefore \int y dx = x - \sin. x$, the area required.

Again,

$$\frac{dy}{dx} = \sin. x$$

$$\frac{d^2 y}{dx^2} = \cos. x$$

which $= 0$, when $x = \frac{\pi}{2}, \frac{3\pi}{2} \ldots\ldots \frac{(2n-1)\pi}{2}$)

or when $y = 1$, which values of the co-ordinates give the positions of the several points of contrary flexure.

159. By Problem 132, Vol. II., the area (T) of a plane $\triangle$ expressed in terms of its sides a, b, c and semi-perimeter $\frac{S}{2}$, is

$$T = \sqrt{\frac{S}{2} \cdot \left(\frac{S}{2} - a\right)\left(\frac{S}{2} - b\right) \cdot \left(\frac{S}{2} - c\right)}$$

But when the right-angled (c being the hypothenuse),

$$\left(\frac{S}{2} - a\right) \cdot \left(\frac{S}{2} - b\right) = \frac{(c + a - b) \cdot (c - \overline{a - b})}{4}$$

$$= \frac{c^2 - (a-b)^2}{4} = \frac{c^2 - \overline{a^2 + b^2} + 2ab}{4}$$

$$= \frac{ab}{2} = T.$$

$$\therefore T^2 = \frac{S}{2} \cdot \left(\frac{S}{2} - c\right) \times T$$

$$\text{and } T = \frac{S}{2} \cdot \left(\frac{S}{2} - c\right). \quad \text{Q. E. D.}$$

THE CUBATURE OF SOLIDS.

160. Every Triangular Prism is decomposable into three equal Pyramids, of bases and altitudes, equal to those of the prism, and the prism = *its base* × *its altitude*, ∴ any triangular pyramid $= \dfrac{\textit{its base} \times \textit{its altitude}}{3}$.

If the base of the pyramid be a polygon, it may be decomposed into triangular pyramids of the same altitude, the sum of the bases being = that of the whole pyramid. Hence the volume of any pyramid whatever $= \frac{1}{3}$ (base × altitude).

161. Let a, b be the axes of the ellipse, then

$$y^2 = \frac{b^2}{a^2}.(2ax - x^2)$$

$$y^2 = \frac{a^2}{b^2}(2bx - x^2)$$

are equations to the ellipse, in the former of which x is measured along a from its origin in the curve; in the latter x is measured along b from its extremity.

Hence the volumes of the solids (A, B,) generated about $2a$, $2b$, are

$$\left.\begin{aligned} A &= \int \pi y^2 dx = \frac{b^2}{a^2}\frac{\pi}{3}.x^2(3a - x) \\ &= \int \pi y^2 dx = \frac{a^2}{b^2}\frac{\pi}{3}.x^2(3b - x) \end{aligned}\right\}$$

respectively. Let $x = 2a$ in the former, and $2b$ in the latter, then the whole volumes are

$$\left.\begin{aligned} A &= \frac{4ab^2\pi}{3} \\ \text{and } B &= \frac{4ba^2\pi}{3} \end{aligned}\right\}$$

Now when $b = a$, the spheroid will be a sphere $S = \frac{4a^3\pi}{3}$.

$$\therefore A \times S = \frac{4ab^2\pi}{3} \times \frac{4a^3\pi}{3}$$

$$= \frac{16a^4b^2\pi^2}{9} = \left(\frac{4}{3} \cdot a^2b\pi\right)^2 = B^2 .$$

$\therefore A : B :: B : S.$ Q. E. D.

162. Let the tangent parallel to the diameter of the semicircle be the line abscissæ, and let the abscissæ x originate at the point of contact, then by property of the $\odot$, we have

$$(r - y)^2 = r^2 - x^2$$

$$\therefore y, = r - \sqrt{r^2 - x^2}$$

$$\therefore y^2 = 2r^2 - 2r\sqrt{r^2 - x^2} - x^2$$

$\therefore$ the volume V, generated between the convex part of the $\frac{1}{2}\odot$, and the tangent, is expressed by

$$V = \int y^2dx = 2\pi r^2x - 2\pi r\int dx\sqrt{r^2 - x^2} - \frac{\pi x^3}{3}$$

$$= 2\pi r^2x - \frac{\pi x^3}{3} - r\pi x\sqrt{r^2 - x^2} - r^2\pi \sin.^{-r}x.$$

Let $x = r$; then the whole exterior volume $2V = \frac{r^3}{6} \times (10 - 3\pi)$,

And subtracting this from the cylinder described by the diameter about the tangent which $= \pi r^2 \times 2r = 2r^3\pi$, we get the volume V′, generated by the $\frac{1}{2}\odot$, expressed by

$$V' = \frac{\pi r^3}{6} \cdot (3\pi - 4).$$

Otherwise,

By Guldin's Theorem, the volume $V' =$ area of the $\frac{1}{2}\odot \times$ circumference of the circle described in the revolution by its cen-

tre of gravity. But the radius of this circle $= \frac{3\pi - 4}{3\pi} \times r$, $\therefore$ its circumference $= \frac{3\pi - 4}{3} \times r$. Hence

$$V' = \frac{\pi r^2}{2} \times \frac{2\pi - 4}{3} \times r = \frac{\pi r^3}{6} . (3\pi - 4) \text{ as before.}$$

163. Let r be the radius of the circle, then by prob. 161, the volume of the spheroid $V, = \frac{4\pi r3}{3}$, and that described by the $\triangle$ V' will evidently consist of two cones of the same base πr^2 and altitude r, each of which is known to equal $\pi r^2 \times \frac{r}{3}$. Hence

$$V' = \frac{2\pi r^3}{3}.$$

$$\therefore V : V' :: 4 : 2 :: 2 : 1.$$

Again, to compare their surfaces S, S′, we have

$$y = \sqrt{2rx - x^2}$$

$$\therefore S = 2\pi \int y dx \sqrt{1 + \frac{dy^2}{dx^2}} = 4\pi rx$$

$$= 4\pi r^2 \text{ for the whole sphere.}$$

Now the surface of a right cone $= \frac{1}{2}$ circumference of its base $\times$ slant side (as is known)

$$\therefore S' = \pi r \times r \sqrt{2} + \pi r \times r \sqrt{2}$$

$$= 2\sqrt{2}\, \pi r^2$$

$$\therefore S : S' :: \sqrt{2} : 1.$$

164. Let $Aa = 2a'$ (Fig. 68,) be the given diameter, and $Bb = 2b'$ its conjugate, then PM being any ordinate, if through it a plane $\perp$ plane ABC be made to pass, cutting from the spheroid the segment Ap PP′, it is required to find the volume of that segment.

By way of lemma, let us first prove that *all sections of a spheroid, made by planes mutually parallel, are similar ellipses*, which may be effected either by the formulæ in page 11, or more simply thus;

Let $P'p'$ be the line of intersection of the planes of $PP'p$ with a circle Dd described by the revolution of any point D in the generating ellipse; then since $DP'd$ and $PP'p$ are $\perp$ plane $AbaB$, PM' is $\perp$ lines Pp, Dd, and Dd is evidently the diameter of the circle. Hence

$$PM'^2 = DM' \times M'd,$$

But by a well known property of the ellipse

$$DM' \times M'd : PM' \times M'p :: CB'^2 : CB^2,$$

$$\text{or } y_{,}^2 : 2a_{,}x_{,} - x_{,}^2 :: b^2 : b'^2.$$

putting CB', the semi-conjugate axis $= b$.

$$PM' = y_{,}\,,\ Pp = 2a_{,},\ \text{and } PM' = x_{,}$$

$$\therefore y_{,}^2 = \frac{\frac{b^2}{b'^2} \cdot a_{,}^2}{a_{,}^2} \cdot (2a_{,}x_{,} - x_{,}^2) \ \ldots\ldots \ (1)$$

$\therefore$ the section PpP' is an ellipse whose axes are $\frac{b}{b'}\,a_{,}$ and $a_{,}$ and are $\therefore$ in the constant ratio of $b : b'$, and it is evident that the axes of all sections parallel to PpP' are in the same ratio. Hence all sections parallel to one another are similar. (See Prob. 68).

Now, making $CM = x$, $PM = y$, and drawing $CN \perp Pp$, since we may conceive the volume $BbAB$ to be generated by the parallel motion of the ellipse Bb, varying in magnitude but not in form, we shall have

$$V = \int \text{area of the ellipse} \times d \,.\, CN.$$

But since the area of an ellipse $= \pi \times$ (the product of its diameters,) and that of $BCb = \pi \times bb'$, (as we learn from equation 1,) and $\therefore$ the area of the similar ellipse $Pp = \pi bb' \times \frac{PM^2}{b'} =$

$$\frac{\pi bb'}{a'^2} \cdot (a'^2 - x^2),$$

$$V = \frac{\pi bb'}{a'^2} \int (a'^2 - x^2)\, dCN = \frac{\pi bb'}{a'^2} \sin. C \int (a'^2 - x^2)\, dx$$

$$\text{or } V = \frac{\pi bb'}{a'^2} \sin. C \times \left(a'^2 x - \frac{x^3}{3}\right) \ \ldots\ldots \ (2)$$

Let $x = a'$. Then, since $ab = a'b'$ sin. C,

$$V' = \frac{2\pi bb'a' \sin. C}{3} = \frac{2\pi b^2 a}{3} \quad \ldots\ldots\ldots (3)$$

which is known from other principles to be $= \frac{1}{2}$ spheroid.

Hence $V' - V = \frac{\pi bb' \sin. C}{3a'^2} \times (2a'^3 + x^3 - 3a'^2 x) = PpA$ the volume required.

By this method may be found the volumes of all solids whose parallel sections are similar figures.

165. The base B of the solid may be decomposed into triangles, and consequently the whole volume into as many pyramids of the same altitude h, each of which being known to $= \frac{1}{3}$ its base $\times$ altitude. Hence the volume required $= \frac{1}{3} B \times h$.

Otherwise.

As in the preceding problem it may be easily shewn that all sections parallel to the base B are similar to it, and also that (h, x being dist. of B, and B′ from the given point) the volume corresponding to B′ is expressed by

$$V' = \int \frac{Bx^2}{h^2} dx = \frac{Bx^3}{3h^2}$$

$$\therefore V = \frac{Bh^3}{3h^2} = \frac{Bh}{3}.$$

166. The bases of the whole cone and part cut off, are

πb^2, πc^2;

$\therefore$ if h, h' be their altitudes, the frustum F will be expressed by

$$F = \frac{\pi (b^2 h - c^2 h')}{3}.$$

$$\text{But } h = a + h' = h' \cdot \frac{b}{c}$$

$$\therefore k' = \frac{ac}{b-c}, \text{ and } h = \frac{ab}{b-c}$$

Hence

$$F = \frac{\pi}{3} a \cdot \frac{b^3 - c^3}{b-c}$$

$$= \frac{\pi a}{3} \cdot (b^2 + bc + c^2).$$

167. Let the equation to the generating parabola be

$$y^2 = mx.$$

Then the frustum F will be expressed by

$$F = \frac{\pi m}{2} (x^2 - x'^2)$$

$$= \frac{\pi m}{2} \cdot (x + x')(x - x') \text{ generally.}$$

Let $x - x' = a$, $y = b$, $y' = c$

Then $F = \frac{\pi m a}{2} \frac{b^2 + c^2}{m} = \frac{\pi a}{2} \cdot (b^2 + c^2).$

168. By the question

$$\int y dx : yx :: m : n$$

$$\therefore y dx = \frac{m}{n} y dx + \frac{m}{n} x dy$$

$$\therefore \frac{dy}{y} = \frac{n-m}{m} \cdot \frac{dx}{x}$$

$$\therefore ly = l.\, x^{\frac{n-m}{m}}$$

$$\therefore y = x^{\frac{n-m}{m}} = x^r$$

the equation to the parabolic curve.

Hence $V = \int \pi y^2 dx = \frac{\pi x^{2r+1}}{2r+1}$ the paraboloid.

Now the cylinder (V′) of same base and altitude $= \pi y^2 x = \pi x^{2r+1}$.

$$\therefore V : V' :: 1 : 2r + 1 :: m : 2n - m.$$

169. Let PM (Fig. 59,) $= Am = x$ and $Pm = y$, (Amt being the tangent parallel to the base TV) and AB the axis of the cycloid $= 2r$ &c. &c.

Then $x = AQ + MQ = \text{vers.}^{-1} y + \sqrt{2ry - y^2}$ the equation to the cycloid.

Hence the volume V generated by APm is expressed by

$$V = \pi \int y^2 dx = \pi \int \left\{ \frac{ry^2 dy}{\sqrt{2ry - y^2}} + \frac{y(r - y)\, dy}{\sqrt{2ry - y^2}} \right\}$$

$$= \pi \int y dy \sqrt{2ry - y^2}$$

$$= \pi r \int dy \sqrt{2ry - y^2} - \pi \int (r - y)\, dy \sqrt{2ry - y^2}$$

$$= \pi r \times \text{area } AMQ - \frac{2\pi}{3} \cdot (2ry - y^2)^{\frac{3}{2}}.$$

Let $x = 2r$. Then the whole volume described by $2ATt$ is

$$V = 2\pi r \times \frac{1}{2} \odot AQB = \pi^2 r^3.$$

Again, the volume of the cylinder described by Vt is

$$V' = \pi (2r)^2 \times TV = 4\pi r^2 \times 2r\pi$$

$$= 8\pi^2 r^3.$$

$\therefore$ the volume of the solid described by the cycloidal area VAT is expressed by

$$V' - V = 7\pi^2 r^3.$$

170. Let a be the part of the axis of the cylinder intercepted by the parallel bases; these bases being equal, if d be the distance of their planes, it is evident that the volume comprised between them, is expressed by

$$V = d \times \text{Base} = d \times B.$$

Now a is $\perp$ circular ends, and d is $\perp$ elliptical ones, and they meet in the centre; $\therefore$ if r be the radius of the circular ends, and $\theta =$ inclination of B to them, we shall have, by projections,

$$B = \frac{\pi r^2}{\cos.\theta}.$$

But it is also evident that $d = a \cdot \cos.\theta$,

$$\therefore V = \pi r^2 a,$$

which being independent of the magnitude and inclination of the ends, gives the proof required.

171. Since $y^2 = \frac{b^2x}{a-x} = -b^2 \cdot \frac{a-x+a}{a-x}$

$$= -b^2 + \frac{ab^2}{a-x}$$

we have

$$V = \int \pi y^2 dx = C - \pi b^2 x - ab^2\pi\, l.\,(a-x)$$

$$= ab^2\pi \,.\, l.\,\frac{1}{a-x} - b^2\pi x.$$

172. Let r be the radius of the sphere, then the area of a great circle is πr^2, and the volume of a cone of that base and altitude $= 2r$ is $\pi r^2 \times \frac{2r}{3} = \frac{2r^3\pi}{3} = \frac{1}{2} \cdot \frac{4r^3\pi}{3} = \frac{1}{2}$ the sphere. (See 161).

173. The octagon is composed of 8 equilateral-triangular faces.

Let A be any one of the equal angles of corresponding spherical equilateral $\triangle$, of the circumscribing sphere whose radius is 1, then the surface of any one of them is expressed by

$$3A - 2 \text{ right } \angle = 3A - \pi$$

and eight of those $\triangle$ cover the sphere which is $=$ 4 great circles,

$$\therefore 8 \times (3A - \pi) = 4\pi$$

$$\therefore A = \frac{\pi}{2}$$

Now a being a side of the spherical $\triangle$, we have by the fundamental theorem of spherics

$$\text{Cos. A} = \frac{\cos. a - \cos.^2 a}{\sin.^2 a} = \cos. \frac{\pi}{2} = 0$$

$$\therefore \cos. a = 0, \text{ or } a = \frac{\pi}{2}$$

Hence the side (s) of the octagon being the chord of a, is expressed by

$$s = 2 \sin. \frac{a}{2} = 2 \sin. \frac{\pi}{4} = \sqrt{2}$$

$= R \times \sqrt{2}$, if instead of being $=$ unity the radius of the sphere is $= R$.

Hence the area of each face is $\frac{R \times \sqrt{2}}{2} \times \sqrt{2R^2 - \frac{R^2}{2}} = \frac{R^2}{2} \cdot \sqrt{3}$, and the distance of the face from the centre, or the altitude of the equal prisms whose bases are the equal faces of the octaëdron and common vertex at the centre, is

$$\sqrt{R^2 - \frac{2}{3}R^2} = R \times \sqrt{\frac{1}{3}}.$$

Hence each of these prisms

$= \frac{1}{3} \times \frac{R^2}{2}\sqrt{3} \times R \times \sqrt{\frac{1}{3}} = \frac{1}{6} R^3$, and the whole octaëdron is therefore expressed by

$$\frac{4}{3} R^3. \qquad \text{Q. E. D.}$$

174. Let b be the radius of the base common to the hemispheroid (H) and paraboloid (P), and A their height, then

$$H : P :: 4 : 3.$$

$$\text{For } H = \pi . \int y^2 dx = \frac{\pi b^2}{a^2} \int (2ax - x^2)\, dx$$

$$= \frac{\pi b^2}{a^2} . \left(ax^2 - \frac{x^3}{3}\right) = \frac{2\pi b^2 a}{3}$$

when $x = a$, or $y = b$.

$$\text{and } P = \pi \int pxdx = \frac{\pi p x^2}{2}$$

$$\text{But since } y^2 = \frac{b^2}{a^2} . (2ax - x^2) = px$$

$$\therefore p = \frac{b^2}{a^2} . (2a - x)$$

$$\therefore P = \frac{\pi b^2}{2a^2} . (2ax^2 - x^3) = \frac{\pi b^2 a}{2} \text{ when } x = a.$$

$$\therefore H : P :: \frac{2\pi b^2 a}{3} : \frac{\pi b^2 a}{2} :: 4 : 3.$$

175. Let PQ (Fig. 69,) the transverse axis of the ellipse $= 2a$, and its conjugate (to be determined) $= 2b$, and put $\triangle$ ABC $=$ T, $\triangle$ APQ $=$ T′, cone ABC $=$ C, and segment APQ $=$ C′, then $C : C' :: T^{\frac{3}{2}} : T'^{\frac{3}{2}}$.

For since PQ is the axis of the ellipse, its plane is $\perp$ plane ABC which passes through the axis of the cone.

Hence pm being the intersection of the $\odot$ B′C′ with the ellipse PQ, and B′C′ that with the plane ABC, it is easily shewn that

$$pm^2 = B'm \times mC'.$$

$$\left.\begin{aligned} &\text{But } Pm : B'm :: \sin. B : \sin. P \\ &\text{and } mQ : mC' :: \sin. B : \sin. Q \end{aligned}\right\}$$

$$\therefore B'm \times mC' = Pm \times mQ \times \frac{\sin. P \times \sin. Q}{\sin.^2 B}$$

$$= Pm \times mQ \times \frac{p'^2}{p^2} \times \frac{T'}{T} = pm^2,$$ p and p' being the perpendiculars let fall from A upon BC and PQ respectively.

$$\text{Let } mQ = Pm = \frac{1}{2} . 2b = b.$$

Then

$$b^2 = a^2 \times \frac{p'^2}{p^2} \times \frac{T}{T'} = \frac{TT'}{p^2}$$

and the area of the ellipse is $\therefore$ expressed by

$$\pi ab = \frac{a}{p} \sqrt{TT'}.$$

Now it is evident that C′ may be decomposed into triangular prisms of the same altitude p', and the limit of the sum of whose bases $=$ the base of C′, hence

$$C' = \frac{1}{3} p' \times \frac{a}{p} \sqrt{TT'} = \frac{1}{3p} . T'^{\frac{3}{2}} \sqrt{T} \,..... (1)$$

Let T′ $=$ T, then C′ $=$ C, and we have

$$C = \frac{1}{8p} \times T^3 \ldots\ldots\ldots (2)$$

$$\therefore C : C' :: T^{\frac{3}{2}} : T'^{\frac{3}{2}}.$$

Again, to find the equation to the surface of the cone, we will suppose the rectangular axes of x, y, z, to originate in its vertex, and the axis of z to be that of the cone itself, as in Fig. 69. Then, L being any point in the surface, let OLA be that position of the generating $\triangle$ in which it passes through L, and draw LN $\perp$ AZ, LL′ $\perp$ plane of (x, y,) and L′M $\perp$ AX. Let, therefore, AN $= z$, AM $= x$, and L′M $= y$, and we have

$$Z : NL = AL' = \sqrt{x^2 + y^2} :: OA : OC :: 1 : \tan. \frac{A}{2}$$

A being the angle at the vertex of the cone.

$$\therefore Z \times \tan. \frac{A}{2} = \sqrt{x^2 + y^2}$$

the equation of the surface of the cone, whose rectangular co-ordinates originate in its vertex, and that of z coincides with its axis, which, in practice, will be found the most commodious way of considering the question.

THE RECTIFICATION OF CURVES.

176. This is a particular case of the Theorem of Fagnani, which we will first demonstrate generally.

Let 1, and b, be the semi-axes of the ellipse, and e its eccentricity. Then the equation of the curve referred to the centre by rectangular co-ordinates x, y, is

$$y = b\sqrt{1-x^2}.$$

Hence, denoting by E_x that part of the elliptic arc measured from the extremity of b, whose abscissa is x, we have

$$E_x = \int dx \sqrt{1+\frac{dy^2}{dx^2}} = \int dx \sqrt{\frac{1-e^2x^2}{1-x^2}}$$

and putting

$$x = \sqrt{\frac{1-u^2}{1-e^2u^2}} \text{ or } \frac{x\sqrt{1-x^2}}{\sqrt{1-e^2x^2}} = \frac{u\sqrt{1-u^2}}{\sqrt{1-e^2u^2}}$$

we get

$$d\,E_x + du\,.\sqrt{\frac{1-e^2u^2}{1-u^2}} = e^2 d.\,\frac{u\sqrt{1-u^2}}{\sqrt{1-e^2u^2}}$$

$$\therefore E_x + E_u = \frac{e^2 u\sqrt{1-u^2}}{\sqrt{1-e^2u^2}} + C.$$

But $u = 1$ when $x = 0$. $\therefore C = E_1$

$$\text{and } E_x + E_u - E_1 = \frac{e^2u\sqrt{1-u^2}}{\sqrt{1-e^2u^2}} \ldots\ldots\ (1)$$

which is the theorem of Fagnani.

Let $u = x = \dfrac{1}{\sqrt{1+b}}$ (as we readily find by substituting $x = \sqrt{\dfrac{1-u^2}{1-e^2u^2}}$.)

Then equation (1) becomes

$$2E_{\frac{1}{\sqrt{1+b}}} - E_1 = \frac{e^2}{1+b} = 1 - b,$$

which solves the problem.

177. Let 2, 2*b*, and *e*, be the transverse axis, conjugate axis, and eccentricity of the ellipse, then its equation, referred to the centre, is expressed by

$$y = b\sqrt{1-x^2}$$

and the length of any arc by

$$E_x = \int dx \sqrt{1 + \frac{dy^2}{dx^2}} = \int dx \sqrt{\frac{1-e^2x^2}{1-x^2}} \;\ldots\ldots (1)$$

Let

$$u = \frac{x}{\sqrt{1+x^2}} \text{ or } x^2 = \frac{u^2}{1+u^2}.$$

Then

$$1 - x^2 = \frac{1}{1+u^2},\ 1 - e^2x^2 = \frac{1+b^2u^2}{1+u^2},$$

$$\text{and } dx = \frac{du}{(1+u^2)^{\frac{3}{2}}},$$

Hence

$$d\,E_x = \frac{du\,.\sqrt{1+b^2u^2}}{(1+u^2)^{\frac{3}{2}}} \;\ldots\ldots\ldots (2)$$

which, by putting

$$P = u\sqrt{\frac{1+b^2u^2}{1+u^2}}$$

and making the requisite substitutions transforms to

$$d\,E_x = dP - \frac{b^2u^2du}{\sqrt{(1+u^2)(1+b^2u^2)}} \;\ldots\ldots (3)$$

Again, make

$$v = u\sqrt{\frac{1+u^2}{1+b^2u^2}}.$$

Then

$$u^2 = \frac{b^2v^2 - 1 + \sqrt{1+2\,.\,(2-b^2)\,v^2 + b^4v^4}}{2},$$

and $\dfrac{du}{\sqrt{(1+u^2).(1+b^2u^2)}} = \dfrac{1+b^2u^2}{1+2u^2+b^2u^4}\,dv = \dfrac{dv}{1+2u^2-v^2}$

$= \dfrac{dv}{\sqrt{1+2.(2-b^2)\,v^2+b^4v^4}} = \dfrac{dv}{\sqrt{\{1+(1+e)^2v^2\}\times\{1+(1-e)^2v^2\}}}$

Let $(1+e)\,v = u_1$, $(1-e)\,v = b_1\,u_1$, or $b_1 = \dfrac{1-e}{1+e}$

and $\therefore\ e = \dfrac{1-b_1}{1+b_1}$ and $1+e = \dfrac{2}{1+b_1}$;

also let $\sqrt{(1+u^2)(1+b^2u^2)} = \mathrm{U}$, $\sqrt{(1+u_1^2)\,(1+b^2u_1^2)} = \mathrm{U}_1$

$$\text{Then } \frac{du}{\mathrm{U}} = \frac{1+b_1}{2}\cdot\frac{du_1}{\mathrm{U}_1}\ \ldots\ldots\ldots\ (4)$$

and $\therefore$ after proper reductions,

$$\frac{b^2u^2du}{\mathrm{U}} = \frac{b^2}{4}\mathrm{B}_1du_1 - \frac{b^2}{4}\mathrm{B}_1\frac{du_1}{\mathrm{U}_1} + \frac{b_1^2\,u_1^2\,du_1}{\mathrm{B}_1\mathrm{U}_1}\ (5)$$

B_1 being put $= 1 + b_1$.

Hence it is evident that by repeating the foregoing operations, and making successively

$$u_1 = 2u_1\sqrt{\frac{1+u_1^2}{1+b_1^2u_1^2}},\ b_2 = \frac{1-\sqrt{1-b_1^2}}{1+\sqrt{1-b_1^2}}$$

$$2u_2 = 2u_2\sqrt{\frac{1+u_2^2}{1+b_2^2u_2^2}},\ b_3 = \frac{1-\sqrt{1-b_2^2}}{1+\sqrt{1-b_2^2}}$$

&c. = &c. &c. = &c.

$$u_n = 2u_{n-1}\sqrt{\frac{1+u^2_{n-1}}{1+b^2_{n-1}u^2_{n-1}}},\ b_n = \frac{1-\sqrt{1-b^2_{n-1}}}{1+\sqrt{1-b^2_{n-1}}}$$

and $\mathrm{U}_2 = \sqrt{(1+u_2^2)\,(1+b_2^2u_2^2)}$, ... $\mathrm{U}_n = \sqrt{(1+u_n^2)(1+b_n^2u_n^2)}$
we shall obtain results symmetrical with the former scil.

$$\frac{b_1^2\,u_1^2du_1^2}{\mathrm{U}_1} = \frac{b_1^2}{4}\,.\,\mathrm{B}_2\,du_2 - \frac{b_1^2}{4}\mathrm{B}_2\frac{du_2}{\mathrm{U}_2} + \frac{b_2^2}{\mathrm{B}_2}\cdot\frac{u_2^2\,du_2}{\mathrm{U}_2}$$

$$\frac{b_2^2\,u_2^2\,du_2}{\mathrm{U}_2} = \frac{b_2^2}{4}\mathrm{B}_3\,du_3 - \frac{b_2^2}{4}\mathrm{B}_3\frac{du_3}{\mathrm{U}_3} + \frac{b_3^2}{\mathrm{B}_3}\cdot\frac{u_3^2\,du_3}{\mathrm{U}_3}$$

&c. = &c.

which, by aid of the formula (4), which evidently expands into

$$\frac{du}{\mathrm{U}} = \frac{\mathrm{B}_1}{2}\cdot\frac{\mathrm{B}_2}{2}\cdots\cdots\frac{\mathrm{B}_n}{2}\cdot\frac{du_n}{\mathrm{U}_n}\ \ldots\ldots\ (4')$$

$$\text{give } dE_x = dP - \left\{\frac{b^2}{4} B_1 du_1 + \frac{b_1^2}{4} \cdot \frac{B_2}{B_1} du_2 + \dots \frac{b^2_{n-1}}{4} \times \frac{B_n}{B_1 B_2 \dots B^2_{n-1}} du_n \right\} + \left\{\frac{b^2}{2} \cdot \frac{B_1 B_2 \dots B_n}{2^n} + \frac{b_1^2}{2B_1} \cdot \frac{B_2 B_3 \dots B_n}{2^{n-1}} + \dots \frac{b_n^2}{2B_1 B_2 \dots B_n} \right\} \times \frac{du_n}{U_n} - \frac{b_n^2}{B_1 \dots B_n} \times \frac{u_n^2 du_n}{U_n} \dots (6)$$

Consequently the difficulty of rectifying an ellipse is reduced to that of integrating the expressions

$$\frac{du_n}{U_n} \text{ and } \frac{u_n^2 du^n}{U_n},$$

which, however, can only be effected by approximation.

Now since in every case of eccentricity, and *more especially when* b *is small, or* e *nearly* $= 1$, the values of

$$b_1, b_2, b_3 \dots\dots b_m,$$

become rapidly less and less, we shall soon arrive at b_m, which may be considered $= 0$ without materially affecting the result.

$$\text{Hence } U_m = \sqrt{1 + u_m^2}$$

$$\text{and } \int \frac{du_m}{U_m} = l.\,(u_m + \sqrt{1 + u_m^2})$$

and we have generally

$$\frac{b_n^2}{2B_1 B_2 B_3 \dots B_n} = \frac{b^2 b_1 \dots b_n B_1 B_2 \dots B_n}{2^{2n+1}}$$

$$\text{since } b_n = \frac{b^2_{n-1}}{4},\; b_{n-1} = \frac{b^2_{n-2}}{4}, \text{ \&c., nearly.}$$

∴ equation (6) becomes, by substitution and integration

$$E_x = P - \left\{\frac{b^2 B_1}{2^2} u_1 + \frac{b^2 b_1 B_1 B_2}{2^4} u_2 + \text{\&c.} + \frac{b^2 b_1 b_2 \dots b_{m-1} B_1 B_2 B_3 \dots B_m}{2^{2m}} \right\} + \frac{b B_1 \dots B_m}{2^m} \times \left\{\frac{b}{2} + \frac{bb_1}{2^2} + \frac{bb_1 b_2}{2^3} + \dots \frac{bb_1 \dots b_n}{2^{m+1}} \right\} l.\,(u_m + \sqrt{1 + u_m})$$

nearly . . . . (7)

and to obtain the whole quadrant E_1 of the ellipse,

$$\left.\begin{aligned} \text{since } u &= \frac{1}{\sqrt{b}} \\ u_1 &= \frac{1}{\sqrt{b_1}} \\ u_2 &= \frac{1}{\sqrt{b_2}} \\ \&c. &= \&c. \end{aligned}\right\} \text{when } x = \frac{1}{\sqrt{1+b}}$$

and $l. (u_m + \sqrt{1+u_m}) = \frac{b_m}{4} + l. \frac{2}{\sqrt{b_m}}$ nearly $= \frac{b^2_{m-1}}{4^2}$

$+ 2\,l. \frac{2}{\sqrt{b_{m-1}}} = \frac{b^4_{m-2}}{4^4} + 2^2\, l. \frac{2}{\sqrt{b_{m-2}}} = \&c. = 2^{m-m}\, l. \frac{2}{b_m}$ nearly, (m being the index at which b_{m+1} may first be taken $= \frac{b^2_m}{4}$ nearly).

Hence, by substituting in equation (7) and making the necessary reductions, and putting $P_m = B_1 B_2 \ldots B_m$, $Q_m = \frac{b}{2} + \frac{bb_1}{2}$ &c., we have

$$E_{\left(\frac{1}{\sqrt{1+b}}\right)} = 1 - \frac{b}{2}\left\{1 + \frac{b.\,B_1}{2^2} + \frac{bb_1\,B_1B_2}{2^4} + \&c.\right\}$$
$$+ \frac{b}{2^m} P_m Q_m \times l. \frac{4}{b_m}.$$

But by the preceding problem we have

$$E_1 = 2E_{\left(\frac{1}{\sqrt{1+b}}\right)} + b - 1$$

$$\therefore E_1 = 1 - b\left\{\frac{b\,B_1}{2^2} + \frac{bb_1\,B_1B_2}{2^4} + \ldots\right\}$$
$$+ \frac{b}{2^m} P_m Q_m \times l. \frac{4}{b_m}$$

the approximate value of the elliptic quadrant.

In the problem we have

$$e = .99, \therefore b = \sqrt{1-(.99)^2} = .1410673$$
$$b_1 = \frac{1-e}{1+e} = \frac{1}{199} = .0050251$$

$$b_3 = \frac{b_2^2}{(1+\sqrt{1-b_2^2})^2} = .000006813$$

&c. = &c.

But since $\frac{b_1^2}{4} = .0000068129$ differs so little from b_2, we make take $m = 1$. Hence

$$P_1 = B_1 = \frac{200}{199},\ Q_1 = \frac{b}{2} + \frac{bb_1}{2^2} = .0707108$$

and $l.\ \frac{4}{b_1} = l.\ 796 = 6.679599$

$$\therefore\ \frac{b}{2}\,P_1Q_1\,l.\,\frac{4}{b_1} = .0334818,$$

and $\frac{b^2\,B_1}{2^2} = \frac{b^2}{4}\cdot\frac{200}{199} = .005$

Hence $E_1 = 1 - .005 + .0334818 = 1.0284808$

and $4E_1 = 4.1139232$ nearly.

For a very elaborate and profound discussion of this and kindred subjects, the reader is referred to a paper in the *Transactions of the Royal Society* for 1804, by Mr. Woodhouse. He will also find it ably treated by Mr. Ivory, in the 4th volume, and by Mr. Wallace in the 5th volume of the *Edinburgh Transactions*. Euler, moreover, in his *Animadversiones in Rectificationem Ellipseos Opuscul.* Vol. II., may be consulted with advantage.

173. Let the rectangular co-ordinates (x, y) of the hyperbola (H) be referred to its centre, then, 1 and b being its semi-axes

$$y = b\sqrt{x^2 - 1}$$

$\therefore$ denoting by H_x the arc whose abscissa $= x$,

$$d\,H_x = dx\sqrt{1 + \frac{dy^2}{dx^2}} = dx\sqrt{\frac{e^2x^2 - 1}{x^2 - 1}}$$

where e the eccentricity $= \sqrt{1 + b^2}$.

Now since e is > 1, to transform $d\,H_x$ into *algebraical and*

elliptic functions, we must put $\frac{1}{e} = e_1$ (which gives $e_1 < 1$.)

and $x = \sqrt{\frac{1-e_1^2 x_1^2}{1-x_1^2}}$, we get, by substitution,

$$d\,H_x = \frac{(1-e_1^2)\,dx_1}{(1-x_1^2)^{\frac{3}{2}}\ \sqrt{1-e_1^2 x_1^2}} \ldots\ldots\ldots (1)$$

Again, let $P = x_1 \sqrt{\frac{1-e_1^2 x_1^2}{1-x_1^2}}$.

Then $d\,H_x = dP - dx_1 \sqrt{\frac{1-e_1^2 x_1^2}{1-x_1^2}} + \frac{(1-e_1^2)dx_1}{\sqrt{(1-x_1^2)(1-e_1^2 x_1^2)}}$

$$\therefore H_x = P - E_x + \int \frac{(1-e_1^2)\,dx_1}{\sqrt{(1-x_1^2)(1-e_1^2 x_1^2)}}$$

Again, let $e_1 = \frac{1-\sqrt{1-e_2^2}}{1+\sqrt{1-e_2^2}}$, and $x_1 = \frac{2}{1-e_1}.x_2 \times \sqrt{\frac{1-x_2^2}{1-e_2^2 x_2^2}}$

Then, we get

$$\int \frac{(1-e_1^2)\,dx_1}{\sqrt{(1-x_1^2)(1-e_1^2 x_1^2)}} = 2e_1 x_1 - 2\,(1+e_1)\int dx_2 \times \sqrt{\frac{1-e_2^2 x_2^2}{1-x_2^2}} + 2E_{x_1}.$$

Hence $H_x = P + 2e_1 x_1 + E_{x_1} - 2\,.\,(1+e_1)\,E_{x_2}$

which expression shows that the hyperbolic arc H_x may be found by means of E_{x_1} the elliptic arc whose abscissa is x_1 and axes 1, and $b_1 \left(= \sqrt{1-e_1^2} = \sqrt{1-\frac{1}{e^2}} = \frac{b}{\sqrt{1+b^2}}\right)$, and of E_{x_2} whose abscissa is x_2 and axes 1 and $b_2 \left(= \frac{e-1}{e+1} = \frac{(\sqrt{1+b^2}-1)^2}{b^2}\right)$ and of P an algebraic function of x.

179. Since the equation to the *Lemniscata* expressed by rectangular co-ordinates, is

$$(x^2+y^2)^2 = x^2 - y^2,$$

if we make their origin A (Fig. 1,) the pole of the curve, the $\angle$ subtended by $y = \theta$ and the radius vector AM $= \rho$, we have

$$y = \rho \cdot \sin. \theta, \; x = \rho \cdot \cos. \theta$$

$$\text{and } \therefore \rho^2 = \cos.^2 \theta - \sin.^2 \theta = \cos. 2\theta$$

$$\text{or } \rho = \sqrt{\cos. 2\theta} \; \ldots\ldots\ldots \; (1)$$

the polar equation to the curve.

$$\text{Hence } ds = d\rho \sqrt{1 + \rho^2 \frac{d\theta^2}{d\rho^2}} = \frac{d\rho}{\sqrt{1-\rho^4}} \text{ or } = \frac{d\theta}{\sqrt{\cos. 2\theta}}$$

and measuring s' from any given point B′ of the branch AB, towards A, we have ds' negative, or

$$ds' = \frac{-d\rho'}{\sqrt{1-\rho'^4}} \text{ or } = \frac{-d\theta'}{\sqrt{\cos. 2\theta'}}.$$

Now by the question $s = s'$.

$\therefore ds = ds'$ and

$$\left.\begin{array}{l} \dfrac{d\rho}{\sqrt{1-\rho^4}} + \dfrac{d\rho'}{\sqrt{1-\rho'^4}} = 0 \\ \text{or } \dfrac{d\theta}{\sqrt{\cos. 2\theta}} + \dfrac{d\theta'}{\sqrt{\cos. 2\theta'}} = 0 \end{array}\right\} \text{either of which equa-}$$

tions being integrated will give the solution required.

To integrate the latter we have

$$\frac{d\theta}{ds} = \sqrt{\cos. 2\theta}, \text{ and } \frac{d\theta'}{ds} = \sqrt{\cos. 2\theta'}$$

$$\therefore \frac{d\theta^2}{ds^2} = \cos. 2\theta, \text{ and } \frac{d\theta'^2}{ds^2} = \cos. 2\theta'$$

$$\therefore \frac{d^2\theta}{ds^2} = -\sin. 2\theta, \text{ and } \frac{d^2\theta'}{ds^2} = -\sin. 2\theta'$$

and putting $\theta + \theta' = u$, $\theta - \theta' = v$, we have

$$\left.\begin{array}{l} \dfrac{d^2u}{ds^2} = -(\sin. 2\theta + \sin. 2\theta') = -2 \sin. u . \cos. v \\ \dfrac{d^2v}{ds^2} = -(\sin. 2\theta - \sin. 2\theta') = -2 \sin. v . \cos. u \end{array}\right\} (2)$$

$$\text{But } \frac{du}{ds} \cdot \frac{dv}{ds} = \frac{d\theta^2}{ds^2} - \frac{d\theta'^2}{ds^2} = \cos. 2\theta - \cos. 2\theta'$$

$$= -2 \sin. u . \sin. v \;\ldots\ldots\; (2)$$

$\therefore$ dividing each of equations (1) by equation (2), we get

$$\left.\begin{aligned} \frac{d^2u}{du\,.\,dv} &= \frac{\cos. v}{\sin. v}, \text{ or } \frac{\left(\frac{d^2u}{ds}\right)}{\left(\frac{du}{ds}\right)} = \frac{dv \cos. v}{\sin. v} \\ \text{and } \frac{d^2v}{dv\ du} &= \frac{\cos. u}{\sin. u}, \text{ or } \frac{\left(\frac{d^2v}{ds}\right)}{\left(\frac{dv}{ds}\right)} = \frac{du \cos. u}{\sin. u} \end{aligned}\right\}$$

and integrating

$$\left.\begin{aligned} l.\ \frac{du}{ds} &= l. \sin. v + lc \\ l.\ \frac{dv}{ds} &= l. \sin. u + lc' \end{aligned}\right\}$$

$$\therefore\ \frac{du}{ds} = c \sin. v, \text{ and } \frac{dv}{ds} = c' \sin. u,$$

and substituting for $\frac{du}{ds}$, $\frac{dv}{ds}$, v and u, we get the integrals

$$\left.\begin{aligned} \sqrt{\cos. 2\theta} + \sqrt{\cos. 2\theta'} &= c\,.\sin. (\theta - \theta') \\ \sqrt{\cos. 2\theta} - \sqrt{\cos. 2\theta'} &= c'\,.\sin. (\theta + \theta') \end{aligned}\right\} (3)$$

Now supposing α to denote the given $\angle$ B′AB, we have $\theta' = \alpha$ when $\theta = 45°$, and $\therefore$

$$c = \frac{\sqrt{\cos. 2\alpha}}{\sin. (45 - \alpha)} = \sqrt{2 \tan.(45° + \alpha)}$$

$$c' = -\frac{\sqrt{\cos. 2\alpha}}{\sin. (45 + \alpha)} = -\sqrt{2 \tan.(45° - \alpha)}.$$

Substituting these values in equations (3) and taking their difference, we get, after the necessary reductions,

$$\sqrt{\cos. 2\theta'} = \sqrt{\frac{2}{1 - \tan.^2 \alpha}} \times (\sin.\theta \cos.\theta' + \tan.\alpha \times \sin.\theta' \cos.\theta)$$

which affords a general solution to the problem.

Hence any given arc of the lemniscata AB′ (measured from the double point A) may be bisected.

For then $\theta' = \theta$, and $\therefore$

$$\sqrt{\cos. 2\theta} = \sqrt{\frac{2}{1-\tan.^2\alpha}} \times (1+\tan.\alpha)\sin.\theta\cos.\theta$$

$$\therefore \cos. 2\theta = \frac{\tan.(45+\alpha)}{2}\sin.^2 2\theta,$$

and finally

$$\cos. 2\theta = -\tan.(45^\circ - \alpha) + \sec.(45 - \alpha).$$

and for the whole arc AB ($\alpha = 0$)

$$\cos. 2\theta = -1 + \sqrt{2}$$

$$\text{and } \therefore \cos.^2\theta = \frac{1}{\sqrt{2}}.$$

By aid of the equations (8) we may also bisect any given arc A′B′; the only difference in the processes consisting in the determination of c, c'.

The reader will find solutions to this problem by Mr. Ivory, and Mr. Wallace, in Leybourne's *Math. Rep.* pp. 204, ... vol. i; which, however, have the disadvantage of being neither *direct* nor sufficiently general.

180. Since $y = l.\dfrac{e^x+1}{e^x-1} = l.(e^x+1) - l.(e^x-1)$

$$\therefore \frac{dy}{dx} = -\frac{2e^x}{e^{2x}-1}.$$

$$\text{and } s = \int dx\sqrt{1+\frac{dy^2}{dx^2}} = \int dx.\frac{e^{2x}+1}{e^{2x}-1}$$

Let $\dfrac{e^{2x}+1}{e^{2x}-1} = u$. Then $dx = -\dfrac{du}{u^2-1}$ and $s = c$

$$-\frac{1}{2}.l.(u^2-1) = c - l.\frac{2e^x}{e^{2x}-1}.$$

Let $x = \infty$. Then since y is possible, $s = \infty$, and e being > 1

$$l.\frac{2e^\infty}{e^{2\infty}-1} = l.\frac{2}{e^\infty} = l.2 - \infty$$

$$\therefore c = l.2.$$

Hence

$$s = l.2 - l.\frac{2e^x}{e^{2x}-1} = l.\frac{e^{2x}-1}{e^x}.$$

181. Let r be the radius of the generating ⊙ of the cycloid, and referring the abscissa x to the vertex and axis, we have

$$y = \sqrt{2rx - x^2} + \text{vers.}^{-r} x.$$

$$\therefore \frac{dy}{dx} = \frac{\sqrt{2r - x}}{\sqrt{x}}$$

$$\text{and } s = \int dx \sqrt{1 + \frac{dy^2}{dx^2}} = \int \frac{2rdx}{x} = 2\sqrt{2rx}$$

But x : chord of gen. ⊙ $= m :: m : 2r$
Hence $s = 2m$.
$\therefore$ the semi-cycloidal arc $= 4r$
Let $2m = \frac{1}{n} \times 4r$. Then

$$s = \frac{1}{n} \text{ semi-cycloidal arc,}$$

which more than solves the problem.

182. By Mechanics we have (x abscissa from lowest point)

$$s = a \cdot \frac{dx}{dy}.$$

$$\therefore a \frac{d^2 x}{dy} = ds = dy \sqrt{1 + \frac{dx^2}{dy^2}}$$

$$\text{or } dy = \frac{a \cdot d \cdot \frac{dx}{dy}}{\sqrt{1 + \frac{dx^2}{dy^2}}}$$

$$\therefore y = a \cdot l \cdot \left(\frac{dx}{dy} + \sqrt{1 + \frac{dx^2}{dy^2}}\right)$$

$$\text{and } e^{\frac{y}{a}} = \frac{dx}{dy} + \sqrt{1 + \frac{dx^2}{dy^2}}$$

Hence solving the above equation when reduced to the form of a quadratic, we get

$$\frac{dx}{dy} = \frac{e^{\frac{2y}{a}} - 1}{2e^{\frac{y}{a}}}$$

$$\therefore s = \int dy \sqrt{1 + \frac{dx^2}{dy^2}} = \int \frac{e^{\frac{y}{a}} - e^{-\frac{y}{a}}}{2} y$$

$$= \frac{a}{2} \left(e^{\frac{y}{a}} - e^{-\frac{y}{a}}\right)$$

Let $y = D$ the greatest ordinate.

$$\text{Then } 2s = a \cdot \left(e^{\frac{D}{a}} - e^{-\frac{D}{a}}\right)$$

See Whewell's Mechanics, page 171.

183. By the question, we have

$$y \times x = \frac{ydx}{dy} \times a$$

$$1 + \frac{dy^2}{dx^2} = \frac{a^2}{x^2} + 1 = \frac{a^2 + x^2}{x^2}$$

$$\text{and } s = \int dx \sqrt{1 + \frac{dy^2}{dx^2}} = \int \frac{dx}{x} \sqrt{a^2 + x^2}$$

$$= \int \frac{a^2 dx}{x \sqrt{a^2 + x^2}} + \int \frac{x dx}{\sqrt{a^2 + x^2}}$$

$$= \frac{a}{2} l. \frac{\sqrt{a^2 + x^2} - a}{\sqrt{a^2 + x^2} - a} + l. \sqrt{a^2 + x^2} + C$$

$$= a . l. \frac{\sqrt{a^2 + x^2} - a}{x} + l. \sqrt{a^2 + x^2} + C$$

C being any constant whatever.

QUADRATURE OF SURFACES.

184. If s denote the arc, $\pi = 3.141$, &c.; then the surface (S) generated by the revolution of s about its line of abscissæ (x) will be represented by

$$S = 2\pi \int y ds$$

Now in the sphere referred to its centre,

$$y = \sqrt{r^2 - x^2}$$

$$\therefore \frac{dy}{dx} = \frac{-x}{\sqrt{r^2 - x^2}}$$

$$\therefore dS = \frac{2\pi r y dx}{\sqrt{r^2 - x^2}} = 2\pi r dx$$

$\therefore$ S $= 2\pi r x =$ surface of the corresponding portion of the cylinder. $\therefore$ &c.

185. By the preceding problem we have the surface (S) of a sphere expressed by

$$S = 4\pi r^2$$

and the area of the ends $= 2 \times \pi r^2$

$\therefore$ the whole surface S′ of the cylinder is

$$S' = 4\pi r^2 + 2\pi r^2 = 6\pi r^2,$$

$$\therefore S = \frac{2}{3} S'.$$

Again, the volumes V, V′, of the sphere and cylinder are expressed by

$$V = \frac{4}{3} \pi r^3, \; V' = 2r \times \pi r^2 = 2\pi r^3$$

$$\therefore V = \frac{2}{3} V'.$$

186. The surface of a spherical Δ whose angles are A, B, C, is generally expressed by (Woodhouse)

$$S = \frac{A+B+C-180}{180} \cdot r^2\pi$$

r being the radius of the sphere.

But $A = 160$, $B = 150$, $C = 140$.

$$\therefore S = \frac{270}{180} r^2\pi = \frac{3r^2\pi}{2}$$

$$= \frac{3}{2} \cdot \text{area of great} \odot. \quad \text{Q. E. D.}$$

187. The surface (S) of a cylinder is

$$S = 4\pi r^2 = 4 \text{ area of its base.}$$

Hence the whole surface $= 6$ area of the base.

188. By prob. 184, the surface S_x of a sphere corresponding to abscissa x, is

$$S_x = 2\pi r x$$

$$\therefore \text{the whole surface } S_{\varpi} = 4\pi r^2$$

$$\therefore S_x : S_{\varpi} :: x : 2r.$$

189. The equation to the cycloid being

$$y = \sqrt{2rx - x^2} + \text{vers.}^{-r} x$$

$$\sqrt{1 + \frac{dy^2}{dx^2}} = \sqrt{2r} \cdot \frac{1}{\sqrt{x}}$$

$$\therefore S = 2\pi \int y dx \sqrt{1 + \frac{dy^2}{dx^2}} = C - \frac{4}{3} \sqrt{2r} \cdot \pi (2r - x)^{\frac{3}{2}} +$$

$$2\pi \sqrt{2r} \int \frac{dx}{\sqrt{x}} \text{vers.}^{-r} x$$

$$= C - \frac{4\pi \sqrt{2r}}{3} (8r - x) \sqrt{2r - x} + 8\pi \sqrt{2r} (\sqrt{x} \text{vers.}^{-r} x)$$

Let $x = 0$. Then $C = \dfrac{64r^2\pi}{3}$.

Hence, &c. &c.

190. The surface (S) of any portion of a sphere corresponding to abscissa x (from centre) is (prob. 184).

$$S = 2\pi r x$$

$$\therefore \frac{1}{2} \text{ whole surface} = 2\pi r^2.$$

$\therefore$ the surface (S′) of the spherical segment $= 2\pi r \, . \, (r - x)$.

Now the dist. of the pole of the circle whose rad. is y, is $= \sqrt{y^2 + (r-x)^2} = \sqrt{2r \, . \, (r-x)}$.

$\therefore$ the area (C) of the ⊙ whose rad. is that dist. $= 2\pi r \, (r - x) = S'$. Q. E. D.

MISCELLANIES.

191. Let Aa (Fig. 70,) be the axis of the paraboloid, and of Dc the perforating cylinder. Then, PMAM being the co-ordinates of the point P of the generating parabola ($y^2 = px$), it may easily be shewn that the intersection of the surfaces PMp is a ⊙ whose rad, is PM. Hence the volume (V) of the paraboloid APp is

$$V = \pi \int y^2 dx = \frac{p\pi x^2}{2}$$

Let $x = Aa = a$. Then

$$V' = \frac{p\pi a^2}{2}.$$

$$\text{and the cylinder Pc} = \pi y^2 \times (Aa - MA)$$
$$= p\pi x (a - x).$$

Hence the part bored away $= V + p\pi x (a - x) = \frac{p\pi}{2} x \times (2a - x) = \frac{1}{2} . V' = \frac{p\pi a^2}{4}$, by the question.

$$\therefore x . (2a - x) = \frac{a^2}{2}$$

$$\text{and } x = a - \frac{a}{\sqrt{2}}$$

which gives the diameter of the cylinder $= 2y = 2\sqrt{px} = \sqrt{2ap . (\sqrt{2} - 1)}$, and its length $= a - x = \frac{a}{\sqrt{2}}$.

192. For the investigation see *Translation of Lacroix*, p. 176, where it appears that the equation to a plane touching a surface at any point ($x, y, z,$) is

$$z' - z = \frac{dz}{dx}(x' - x) + \frac{dz}{dy}(y' - y) \ldots . (1)$$

x', y', z', being the co-ordinates of the plane.

Now, since

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 = 1$$

we have

$$\frac{dz}{dx} = -\frac{c^2x}{a^2z}, \frac{dz}{dy} = -\frac{c^2y}{b^2z}.$$

$\therefore$ by substituting in equation (1), we get

$$z' - z = -\frac{c^2}{a^2}\frac{x}{z}(x' - x) - \frac{c^2}{b^2}\cdot\frac{y}{z}(y' - y)$$

$$\text{or } \frac{xx'}{a^2} + \frac{yy'}{b^2} + \frac{zz'}{c^2} = 1$$

for the equation determining the position of a tangent plane $(x', y', z',)$ to an ellipsoid, at the point $(x, y, z,)$ of its surface.

193. Since the equation to the elliptic paraboloid is

$$ax^2 + by^2 + abz = abc$$

we have the partial differential coefficients

$$\frac{dz}{dx} = \frac{2x}{b}, \frac{dz}{dy} = \frac{2y}{a}$$

which being substituted in the equation to the *normal*, (*Lac. Trans.* page 177,)

$$\left.\begin{aligned} x' - x + \frac{dz}{dx}(z' - z) &= 0 \\ y' - y + \frac{dz}{dy}(z' - z) &= 0 \end{aligned}\right\}$$

produce

$$\left.\begin{aligned} x' - x + \frac{2x}{b}(z' - z) &= 0 \\ y' - y + \frac{2y}{a}(z' - z) &= 0 \end{aligned}\right\}$$

thereby determining the position of the normal to any point $(x, y, z,)$ of the surface.

Now, when this normal meets the plane of (xy), or (xz), or (yz), we evidently have $z' = 0$, or $y' = 0$, $x' = 0$.

$$\left.\begin{aligned} \therefore x' &= \frac{x}{b}\cdot(2z+b) \\ y' &= \frac{y}{a}\cdot(2z+a) \end{aligned}\right\}, \quad \left.\begin{aligned} x' &= x+\frac{a}{b} \\ z' &= z+\frac{a}{2} \end{aligned}\right\} \text{ and } \left.\begin{aligned} y' &= y+\frac{b}{a} \\ z' &= z+\frac{b}{2} \end{aligned}\right\}$$

determine the points of intersection required.

Again, to find the volume we have (*Lac. Trans.* p. 311.)

$$V = \int_x dy \int_y z dx$$

the symbols $\int_x, \int_y$, denoting that the integral is taken on the supposition that x or y is constant.

But $\int_y z dx = \frac{x}{a}(ac - y^2)^{\frac{1}{2}} - \frac{x^3}{3b}$, which being taken between the limits 0 and $\sqrt{\frac{b}{a}}\cdot\sqrt{ac-y^2}$ of x, gives, after reduction,

$$\therefore \int_y z dx = \frac{2}{3}\sqrt{\frac{b}{a^3}}(ac-y^2)^{\frac{3}{2}}$$

for the area of the section of the solid made by a plane parallel to that of (x, z).

Hence the volume is generally expressed by

$$V = \int dy \int_y z dx = \frac{2}{3}\sqrt{\frac{b}{a^3}} \times \int (ac - y^2)^{\frac{3}{2}} dy, \text{ which, by}$$

means of two substitutions

$$P = (ac - y^2)^{\frac{3}{2}} y, \; P' = (ac - y^2)^{\frac{1}{2}} y$$

and proper reductions, becomes

$$V = \frac{1}{6}\sqrt{\frac{b}{a^3}}P + \frac{1}{4}c\sqrt{\frac{b}{a}}P' + \frac{1}{4}c\sqrt{bc}\sin.^{-1}\frac{y}{\sqrt{ac}}$$

Let $y = \sqrt{ac}$ (or $x = 0$, $z = 0$). Then the whole volume comprehended by the planes (x, y), (y, z), (x, z), is expressed by

$$V = \frac{1}{4}c\sqrt{bc} \times \frac{\pi}{2}.$$

194. Let the radius of the base of either cylinder $= r$, and suppose $(x, y,)$ originating in the intersection of the axes of

the cylinders, to be measured along those axes. Then the equations to the surfaces of the cylinders being

$$\left.\begin{array}{r} x^2+z^2=r^2 \\ \text{and } y'^2+z'^2=r^2 \end{array}\right\} \ldots\ldots\ldots\ldots (1)$$

if a section of the part common to both be made by a plane parallel to that of (x, y), we shall have $z=z'$, and $\therefore y'=x$. Hence it is evident that the section is a square whose side is $2x$. Consequently the *solid common to both cylinders* may be generated by the motion of that square parallel to itself, and its volume will therefore be expressible by

$$V=\int 4x^2dz$$

But $dz=-\dfrac{xdx}{\sqrt{r^2-x^2}}$, by equation (1).

$$\therefore V=-4\int\frac{x^3dx}{\sqrt{r^2-x^2}}=4x^2\sqrt{r^2-x^2}+\frac{8}{3}(r^2-x^2)^{\frac{3}{2}}$$

Let $z=r$, or $x=0$.

Then the whole volume of the part common to both cylinders will be expressed by

$$2V=\frac{16}{3}\cdot r^3.$$

This solid is evidently the common groin.

Mr. Peacock (A Collection of Examples, &c., page 451,) has arrived at the above result, but by a process of reasoning obviously incorrect. This work, however, upon the whole, is worthy our best recommendation.

195. Let r be the radius of the base of the cone, and $\therefore \dfrac{r}{2}$ that of the base of the cylinder. Then referring the co-ordinates to the centre of the base of the cone, its equation will be ($a=$ its alt.)

$$z=\frac{a}{r}(r-\sqrt{x^2+y^2})\ldots\ldots (1)$$

and that of the cylinder

$$y'^2=rx'-x'^2\ldots\ldots\ldots\ldots (2$$

Hence the equations to the curve of their intersection ($x = x'$, $y = y'$), are

$$\left.\begin{aligned} z &= \frac{a}{\sqrt{r}} \cdot (\sqrt{r} - \sqrt{x}) \\ y &= \sqrt{rx - x^2} \end{aligned}\right\} \ldots\ldots\ldots (8)$$

$$\therefore \frac{dz}{dy} = -\frac{a}{\sqrt{r}} \cdot \frac{\sqrt{r-x}}{r-2x}, \frac{dx}{dy} = \frac{2\sqrt{rx-x^2}}{r-2x}.$$

But projecting the tangent and curve upon the planes (x, y), (y, z), we have (*Lac. Trans.* p. 171.)

$$\left.\begin{aligned} x' - x &= \frac{dx}{dy} \cdot (y' - y) \\ z' - z &= \frac{dz}{dy} \cdot (y' - y') \end{aligned}\right\}$$

$$\text{or } \left.\begin{aligned} x' &= x + \frac{2\sqrt{rx-x^2}}{r-2x} \cdot (y' - y) \\ z' &= z - \frac{a}{\sqrt{r}} \cdot \frac{\sqrt{r-x}}{r-2x} \cdot (y' - y) \end{aligned}\right\}$$

which determine the position of the tangent.

196. Since $yz = \varphi \cdot \left(\frac{y}{x}\right)$ is the equation to the curve surface, we have

$$dz = -\frac{dy}{y^2} \cdot \varphi\left(\frac{y}{x}\right) + \frac{1}{y} \cdot d \cdot \varphi\left(\frac{y}{x}\right).$$

But $d \cdot \varphi\left(\frac{y}{x}\right) = d \cdot \frac{y}{x} \cdot \varphi'\left(\frac{y}{x}\right) = \frac{xdy - ydx}{x^2} \varphi'\left(\frac{y}{x}\right)$.

where $\varphi'\left(\frac{y}{x}\right)$ denotes a certain function of $\left(\frac{y}{x}\right)$, of which it is not necessary to know the precise form.

$$\therefore dz = -\frac{dy}{y} \cdot z \times \frac{xdy - ydx}{x^2} \varphi' \cdot \left(\frac{y}{x}\right).$$

Let $dx = 0$. Then $\frac{dz}{dy} = -\frac{z}{y} + \frac{1}{x} \cdot \varphi'\left(\frac{y}{x}\right)$

Let $dy = 0$. Then $\frac{dz}{dx} = -\frac{y}{x^2}.\varphi'\left(\frac{y}{x}\right)$.

Hence the equation to the *tangent plane*. (*Lac. Trans.* p. 176.)

$$z' - z = \frac{dz}{dx}.(x' - x) + \frac{dz}{dy}\ y' - y) \text{ becomes}$$

$$z' - z = -\frac{y}{x^2}.\varphi'.\left(\frac{y}{x}\right).(x'-x) - \left(\frac{z}{y} - \frac{1}{x}\varphi'\frac{y}{x}\right)(y' - y)$$

Let $x' = y' = 0$. Then

$$z' - z = \frac{y}{x}.\varphi'\left(\frac{y}{x}\right) + z - \frac{y}{x}\varphi'\left(\frac{y}{x}\right)$$

$\therefore\ z' = 2z$. Q. E. D.

197. Let r be the radius of the sphere, y the radius of the base of the cone, and $x + r$ its altitude. Then its volume V is expressed by

$$V = \frac{1}{3}\pi y^2(r + x) = \max.$$

$$\therefore\ y^2.(r + x) = (r^2 - x^2).(r + x) = \max.$$

$$\therefore\ -2xdx.(r + x) + dx.(r^2 - x^2) = 0,$$

whence $x = \frac{r}{3}$ and $y = \sqrt{r^2 - x^2} = \frac{2\sqrt{2}}{3}.r.$

$$\therefore\ V = \frac{32}{81}.\pi r^3.$$

198. Let APB (Fig. 71,) be the circular arc whose centre is C. Then D being the middle point of the radius CD, describe a circle passing through C, D, and touching the arc AB (p. 7, Vol. I.) in the point P. P is the point required.

For the $\angle$ CPD = $\angle$ CQD > CPD.

Now $\because$ the circles touch at P, they have a common tangent, which is $\perp$ their radii, and $\therefore$ $\perp$ CP. Hence CP is the diameter of $\odot$ CDP, and $\therefore$ CDP is a right $\angle$.

$$\therefore\ CD = CP.\sin.P = \frac{1}{2}CP.$$

$\therefore$ sin. P $= \frac{1}{2}$, or the max. value of P is 30°.

199. Let y be the radius of the base of a cylinder inscribed in the sphere, whose radius suppose $= r$, and $2x$ its altitude. Then it easily appears that the volume (V) is expressed by

$$V = \pi y^2 \times 2x = \text{max.}$$

$$\therefore y^2 x = (r^2 - x^2)\, x = \text{max.}$$

$$\text{or } dx \,.\, (r^2 - x^2) - 2x^2 dx = 0$$

$$\therefore x = \frac{r}{\sqrt{3}}, \text{ and } y = \sqrt{\frac{2}{3}} \times r.$$

$$\therefore \text{Max. of } V = \frac{4}{3\sqrt{3}} \,.\, \pi r^3.$$

200. If y and x be the co-ordinates of the ⊙ referred to its centre, and r its rad. then

$$y = \sqrt{r^2 - x^2}$$

and by the question

$$2y \times 2x = \text{max.}$$

$$\therefore y dx + x dy = 0; \text{ or } \frac{dy}{dx} = -\frac{y}{x}.$$

$$\text{Also } \frac{dy}{dx} = -\frac{x}{y} = -\frac{y}{x}.$$

$$\therefore x^2 = y^2 = r^2 - x^2$$

$$\therefore x = \frac{r}{\sqrt{2}}, \text{ and max.} = 4 \,.\, \frac{r^2}{2} = 2r^2.$$

201. Let AC, BC, be the given lines, and P the point between them. Draw APB $\perp$ CB, and let Q P q be any other line passing through P, &c.

Then, putting PA $= b$, PB, $= \beta$, CB $= \alpha$, and $\angle q$QC $= \theta$, we have

$$PQ = \frac{b \,.\, \cos.\, C}{\sin.\, \theta},\ Pq = \frac{\beta}{\sin.\, (\theta + C)}$$

and $\therefore$ by the question

$$\frac{b \cos. C}{\sin. \theta} \times \frac{\beta}{\sin. (\theta + C)} = \max.$$

$$\therefore \sin. \theta . \sin. (\theta + C) = \min.$$

$$\therefore d\theta . \cos. \theta - \sin. (\theta + C) + d\theta . \cos. (\theta + C) \sin. \theta = 0$$

$$\text{or } \sin. (2\theta + C) = 0$$

$$\therefore 2\theta + C = \pi, \text{ and } \theta = \frac{\pi}{2} - \frac{C}{2}.$$

Hence max. $= 2b\beta \times \cot. C$.

202. Let a, b, be the axes of the generating ellipse of the spheroid; then its equation, referred to the centre, is

$$y^2 = \frac{b^2}{a^2} . (a^2 - x^2)$$

and πy^2 will be the base of a cone inscribed, whose altitude is $a + x$.

$$\therefore \frac{1}{3} . \pi y^2 . (a + x) = \max.$$

$$\therefore y^2 . (a + x) = \max.$$

$$\text{and } \therefore 2ydy . (a + x) + y^2dx = 0$$

$$\therefore \frac{dy}{dx} = - \frac{\frac{b}{a} . \sqrt{a^2 - x^2}}{2 . (a+x)}$$

$$\text{But } \frac{dy}{dx} = - \frac{b}{a} . \frac{x}{\sqrt{a^2 - x^2}} = - \frac{b}{a} . \frac{\sqrt{a^2 - x^2}}{2 (a + x)}$$

$$\therefore x = \frac{a}{3}$$

and the greatest cone required $= \frac{8\pi}{27} . ab^2$.

203. Let a be the length of the line, and $x, a - x$, the parts required. Then by the question

$$u = x . (a - x) . (a + 2x) = \max.$$

$$\therefore du = 0, \text{ gives}$$

$$a^2 + 2ax - 6x^2 = 0,$$

$$\therefore x = \frac{a(1 \pm \sqrt{7})}{6}.$$

$$\therefore u = \frac{10 \pm 7\sqrt{7}}{54} \times a^3.$$

204. Let α, β, γ, be the inclinations of the given plane figure (S) to the co-ordinate planes (x, y), (y, z), (x, z). Then it is well known that

$$S' = S\cos.\alpha,\ S'' = S.\cos.\beta,\ S''' = S\cos.\gamma$$

$$\therefore S'^2 + S''^2 + S'''^2 = (\cos.^2\alpha + \cos.^2\beta + \cos.^2\gamma).S^2$$

$$\text{But } \cos.^2\alpha + \cos.^2\beta + \cos.^2\gamma = 1$$

$$\therefore S'^2 + S''^2 + S'''^2 = S^2.$$

205. Let b be the given base, x, y the other sides, whose sum, being given, suppose $= m$. Then S denoting, as usual, the semiperimeter of the $\triangle$, which is also given, we have, by a well known expression,

$$u = \sqrt{S.(S-b).(S-x).(S-y)} = \max.$$

$$\therefore (S-x).(S-y) = \max.$$

and $du = 0$, gives

$$dx.(S-y) + dy.(S-x) = 0$$

But since $x + y = m$, $dy = -dx$,

$$\therefore S - y = S - x,\ \therefore x = y,$$

or the required $\triangle$ is isosceles,

$$\text{and its area} = \left(S - \frac{m}{2}\right)\sqrt{S.(S-b)} = \frac{b}{4}.\sqrt{m^2-b^2}.$$

206. Let y be the radius of the base of the cone, and x its altitude. Then its surface, which is given, will be expressed by

$$m = \frac{2\pi y}{2} \times \text{slant side} = \pi y\sqrt{x^2 + y^2} \ldots\ldots (1)$$

$$\text{and its volume} = \frac{1}{3}.\pi y^2 x = \max.$$

$\therefore y^2x = \max.$

$\therefore 2yxdy + y^2dx = 0$, or $\frac{dy}{dx} = -\frac{y}{2x}$.

Also from eq. 1, we have

$$\frac{dy}{dx} = -\frac{yx}{x^2 + 2y^2}.$$

$$\therefore \frac{y}{2x} = \frac{yx}{x^2 + 2y^2}$$

which gives $x^2 = 2y^2$,

and $\therefore m = \pi y \sqrt{3y^2}$ or $y = \sqrt{\frac{m}{\pi\sqrt{3}}}$

and $x = y\sqrt{2} = \sqrt{\frac{2m}{\pi\sqrt{3}}}$.

Hence the cone required $= \frac{m\sqrt{2m}}{3\sqrt{3\sqrt{3}}}$.

207. For the investigation see *Lac. Trans.* or *Simpson*, by which we learn that the volume of any solid of revolution is expressible by

$$V = \pi\int y^2dx$$

x being the axis of rotation.

Now in the problem before us

$$y = l.(1 + x)$$

$$V = \pi\int dx.(l.\overline{1+x})^2$$

But $\int dx.(l.\overline{1+x})^2 = x.(l.\overline{1+x})^2 - 2\int\frac{xdx}{1+x}.l.\overline{1+x}$

and continuing to integrate by parts we finally obtain

$$V = \pi.(x-1)(l.\overline{1+x})^2 - 2\pi.(x - l.\overline{1+x}) \times (l.\overline{1+x} - 1)$$

$$= \pi.(1+x)(l.\overline{1+x})^2 + 2\pi.(1-x.l.(1+x) + 2\pi x.$$

208. Let BC (Fig. 73,) be the base of the $\triangle$, and BM the common altitude of all equal $\triangle$ equal to it, and MN parallel to BC. Then ABC being that $\triangle$ which is isosceles, and A′BC

any other whatever, if BA be produced making AC′ = AC, and A′C, A′C′ be joined, it is easily shewn that A′C = A′C′.

$\therefore$ BA + AC = BC′ < BA′ + A′C′ < BA′ + A′C, i. e., BAC is less in perimeter than any other $\triangle$ BA′C of the same base and area.

209. Let the two given sides of the $\triangle$ be a, b. Then its area $= ab$ sin. C = max. which evidently is the case when sin C = max. or when C = 90°.

210. Let θ, A − θ, be the parts required of the given $\angle$ A. Then by the question

$$\sec. (A - \theta) . \sec.^2 \theta = \max.$$
$$\text{or } \cos. (A - \theta) . \cos.^2 \theta = \min.$$
$$\therefore d\theta . \sin. (A-\theta) \cos.^2 \theta - 2d\theta \cos. \theta \sin. \theta . \cos. (A-\theta) = 0$$
$$\text{whence } \tan.^2 \theta + \frac{3}{2 \tan. A} . \tan. \theta = \frac{1}{2}$$

and solving the equation in tan. θ, we have

$$\tan. \theta = \frac{-3 \pm \sqrt{9+8 \tan.^2 A}}{4 \tan. A .} \ldots\ldots\ldots (a)$$

which involving no imaginary quantities, shews that the restriction in the enunciation with regard to the limits of A, is by no means necessary. Indeed, when

$$A = 90^\circ, \tan. \theta = \pm 2 \sqrt{2} \frac{\tan. A.}{4 \tan. A} = \frac{1}{\sqrt{2}}.$$

Having found θ from eq. (a) by means of the tables, we shall easily obtain A − θ, and sec. (A − θ) . sec.^{2} θ.

211. By the equation to the ellipse referred to its focus as a pole, we have (Pp = the line required)

$$Pp = \rho + \rho' = \frac{a . (1 - e^2)}{1+e . \cos. \theta} + \frac{a . (1 - e^2)}{1+\cos. (\pi+\theta)}$$
$$= \frac{2 a . (1 - e^2)}{1 - e^2 . \cos.^2 \theta} = \max.$$

or min. according as $\theta = 0$, or $\frac{\pi}{2}$, i. e., according as Pp coincides with or is $\perp$ to the axis a.

212. Let θ be the required angle, and x, y, the corresponding co-ordinates measured from the vertex. Then it will easily appear that

$$\sqrt{x^2+y^2} : x :: \frac{dy}{dx} : \sin.\theta.$$

But by the equation to the parabola

$$y^2 = px.$$

$$\therefore \text{ we have } \frac{dy}{dx} = \frac{\sqrt{p}}{\sqrt{p+4x}}$$

$$\therefore \sin.\theta = \frac{\sqrt{px}}{\sqrt{(4x+p).(x+p)}}.$$

But since $\theta = \max.$ $d\theta = 0$.

$$\therefore \frac{d.\sin.\theta}{\cos.\theta} = d\theta = 0,$$

$$\therefore d.\sin.\theta = 0$$

which will be found to give

$$x = \frac{p}{2}, \text{ and } y = \frac{p}{\sqrt{2}}$$

which determine the position of the point required.

213. Let A be the given $\angle$, and θ, A $-$ θ the parts required. Then by the question,

$$u = \sin.^3\theta . \sin.^2(A - \theta) = \max.$$

$$\therefore du = 0, \text{ gives}$$

$$2\tan.\theta = 3\tan.(A-\theta),$$

$$\text{whence } \tan.^2\theta + \frac{5}{2\tan.A}.\tan.\theta = \frac{3}{2},$$

which, being solved, gives

$$\tan.\theta = \frac{-5 \pm \sqrt{25+24\tan.^2 A}}{4\tan.A}.$$

The construction is easily effected as follows:

In the line Aa (Fig. 74,) take $AN = 1$, and $Aa = 5 \times AN = 5$; then BAa being the given $\angle$, we have

$$PN\ (\perp Aa) = AN.\ \text{Tan. } A = \tan. A.$$

Again, take $aM\ (\perp Aa) = \sqrt{24} \times \tan. A$, join AM. Then $AM = \sqrt{25 + 24 \tan.^2 A}$, and if $AQ = Aa = 5$, we have $qM = -5 + \sqrt{25 + 24 \tan.^2 A}$, $qm = 5 + \sqrt{25 + 24 \tan.^2 A}$. Finally, taking $AM' = 4 \tan. A$, and $Q'M' \perp Aa$ and $= QM$, $q'M \perp Aa$ and $= qm$ taken negatively, we obtain

$$\theta = Q'AM', \text{ or } = q'AM'.$$

214. The least polygon circumscribing a ⊙ and the greatest inscribed in the ⊙ of the same number of sides, are known to be those which are equilateral and equiangular. Hence it is evident that the sides of these circumscribed and inscribed polygons subtend equal $\angle$ (θ for instance) at the centre, and moreover the polygons (P, P') are similar figures. Consequently, s, s' being any two sides, we have

$$P : P' :: s^2 : s'^2 :: r^2 : \cos.^2 \frac{\theta}{2}$$

as we easily learn from similar Δ, &c.

215. This problem may be generalized by stating it, "Required the shortest line drawn from a given point to the circumference of an ellipse, given in position, form and magnitude."

Let a, b, be the semiaxes of the ellipse, and $y = \frac{b}{a} \cdot \sqrt{a^2 - x^2}$ its equation referred to the centre, and a as an axis of x. Then, α, β, being the co-ordinates of the given point referred to the same origin, we have

$$u = (y - \beta)^2 + (x - \alpha)^2 = \min.$$

and $du = 0$, gives

$$\frac{dy}{dx} = -\frac{x - \alpha}{y - \beta}.\quad \text{But } \frac{dy}{dx} = -\frac{b}{a} \cdot \frac{x}{y}$$

$$\therefore y = \frac{b\beta x}{a\alpha - (a-b)x} = \frac{b}{a} \cdot \sqrt{a^2 - x^2}$$

whence $x^4 - \frac{2a\alpha}{a-b} x^3 + \frac{\alpha^2 + \beta^2 - (a-b)^2}{(a-b)^2} a^2 x^2 + \frac{2a^3\alpha}{a-b} x -$

$\frac{a^4\alpha^2}{(a-b)^2} = 0$, by the resolution of which, x and $\therefore y$ may be found, which will determine the position of the line required.

In the problem $\beta = 0$; whence

$$x^4 - \frac{2a\alpha}{a-b} x^3 + \frac{\alpha^2 - (a-b)^2}{(a-b)^2} a^2 x^2 + \frac{2a^3\alpha}{a-b} x - \frac{a^4\alpha^2}{(a-b)^2} = 0$$

whose four roots are a, $\frac{a\alpha}{a-b}$, and

$\frac{a}{2(a-b)} \cdot (b - a - \alpha \pm \sqrt{(a-b)^2 + 6(a-b)\alpha + \alpha^2})$, two of which correspond to maxima, and two to minima, between which the student may easily distinguish by the usual test.

216. Let a be the extreme value of x (measured from the vertex) of the given parabola. Then since the area of a parabola $= \frac{2}{3} xy$, that of the inscribed parabola will evidently be

$$u = \frac{2}{3} \cdot (a - x)\sqrt{px} = \text{max.}$$

$$\therefore (a-x)^2 \cdot x = \text{max.} = u'$$

and $\frac{du}{dx} = 0$ will give

$$x = \frac{a}{3}, \text{ and } \therefore y = \sqrt{\frac{pa}{3}}.$$

$$\therefore \text{the maximum area} = \frac{4a}{9}\sqrt{\frac{pa}{3}}.$$

217. Let Q^3 be the given volume, then x being the radius of the base and z the altitude of the cylinder, we have

$$\pi x^2 \times z = Q^3$$

and by the question

$$u = 2\pi x \times z + \pi x^2 = \text{surface} = \text{min.}$$

$$\therefore \frac{2Q^3}{x} + x^2 = \text{min.}$$

and $\frac{du}{dx} = 0$, gives

$$x = Q, \text{ and } \therefore z = \frac{Q}{\pi}.$$

$$\therefore \text{ the minimum surface} = (2 + \pi)\, Q^2.$$

218. It very easily appears that the area of any rectangle inscribed in a $\frac{1}{2} \odot = 2xy$ (x being measured from the centre).

$$\therefore xy = \text{max.}$$

$$\therefore xdy + ydx = 0, \text{ or } \frac{dy}{dx} = -\frac{y}{x}.$$

$$\text{But } y = \sqrt{r^2 - x^2}, \ \therefore \frac{dy}{dx} = -\frac{x}{y} = -\frac{y}{x}$$

$$\therefore x = y. \quad \therefore x^2 = r^2 - x^2, \ \therefore x = \frac{r}{\sqrt{2}}.$$

$$\text{and the maximum rectangle} = 2 \,.\, \frac{r^2}{2} = r^2.$$

219. Let (y, x,) be the co-ordinates of the cycloid measured from its vertex, and (y', x,) those of the locus required,

$$\text{then since } y = \sqrt{2rx - x^2} + \text{vers.}^{-1}x$$

$$\therefore y' = \int \sqrt{dx^2 + dy^2} = \int dx \,.\, \sqrt{\frac{2r}{x}} = 2\sqrt{2r} \,.\, \sqrt{x}$$

$$\text{or } y'^2 = 8rx$$

and the locus is $\therefore$ a parabola whose latus-rectum $= 8r$.

220. By the question

$$(\tan. P)^m \,.\, (\tan. Q)^n = \text{max.}$$

$$\text{and} \therefore m \, . \, l. \tan. P + n \, . \, l. \tan. Q = \max.$$

$$\therefore m \, . \, \frac{dP}{\cos.^2 P. \tan. P} + n \, . \, \frac{dQ}{\cos.^2 Q \tan. Q} = 0$$

But since $P + Q$ is given, or

$P + Q = a$, $dQ = - dP$.

$$\therefore m. \cos.^2 Q \tan. Q = n \cos.^2 P \tan. P$$

which becomes by reduction

$$\tan. 2P = \frac{m \sin. 2a}{n + m \cos. 2a}$$

Whence, by means of the tables, we have P, and $\therefore$ Q.

221. The radius of the base of any cylinder inscribed in the solid being y, and $\therefore$ its area $= \pi y^2 =$

$$\pi \, . \, a^{\frac{2m}{m+n}} x^{\frac{2n}{m+n}} \text{ (since } a^m x^n = y^{m+n})$$

and the altitude of the cylinder being $b - x$, we have the volume expressed by

$$V = \pi \, . \, a^{\frac{2m}{m+n}} . x^{\frac{2n}{m+n}} . (b - n) = \max.$$

$$\therefore x^{\frac{2n}{m+n}} (b - x) = \max.$$

and putting its differential $= 0$, we get

$$x = \frac{2n}{3n + m} . b$$

whence it is easy to find y and the maximum volume.

222. The equation to the ellipse referred to the focus (S) by the radius vector and $\perp$ upon the tangent, being

$$p^2 = \frac{a^2 b^2}{a^2 + b^2 - \rho^2} = \frac{b^2 \rho}{2a - \rho}.$$

$$\therefore \frac{dp}{d\rho} = \frac{ab}{\sqrt{\rho} \, . \, (2a - \rho)^{\frac{3}{2}}}$$

$$\therefore \frac{d\rho}{dp} = \frac{\sqrt{\rho} \, . \, (2a - \rho)^{\frac{3}{2}}}{ab}$$

But $\rho + HP = a$, and $\therefore 2a - \rho = HP$

and $CD = \sqrt{HP \times SP}$.

$$\therefore \frac{d\rho}{dp} = \frac{HP \times CD}{AC \times BC}.$$

223. Let s be the arc to rad. $= r$,

then $ds = \dfrac{r^2 d\,.\,\tan.\,s}{r^2 + \tan.^2\,s}$. Let $s = 45^\circ$.

Then $\tan.\,s = r$, and $\dfrac{ds}{d\,.\,\tan.\,s} = \dfrac{r^2}{2r^2} = \dfrac{1}{2}$

$\therefore d\,.\,\tan.\,s = 2ds.$

224. Let θ be the arc required, then

$u = \sin.\,\theta - \text{vers.}\,\theta = \max.$

$\therefore \dfrac{du}{d\theta} = 0$, gives

$\tan.\,\theta = 1 = \tan.\,45^\circ.$

or $\theta = 45^\circ.$

225. Let AB (Fig. 75,) be any circle whatever on the surface of the sphere, and suppose PQ that great circle which passes through the extremities of the diameter (AB). Then MN, any circle whose plane is $\perp$ plane of PMQN, and whose poles are P, Q, being considered the plane of projection, and $\therefore$ P the point of vision, ab will be the section of the oblique cone PAB (formed by the rays proceeding from every point of the circumference AB) made by the plane MN.

Now PL being the tangent at P, $\angle$ BPL $= \angle$ BAP $=$ $\angle$ abP.

$\therefore$ ab is the subcontrary section of the cone, or it is a $\odot$ (p. 18).

Again, let $ab = 2r$, AB $= 2r'$, P$c = a$, PC $=$ R, and $\angle$ P AB $=$ A and $\angle$ APB $=$ P. Then by similar $\triangle$

$$ab : AB :: Pa : PB :: \frac{Pc}{\sin.(A+P)} : 2R \times \sin. A$$

$$\text{whence } r = \frac{ar'}{2R \sin A . \sin.(A+P)} \ldots\ldots (1)$$

Also $oc = ac + r = \dfrac{a}{2R . \sin. A . \sin(A+P)} \times \{2R \times$ $\sin. A . \sin.(A+P) + r'\} \ldots\ldots\ldots (2)$.

Now, according to the problem we have $a = r' = R$, $P = 90°$.

$\therefore$ equations (1) and (2) become

$$r = \frac{R}{2\sin. A . \cos. A} = \frac{R}{\sin. 2A} \ldots\ldots (3)$$

$$\text{and } oC = \frac{R}{\sin. 2A}(-2\sin.^2 A + 1) = R\cos. 2A \ldots. (4)$$

But when $P = 90°$, $A = \dfrac{\pi}{2} - \angle Pab = \dfrac{\pi}{2} - ABP = \dfrac{\pi}{2}$ $- T - A. \quad \therefore 2A = \dfrac{\pi}{2} - T.$

$$\text{Hence } r = \frac{R}{\cos. T} = \text{sec. T to rad. R}$$

and $oC = R \tan. T =$ tan. T to rad. R. Q. E. D.

226. We generalize this problem by stating it, Let P (Fig. 76,) be the pole of the great ⊙ ABM of a given sphere whose centre is C; then supposing the quadrant AP to revolve uniformly about the axis PC with a given velocity, whilst a point from P moves along the quadrant also with a given uniform velocity, it is required to find the surface APQM traced out by the point P in moving through the whole quadrant.

Let v and nv be the velocities of P and of A (which measures the angular velocity of the quadrant), and let PQP, Pqb be any two successive positions of the quadrant, and Q, q of the point P. Then S denoting the surface APB,

$$dS = QqbB.$$

Now $QBbq$: Zone corresponding to $'x$ ($=$ CN) :: Bb : ABMA $= 2\pi r$.

But the zone $= 2\pi r x$, and $Bb = nd \,.\, PQ \;=\; -\dfrac{nrdx}{\sqrt{r^2 - x^2}}$,

$$\therefore\ dS = \frac{-nrxdx}{\sqrt{r^2 - x^2}}$$

$$\text{and } S = C + nr\sqrt{r^2 - x^2} = nr\sqrt{r^2 - x^2}$$

Let $x = 0$.

Then $S = APQM = nr^2$,

and $APQMA = 2\pi r^2 - nr^2 = (2\pi - n)\, r^2$.

In the problem $n = 4$, when M coincides with A, and the surface of the hemisphere is divided into the two parts $4r^2 = (2r)^2$ and $2 \,.\, (\pi - 2)\, r^2$.

The curve of *double curvature* AQM is called the *Spiral of Pappus.*

227. Let a, b, c, be the sides of the spherical $\triangle$, and α, β, γ, their chords, and suppose A, A′, the $\angle$ included by b, c, and β, γ. Then it is well known that

$$\cos.\ A = \frac{\cos.\ a - \cos.\ b \cos.\ c}{\sin.\ b \sin.\ c}, \text{ and } \cos.\ A' = \frac{\beta^2 + \gamma^2 - \alpha^2}{2\beta\gamma}.$$

$$\text{But } \alpha = 2\sin.\ \frac{a}{2},\ \beta = 2\sin.\ \frac{b}{2},\ \gamma = 2\sin.\ \frac{c}{2}.$$

Hence, by substitution and reduction, we get

$$\cos.\ A' = \cos.\ A + \sin.\ \frac{b}{2} \sin.\ \frac{c}{2},$$

whence cos. A′ is always $>$ cos. A (for $\sin.\ \frac{b}{2} \sin.\ \frac{c}{2}$ can never be negative), and consequently A is always $>$ A′ whether the spherical $\triangle$ be *isosceles* or *not.*

228. Measuring the co-ordinates of the semicircle S, and the sector S′, we have

$$dS = ydx, \text{ and } dS' = \frac{r}{2}\sqrt{dx^2 + dy^2}$$

$$\therefore dS : dS' :: y : \frac{r}{2}\sqrt{1+\frac{dy^2}{dx^2}}$$

But $y = \sqrt{r^2 - x^2}$. $\therefore \frac{dy}{dx} = -\frac{x}{y}$

and $\sqrt{1+\frac{dy^2}{dx^2}} = \sqrt{\frac{x^2+y^2}{y}} = \frac{r}{y}$

$$\begin{aligned}\therefore dS : dS' &:: 2y^2 : r^2\\ &:: 2\sin.^2\theta : r^2\\ &:: r^2 - r\cos. 2\theta : r^2\\ &:: \text{vers. } 2\theta : r. \quad \text{Q. E. D.}\end{aligned}$$

229. By the Arithmetic of Sines

$\sin. (x - z) = \sin. x \cos. z - \cos. x \sin. z,$

$$\therefore \frac{\sin. (x-z)}{\sin. x} = \cos. z - \cos. x \sin. z.$$

But $\cos. z = 1 - \frac{z^2}{2} + \frac{z^4}{2.3.4} - \&c.$

$\sin. z = z - \frac{z^3}{2.3} + \frac{z^5}{2.3.4.5} - \&c.$

and when z is indefinitely diminished, $\cos. z = 1 - \frac{z^2}{2}$, $\sin. z = z$ nearly.

$$\therefore \frac{\sin. (x-z)}{\sin. x} = 1 - \frac{z^2}{2} - \cot. x \times z \text{ nearly.}$$

$$\therefore \sin. (x-z) = \sin. x - z\cos. x - \frac{z^2}{2}\sin. x.$$

230. Let $y^2 = px$ be the equation to the generating parabola whose altitude $= a$. Then the volume of a cone, the radius of whose base is y, and altitude $a - x$, is

$V = \pi y^2 . (a - x) = \pi px . (a - x) = \text{max.}$ and putting $dV = 0$, we get $x = \frac{a}{2}$, which determines the cone required.

231. Let C (Fig. 77,) be the centre of the earth, AB = 9 feet, the height of the person's eyes from the ground, and CA = radius of the earth = 4000 miles. Then if from B the tangent BD be drawn, the arc AD will be the distance required.

Now CB (which is evidently a straight line) = 4000 miles + 3 yards $= 4000 + \frac{3}{1760}$ miles $= \frac{7040003}{1760}$ and cos. $C = \frac{CD}{CB}$

$$= \frac{7040000}{7040003} = 1 - \frac{C^2}{2} + \frac{C^4}{2.3.4} - \&c.$$

But since C is very small compared with 1, we may neglect the higher powers of C.

$$\therefore C^2 = 2\left(1 - \frac{7040000}{7040003}\right) = \frac{6}{7040003}$$

$\therefore AD = 4000 \times C = 4000\sqrt{\frac{6}{7040003}}$ miles, nearly, which may easily be converted into miles, furlongs, poles, yards, &c.

232. Let A (Fig. 78,) be the given $\angle$, and with AD, the given line drawn from the vertex A bisecting the opposite side of the $\triangle$, describe a circular arc MN. Then ABC being any $\triangle$ of which BC is bisected in D by AD, and A*bc* that whose side *bc* is bisected by A*d* at right angles, and $\therefore$ touches MN, we have $\triangle$ A*bc* $>$ ABC. For *r*D*s* being drawn parallel to *bc* meeting the $\perp$ from C and B in *r*, *s*, it is evident that $\triangle$ C*ec* $>$ CD*s* $>$ D*r*B and à fortiori $>$ *b*B*e*, and the part A*be*C is common to the two $\triangle$ A*bc*, ABC; $\therefore$ A*bc* is $>$ ABC. But ABC is any $\triangle$ whatever, $\therefore$ A*bc* is the greatest $\triangle$ required.

233. Let AB. (Fig. 79,) the given line $= a$, and suppose the velocity of the point A moving along AC from A $= 4 \times$ vel. of B moving from B along BA; then B′, A′, being any contemporaneous positions of A, B, required the curve to which A′ B′ is perpetually a tangent.

Let *ba* be the position of A′ B′ immediately successive to A′ B′,

and intersecting it in P. Then P is evidently the point at which B′A′ touches the curve.

Now let the equation to the line B′A′ referred to origin A by co-ordinates x, y, parallel to AC, AB be

$$y = \mathrm{M}x + \mathrm{N} \ldots\ldots\ldots\ldots (1)$$

$$\text{Then } \mathrm{AB}' = y = \mathrm{N}$$

$$\mathrm{AA}' = x = -\frac{\mathrm{N}}{\mathrm{M}} = n\mathrm{BB}' = n.(a - \mathrm{N})$$

$$\therefore \mathrm{N} = \frac{na\mathrm{M}}{\mathrm{M}n-1}$$

and the equation (1) becomes

$$y = \mathrm{M}x + \frac{na\mathrm{M}}{\mathrm{M}n-1} \ldots\ldots\ldots (2)$$

Now at the point P, y and x belong to the curve as well as to the tangent; and since for the next position ab, they may be considered constant, we have

$$x d\mathrm{M} + \frac{na}{\mathrm{M}n-1} d\mathrm{M} - \frac{n^2 a\mathrm{M}d\mathrm{M}}{(\mathrm{M}n-1)^2} = 0$$

$$\text{which gives } \mathrm{M} = \frac{1}{n} \pm \sqrt{\frac{a}{nx}}$$

and substituting in equation (2) we get

$$y = \frac{x}{n} \pm (\sqrt{a} \pm 1)\sqrt{\frac{x}{n}} \pm a$$

the equation to the curve required. See pp. 47, 48.

234. Let a be the given line, and Q^2 the given square, then x being one of the required parts, and $\therefore a - x$ the other, we have

$$(a - x)\, x = \mathrm{Q}^2$$

$\therefore x = \dfrac{a \pm \sqrt{a^2 - 4\mathrm{Q}^2}}{2}$, which determines the rectangle required. Also since $4\mathrm{Q}^2$ cannot exceed a^2, $\left(\dfrac{a}{2}\right)^2$ is the greatest square the rectangle can equal.

235. Let (Fig. C. P. p. 272)

$$\left.\begin{array}{l} AE = a \\ AD = a' \end{array}\right\} \left.\begin{array}{l} BE = b \\ BD = b' \end{array}\right\} \left.\begin{array}{l} CE = c \\ CD = c' \end{array}\right\} \text{and } DE = \theta.$$

Then $\cos. \theta = \cos. a . \cos. a' + \cos. b \cos. b' + \cos. c . \cos. c'$.

For $\cos. DAE = \cos. (DAC - EAC)$

$= \cos. DAC . \cos. EAC + \sin. DAC \sin. EAC.$

$$\text{But } \cos. DAE = \frac{\cos. \theta - \cos. a \cos. a'}{\sin. a \sin. a'}$$

and since the angles at A, B, C, each $= 90^\circ$.

$$\cos. DAC = \frac{\cos. c'}{\sin. a'} \cos. EAC = \frac{\cos. c}{\sin. a}$$

$$\sin. DAC = \cos. DAB = \frac{\cos. b'}{\sin. a'}$$

$$\sin. EAC = \cos. EAB = \frac{\cos. b}{\sin. a'}$$

$\therefore$ by substitution and reduction, we get

$\text{Cos. } \theta = \cos. a . \cos. a' + \cos. b . \cos. b' + \cos. c \cos. c'$.

This Problem, which is in substance the same as "To find the inclination of two straight lines on the same plane given in position," may be otherwise solved, as follows:

$$\text{Let } \left.\begin{array}{l} x = mz \\ y = nz \end{array}\right\} (1) \quad \left.\begin{array}{l} x' = m'z' \\ y' = n'z' \end{array}\right\} (2)$$

be the equations of the given straight lines referred to the point of their intersection by rectangular co-ordinates passing through A, B, C. Then supposing this point to be the centre of the sphere ABC, let the lines intersect its surface in D, E, and make the radius of the sphere $= 1$.

Now the equation to the points E, D, being

$$x^2 + y^2 + z^2 = 1, \; x'^2 + y'^2 + z'^2 = 1.$$

by means of equations (1) and (2) we have,

$$(1 + m^2 + n^2) z^2 = 1, \; (1 + m'^2 + n'^2) z'^2 = 1$$

But DE^2 (chord) $= (x - x')^2 + (y - y')^2 = z - z (xx' + yy' + zz')$, and $\cos. \theta = 1 - \frac{1}{2} DE^2$.

$$\therefore \cos\theta = xx' + yy' + zz'$$
$$= \frac{1 + mm' + nn'}{\sqrt{1+m^2+n^2} \times \sqrt{1+m'^2+n'^2}},$$

Cos. θ also $=$ cos. a cos. a' + cos. b cos. b' + cos. c cos. c', for it readily appears upon drawing the co-ordinates that $x =$ cos. a, $x' =$ cos. a', $y =$ cos. b, $y' =$ cos. b', $z =$ cos. c, $z' =$ cos. c'.

236. Let θ be the $\angle$ at the centre of the generating $\odot$, subtended by that arc which has been in contact with the base, then R, r, being the radii of the base and generating circles, it readily appears (see Principia, vol. i. p. 284, new edit.) that the length of an arc measured from the first point of contact is expressed by

$$s = \frac{4r \,.\, (R \pm r)}{R} \text{ vers. } \frac{\theta}{2} \text{ . . . . } (1)$$

the positive or negative sign prevailing according as the curve is the epicycloid, or hypocycloid.

$$\text{Let } \theta = 360^\circ. \text{ Then vers. } \frac{\theta}{2} = 2$$

and the whole arc

$$S = \frac{8r}{R} \,.\, (R \pm r)$$

$\therefore$ supposing R constant and r variable, we have

$$r \,.\, (R - r) = \text{max.}$$
$$\therefore Rdr - 2rdr = 0$$
$$\therefore r = \frac{R}{2}, \text{ or } R : r :: 2 : 1, \text{ the proportion required.}$$

In the epicycloid it is evident that the greater r is, the greater will be the length of the curve.

237. Since the surface of water in equilibrium is always plane, the question is reducible to, "Required to shew that the solid cut off from a cone formed by the revolution of an asymptote to an hyperbola about its axis, by a plane any where touching the corresponding hyperboloid, is a constant quantity."

Let the cutting plane APa (Fig. 50,) touch the hyperboloid in the point Q, and let the revolving asymptote and hyperbola be brought into such a position that their planes may be $\perp$ cutting plane. Then Aa, APaA being the intersections of APa, ACr, and of APa with surface of the cone, we know that APaA is an ellipse whose axes are Aa ($= 2a'$) and

$$\frac{a \times \mathrm{CN}}{\cos. \frac{\mathrm{C}}{2}} \cdot \frac{1}{\sqrt{\mathrm{CA} \cdot \mathrm{C}a}} = \sqrt{\mathrm{T}} \times \sqrt{\tan. \frac{\mathrm{C}}{2}} \; (= 2b')$$

where T = area of $\triangle$ ACa.

Hence the area of the ellipse (Aa) is

$$\pi \times a'b' = \frac{\pi a'}{2} \sqrt{\mathrm{T}} \times \sqrt{\tan. \frac{\mathrm{C}}{2}} \ldots\ldots (1)$$

and the volume of the frustum ACA is

$$\frac{1}{3} \times (\mathrm{AP}a\mathrm{A}) \times \mathrm{CN} = \frac{\pi}{6} \times a' \times \mathrm{CN} \times \sqrt{\mathrm{T}} \times \sqrt{\tan. \frac{\mathrm{C}}{2}}$$

$$= \frac{\pi}{6} \times \mathrm{T} \sqrt{\mathrm{T}} \times \sqrt{\tan. \frac{\mathrm{C}}{2}}.$$

But $\mathrm{T} = \frac{a^2 + b^2}{2} \sin. \frac{\mathrm{C}}{2}$ (a, b, being the semi-axes of the hyperbola). See No. 152.

$$\therefore \text{ Frust. C}\mathrm{A}a = \frac{\pi}{12\sqrt{2}} \cdot (a^2+b^2)^{\frac{3}{2}} \frac{\sin.^2 \frac{\mathrm{C}}{2}}{\sqrt{\cos. \frac{\mathrm{C}}{2}}}, \text{ a}$$

a constant quantity. Q. E. D.

238. The equation to the parabola referred to its focus being

$$\rho = \frac{2a}{1 + \cos. \theta} = \frac{a}{\cos.^2 \frac{\theta}{2}}$$

$$\text{its area} = \int \frac{\rho^2 d\theta}{2} = \int \frac{a^2 d\theta}{\cos.^4 \frac{\theta}{2}}$$

$$= a^2 \cdot \left(\frac{\tan.^3 \frac{\theta}{2}}{3} + \tan. \frac{\theta}{2} \right).$$

239. This is a particular case of the general problem of trajectories, "Supposing a given curve $F(x, y, c,) = 0$ to change position according to a certain law depending on the variation of c, required the trajectory, or curve which cuts it in all positions in a given angle α," &c.

Let θ, θ' be the angles which the tangents of the curve and trajectory at the point of intersection make with the axis of x. Then x', y', (the co-ordinates of the trajectory being measured along the same co-ordinates as $(x, y,)$ we have

$$\tan. \alpha = \tan. (\theta' - \theta) = \frac{\tan. \theta' - \tan. \theta}{1 + \tan. \theta' \tan. \theta}$$

$$= \frac{\frac{dy'}{dx'} - \frac{dy}{dx}}{1 + \frac{dy'}{dx'} \cdot \frac{dy}{dx}}.$$

$$\left.\begin{aligned} \therefore \tan. \alpha &= \frac{\frac{dy'}{dx'} - \frac{dy}{dx}}{1 + \frac{dy'}{dx'} \cdot \frac{dy}{dx}} \\ \text{and } F(x, y, c,) &= 0 \end{aligned}\right\} \ldots\ldots\ldots\ldots (1)$$

which will enable us to eliminate c, thereby obtaining the equation to the trajectory.

If the trajectory be *orthogonal* we determine it from

$$\left.\begin{aligned} 1 + \frac{dy'}{dx'} \cdot \frac{dy}{dx} &= 0 \\ F(x, y, c,) &= 0 \end{aligned}\right\} \ldots\ldots\ldots (2)$$

To illustrate the above process,

(1) Let $y = cx$, required the trajectory cutting this line at the same $\angle\ \alpha$, whatever be its situation round the origin of co-ordinates.

$$\text{Here } \frac{dy}{dx} = c$$

and eliminating c from equations (1) we get ($y' = y$, $x' = x$, at the intersection)

$$y + x \tan. \alpha = \frac{dy}{dx} . (x - y \tan. \alpha)$$

which being homogeneous, gives

$$\tan\alpha \,.\, l.\,(C\,.\,\sqrt{x^2+y^2}) = \tan.^{-1}\frac{y}{x}$$

which is the equation to the logarithmic spiral.

Let $\alpha = 90°$. Then

$$\frac{l.\,C\,.\,\sqrt{x^2+y^2}}{0} = \tan.^{-1}\frac{y}{x}$$

$$\therefore\ l.\,C.\,\sqrt{x^2+y^2} = 0 = l.\ 1$$

$$\text{or } \sqrt{x^2+y^2} = \frac{1}{C} = C'$$

which is the equation to a ⊙.

(2). Let the moving curve be expressed by

$$x^m y^n = c$$

which belongs to parabolas and hyperbolas of every order.

$$\text{Then } \frac{dy}{dx} = -\frac{mx^{m-1}\,y^n}{ny^{n-1}\,x^m} = -\frac{m}{n}\cdot\frac{y}{x}$$

which being substituted in eq. (2) gives

$$my'dy' = nx'dx'$$

$$\therefore\ my'^2 = nx'^2 + C.$$

In the semicubical parabola, we have

$$cy^2 = x^3$$

$\therefore$ the required trajectory is

$$3y'^2 = C - 2x'^2$$

the equation to an ellipse.

240. Let θ = the arc required. Then by the question we have

$$u = \sin.\ \theta\ .\ (\sin.\ \theta - \cos.\ \theta) = \max.$$

$$\therefore\ \frac{du}{d\theta} = 0, \text{ gives}$$

$$\sin.\ 2\theta = \cos.\ 2\theta$$

$$\therefore\ \tan.\ 2\theta = 1, \text{ or } 2\theta = 45°, \text{ and } \theta = \frac{45°}{2}.$$

241. Let α, β, be the co-ordinates of the given point *any*

where situated in the plane of the parabola, whose equation referred to the same co-ordinates originating in its vertex is

$$y^2 = px \ldots\ldots\ldots\ldots (1)$$

Then the distance required is easily shewn to be

$$\sqrt{(y-\beta)^2 + (x-\alpha)^2} = \text{max.}$$

$$(y-\beta)^2 + (x-\alpha)^2 = \text{max.}$$

$$\text{and } 2dy \,.\, (y-\beta) + 2dx\,(x-\alpha) = 0$$

But by eq. (1) we have

$$\frac{dy}{dx} = \frac{\sqrt{p}}{2\sqrt{x}}$$

$$\therefore \frac{\sqrt{p}}{2} \cdot \frac{y-\beta}{\sqrt{x}} + (x-\alpha) = 0$$

whence and by means of eq. (1), we have

$$x^3 - 2\alpha x^2 + \frac{4\alpha^2 - p^2}{4}\,x + \frac{p\sqrt{p}\cdot\beta}{2}\sqrt{x} - \frac{p}{4}\beta^2 = 0.$$

Let $\beta = 0$.

Then $x^3 - 2\alpha x^2 + \dfrac{4\alpha^2 - p^2}{4}\,x = 0$

$$\therefore x = 0, \text{ or } \alpha \pm \frac{p}{2}$$

which more than demonstrates the problem. See No. 215.

242. Since the equation to the curve is

$$y = (x^m + ax^{m-1})^{\frac{1}{m}} = x \,.\, \left(1 + \frac{a}{x}\right)^{\frac{1}{m}}$$

$$= x + m + \frac{a^2}{2m^2}(1-m)\,x^{-1} + Ax^{-2} + \&c.$$

by the Binomial Theorem.

Hence the straight line whose equation is

$$y = x + \frac{a}{m},$$

and the given curve continually approach as x increases, and touch when $x = \alpha$, i.e., the line

$$y = x + \frac{a}{m}$$

is an asymptote to the curve.

This is Sterling's method (See *Lineæ Tertii*, &c., p. 48,) of determining the asymptote, which ought to be adopted in all elementary treatises as the most simple. The reader may verify the above result by the common method.

243. Let AB $= x$, BC $= y$; then TB $= \frac{ydx}{dy}$, tan. BCA $= \frac{x}{y}$, and tan. TCA $=$ tan. (TCB $-$ ACB) $=$

$$\frac{\text{tan. TCB} - \text{tan. ACB}}{1 + \text{tan. TCB} \times \text{tan. ACB}} = \frac{\frac{dx}{dy} - \frac{x}{y}}{1 + \frac{dx}{dy} \times \frac{x}{y}} = n \text{ tan. BCA}$$

$= n \frac{x}{y}$ by the question.

$$\text{Hence } \frac{dx}{dy} = \frac{(n+1) \cdot \frac{x}{y}}{1 - n \cdot \frac{x^2}{y^2}} \ldots\ldots\ldots (1)$$

and putting $\frac{y}{x} = u$, by substitution and reduction, we get

$$\frac{n}{n+1} \cdot \frac{dx}{x} = - \frac{udu}{1+u^2}$$

$$\therefore l.\, x^{\frac{n}{n+1}} = l.\, c - l.\, \sqrt{1+u^2}$$

$$\text{or } x^{\frac{n}{n+1}} = \frac{c}{\sqrt{1+u^2}} = \frac{cx}{\sqrt{x^2+y^2}}$$

$$\text{which reduces to } y^2 = cx^{\frac{2}{n+1}} - x^2$$

the equation to the curve required.

When $n = 1$, $y^2 = Cx - x^2$, the equation to a circle, $\therefore$ in the $\odot$ $\angle$ ACB $=$ $\angle$ ACT, which may easily be demonstrated geometrically.

244. Supposing the cylindrical vessel open at top, let a be the thickness of its bottom and side; let also m = the given

capacity, and $y + a$ the radius of the base, and $x + a$ the altitude of the vessel.

Then the whole volume V is expressed by

$$V = \pi . (y + a)^2 . (x + a) \dots\dots\dots (1)$$

and the hollow part m by

$$m = \pi y^2 x \dots\dots\dots\dots\dots\dots\dots (2)$$

$\therefore$ the material is

$$V - m = \pi . (y + a)^2 (x + a) - m = \text{min.}$$

$$\therefore (y + a)^2 . (x + a) = \text{min.}$$

and putting its differential $= 0$, we get, after reductions,

$$y = \left(\frac{m}{\pi}\right)^{\frac{1}{3}}$$

by the extracting of which, the value of y, and $\therefore$ of x (by eq. 2) determining the cylinder required, may be found.

245. Cavalerius, in his method of Indivisibles, first shews that if a cylinder, cone, and hemisphere have equal bases, and altitudes, and sections in each be made by planes parallel to their bases, cutting the axis of the cone at a distance from its vertex = those from the centres of the bases of the hemisphere and cylinder, the section of the cylinder = the sum of the sections of the hemisphere and cone. Hence, since the method supposes solids to consist of indivisible plane surfaces, the cylinder = hemisphere + cone.

But cylinder $= 3 \times$ cone.

$$\therefore \text{Hemisphere} = \frac{2}{3} \text{ cylinder}$$

$$\text{and the sphere} = \frac{4}{3} . \text{cylinder}$$

$$= \frac{4}{3} . \pi r^3,$$

r being the radius of the sphere.

246. The equation to the ellipse referred to its centre being

$$\rho = \frac{b}{\sqrt{1-e^2\cos.^2\theta}}$$

where b = conjugate axis, and e the eccentricity, we have

$$\frac{d\rho}{d\theta} = -\frac{be^2\cos.\theta\sin.\theta}{(1-e^2\cos.^2\theta)^{\frac{3}{2}}}.$$

But if φ be the $\angle$ between ρ and the tangent, we have

$$\tan.\varphi = \frac{\rho d\theta}{d\rho} = -\frac{1-e^2\cos.^2\theta}{e^2\sin.\theta\cos.\theta}$$

and tan. ($\angle$ between ρ and normal) $= \tan.(\varphi \pm 90^\circ) = -\cot.\varphi$

$$= \frac{e^2\sin.\theta\cos.\theta}{1-e^2\cos.^2\theta} = \max.$$

$$\therefore \frac{\sin.\theta\cos.\theta}{1-e^2\cos.^2\theta} = \max.$$

and putting the differential = 0, we get

$$\cos.\theta = \pm\frac{1}{\sqrt{2-e^2}} = \pm\frac{a}{\sqrt{a^2+b^2}}.$$

$$\text{and } \therefore \rho = \pm\sqrt{\frac{a^2+b^2}{2}}, \text{ and } \tan.(\varphi\pm 90^\circ) = \frac{a^2-b^2}{2ab},$$

which will determine the points at which are situated the maximum angles required.

247. Let A, B, C; a, b, c, be the angles and opposite sides of the spherical Δ, and suppose A and C constant. Then, since $\sin.c = \frac{\sin.C}{\sin.A}.\sin.a$, $dc = \tan.c.\cot.a da$ (1)

and since $\tan.\frac{A}{2}\tan.\frac{C}{2} = \dfrac{\sin.\frac{a+c-b}{2}}{\sin.\frac{a+c+b}{2}}$

$$db = (da+dc)\frac{\sin.b}{\sin.(a+c)},$$

which by means of equation (1), becomes

$$db = \frac{\sin.b}{\sin.a\cos.c}\times da$$

and supposing da positive or a increasing, db is positive or negative, according as cos. c is, i. e., according as c is $<$ or $>$ quadrant. Q. E. D.

248. This problem is equivalent to the following one, " Required to prove, that if two parabolas of the same parameter, revolve round their axes, which are in the same straight line, thereby generating two paraboloids one within the other, and a plane be drawn touching the interior paraboloid in any point whatever of its surface, this plane will cut off from the exterior paraboloid a constant volume."

Let the cutting plane touch the interior paraboloid in p. (Fig. 80,) and suppose the plane of the generating parabolas QAq to be brought into a situation $\perp$ cutting plane $QP'q$, and intersecting it in Qq; also let $Q'q'$, $P'M'$ be the intersections of a plane $\perp$ axis Aa, with those of QAq, $QP'q$. Then since the planes $Q'P'q'$, $QP'q$ are $\perp$ QAq, $P'M'$ is $\perp$ Qq, $Q'q'$ at their point of intersection M'.

Now $Q'P'q'$, $QP'q$ being the intersection of the planes with the surface, we have $Q'P'q'$ a circle from the nature of the revolution, and $\therefore$ $P'M'^2 = Q'M' \times M'q' = QM' \times M'q \times \frac{p}{P}$ (by property of the parabola) p and P being the principal parameter, and that corresponding to ordinates parallel to Qq.

Hence the section $QP'q$ is an ellipse whose $\frac{1}{2}$ axes $(a, b,)$ are expressed by

$$a = \frac{Qq}{2}, b = a. \sqrt{\frac{p}{P}}$$

and $\therefore$ its area by

$$A = \pi . ab = \pi \left(\frac{Qq}{2}\right)^2 \sqrt{\frac{p}{P}}.$$

But, drawing pP parallel to Aa, and putting $Aa = m$, we readily prove that $Pp = m$, and Qq (parallel to tangent at P) is bisected in p. $\therefore$

$$A = \pi (Qp)^2 . \sqrt{\frac{p}{P}} = \pi . P \times (Pp) \sqrt{\frac{p}{P}}$$

$$= m \pi \sqrt{p + P} \dots\dots\dots\dots (1)$$

Again, supposing the segment (V) of the paraboloid to be generated by the motion of the variable ellipse parallel to itself, we have

$$dV' = d . (PN) \times \pi m' \sqrt{pP}$$

$$= \cos. pPN \times \pi \sqrt{pP} . m'dm'$$

$$\therefore V' = \frac{\pi \sqrt{p \times P}}{2} . \cos. pPN . m'^2$$

V' being the segment corresponding to the variable value m' of m.

$$\therefore V = \frac{\pi \sqrt{p \times P}}{2} \cos. pPN . m^2 \dots\dots (2)$$

$$\text{But since } \cos. pPN = \frac{PN}{m}$$

$$\text{and } P = \frac{a^2}{m}$$

$$\therefore V = \frac{\pi m^{\frac{1}{2}} \times \sqrt{p}}{2} . a \times PN \dots\dots\dots (3)$$

and since, as it is well known, any segment (which may be proved very elegantly by the method of exhaustions according to Archimedes) of a parabola $= \frac{2}{3}$ of its circumscribing parallelogram, we have

$$a \times PN = \sqrt{pm} \times m = m^{\frac{3}{2}} \sqrt{p}$$

$$\therefore V = \frac{\pi p m^3}{2} \dots\dots\dots\dots\dots (4)$$

which is a constant quantity.

249. Let the radii of the successive circles be r_1, r_2, r_3,.... C_1, C_2, C_3, ... the areas of the ⊙s, and T_1, T_2, ... the areas of their inscribed Δ:

Then it easily appears on drawing the figure, that

$$r_2 = r_1 \sin. 30 = \frac{r_1}{2}$$

$$r_3 = r_2 \sin. 30 = \frac{r_2}{2} = \frac{r_1}{2^2}$$

&c. = &c.

$$r_n = \frac{r_1}{2^{n-1}}$$

$$T_1 = \frac{3\sqrt{3}}{4} r_1^2$$

$$T_2 = \frac{3\sqrt{3}}{4} . r_2^2 = \frac{3\sqrt{3}}{4} . \frac{r_1^2}{2^2}$$

$$T_3 = \frac{3\sqrt{3}}{4} r_3^2 = \frac{3\sqrt{3}}{4} . \frac{r_1^2}{2^2 \times 2}$$

&c. = &c.

$$T_n = \frac{3\sqrt{3}}{4} . \frac{r_1^2}{2^{2 . (n-1)}}.$$

Hence $C_1 + C_2 + C_3 + \ldots\ldots \infty = \pi . (r_1^2 + r_2^2 + \&c.)$

$$= \pi r_1^2 \left(1 + \frac{1}{2^2} + \frac{1}{2^4} + \frac{1}{2^6} + \&c.\right)$$

$$= \pi r_1^2 \times \frac{1}{1 - \frac{1}{2^2}} = \frac{4\pi r_1^2}{3}, \text{ and}$$

$$T_1 + T_2 + \ldots \infty = \frac{3\sqrt{3}}{4} r_1^2 \left(1 + \frac{1}{2^2} + \frac{1}{2^4} + \&c.\right)$$

$$= \frac{3\sqrt{3}}{4} r_1^2 . \frac{1}{1 - \frac{1}{2^2}}$$

$$= \frac{3\sqrt{3}}{3} r_1^2 = r_1^2 \sqrt{3}.$$

250. It may easily be shewn, that any two opposite faces of the parallelopiped inscribed in a sphere, are parallelograms, inscribed in equal and parallel circles of the sphere; and it is evident, that the greater these faces are for the same distance between them, the greater will be the parallelopiped. But the square is the greatest parallelogram inscriptible in a circle. Hence putting

$y = \frac{1}{2}$ diagonal of the square face, and $x =$ its distance from the centre of the sphere, we very readily have

$$V = 2x \times 2y^2 = 4y^2x = max.$$

$$\therefore 2xydy + y^2dx = 0, \text{ or } \frac{dy}{dx} = -\frac{y}{2x}$$

$$\text{But } y = \sqrt{r^2 - x^2} \quad \therefore \frac{dy}{dx} = -\frac{x}{y}$$

$$\therefore \frac{y}{2x} = \frac{x}{y}, \text{ or } y^2 = 2x^2$$

$$\text{and } y = x\sqrt{2} = \sqrt{r^2 - x^2}$$

$$\text{whence } x = \frac{r}{\sqrt{3}},$$

which shews that the faces are all equal, and they have been shewn to be at right $\angle$ to one another. Hence the parallelopiped required is a cube, whose side is

$$2x = \frac{2r}{\sqrt{3}}, \text{ and}$$

$$\text{volume} = \frac{8r^3}{3\sqrt{3}}.$$

251. Let α be the distance of the given point from the centre of the ellipse, whose equation referred to the centre along the conjugate axis is

$$y^2 = \frac{a^2}{b^2}.(b^2 - x^2),$$

then, by the question, we have

$$(\alpha + x)^2 + y^2 = max.$$

$$\therefore (\alpha + x)^2 + \frac{a^2}{b^2}.(b^2 - x^2) = max.$$

$$\therefore 2(\alpha + x)dx - 2\frac{a^2}{b^2}xdx = 0$$

$$\text{which gives } x = \frac{\alpha b^2}{a^2 - b^2}.$$

whence y may be found, and the maximum required.

252. Let π = circumference of a circle whose diameter is (1) = 3.14159, &c., which may be found *approximately* many ways, and suppose θ° the required arc of the given circle = its radius (r). Then by the question,

$$2r\pi : 360^\circ :: r : \theta^\circ$$

$$\therefore \theta^\circ = \frac{360^\circ}{2\pi} = \frac{180^\circ}{(3.1416)}$$

$$= 57^\circ . 17' . 44'' \text{ nearly.}$$

253. Let a, b be the axes of the ellipse a', b' its conjugate diameters, and θ the $\angle$ between them. Then since, by property of the ellipse,

$$a'b' \sin. \theta = ab$$

$$\therefore \sin. \theta = \frac{ab}{a'b'} = \text{min.}$$

$$\therefore a'b' = \text{max.}$$

and $b'da' + a'db' = 0$, or $\dfrac{da'}{db'} = -\dfrac{a'}{b'}$.

But $a'^2 + b'^2 = a^2 + b^2$

$$\therefore \frac{da'}{db'} = -\frac{b'}{a'}$$

Hence $b' = a'$, and $\sin. \theta = \dfrac{ab}{a'^2} = \dfrac{2ab}{a^2 + b^2}$, which determines the angle required.

254. Let the given surface of the cone be Q^2, x the altitude and y the radius of the base. Then we have

$Q^2 = \dfrac{1}{2}$ slant side $\times$ perimeter of the base,

or $Q^2 = \pi y . \sqrt{x^2+y^2}$..... (1)

and $\pi y^2 \times \dfrac{x}{3} = \text{max.}$

$$\therefore 2\pi y dy . \frac{x}{3} + \frac{\pi y^2}{3} . dx = 0, \text{ or } \frac{dy}{dx} = -\frac{y}{2x}$$

Also from eq. (1) we easily get

$$\frac{dy}{dx} = -\frac{yx}{2y^2+x^2} = \frac{-y}{2x}.$$

Hence $y = \frac{x}{\sqrt{2}}$, and substituting in (1) we have

$$x = Q\sqrt{\frac{2}{\pi\sqrt{3}}}$$

$$\text{and } \therefore y = Q\sqrt{\frac{1}{\pi\sqrt{3}}}$$

which determine the form of the cone required.

MECHANICS.

STATICS.

255. Since the bodies (A,B) move at the same rate, and from the same extremity of the diameter of the circle, they will always be in the same straight line $\perp$ diameter. Hence, x, y being the co-ordinates measured from the centre of the circle of the required path, and y' the ordinate of the circle and a its radius, we have

$$A : B :: y' + y : y' - y$$

$$\therefore y = \frac{A - B}{A + B} \cdot y' = \frac{A - B}{A + B} \cdot \sqrt{a^2 - x^2}$$

$$= \frac{\frac{A - B}{A + B} a}{a} \cdot \sqrt{a^2 - x^2}$$

or the required curve is an *ellipse*, whose semi-axes are a and $\frac{A - B}{A + B} a$.

256. Let v, v' be the uniform velocities of the equal bodies. Then since their centre of gravity is the middle point between them, if s, s' and g be the spaces described by each of the bodies, and the centre of gravity respectively, we have

$$\left.\begin{aligned} s^2 + s'^2 &= 2g^2 + 2m^2 \\ \text{and } \cos. 120^\circ &= \frac{s^2 + s'^2 - 4m^2}{2ss'} \end{aligned}\right\} .$$

$2m$ being the distance between the bodies.

Hence eliminating m, and putting $-\frac{1}{2}$ for cos. 120°, we obtain

$$s^2 + s'^2 - ss' = 4g^2$$

But $s : s' :: v : v'$

$$\therefore s^2\left(1 + \frac{v'^2}{v^2} - \frac{v'}{v}\right) = 4g^2$$

$$\text{and } s'^2\left(1 + \frac{v^2}{v'^2} - \frac{v}{v'}\right) = 4g^2$$

$$\therefore \frac{s}{g} = \frac{2}{\sqrt{1 + \frac{v'^2}{v^2} - \frac{v'}{v}}} \text{ and } \frac{s'}{g} = \frac{2}{\sqrt{1 + \frac{v^2}{v'^2} - \frac{v}{v'}}}$$

the relations required.

257. The centre of gravity of the bodies at the base of the Δ is the middle point between them; and supposing the sum of them placed in that point, the common centre of gravity of the three equal bodies will divide the line drawn from it to the vertex in the ratio of $1 : 2$. Hence $2a$ being the base, since the angle at the vertex is a right $\angle$, it easily appears that the centre of gravity required is distant from the vertex by $\frac{2a}{3}$.

258. Let the body A be fixed, and B be moveable. Then (x, y), (x', y') being the co-ordinates of the centre of gravity, and the body B measured from A along the diameter $(2r)$ it readily appears that

$$y : y' :: x : x' :: B : A + B$$

$$\text{But } x^2 : y^2 :: x'^2 : y'^2 = 2rx' - x'^2$$

$$:: x' : 2r - x'$$

$$\therefore x^2 : x^2 + y^2 :: x' : 2r :: x . \frac{A + B}{B} : 2r$$

Hence

$$y^2 = \frac{2 . rB}{A + B} x - x^2,$$

or the curve required is a circle whose radius is $\frac{rB}{A + B}$.

259. Let p be the perpendicular distance between the two sides (a, b) x, y the parts of it made by the centre of gravity of the trapezium.

Then dividing the trapezium into two $\triangle$ (A, B) whose bases are (a, b), and finding their centres of gravity which are known to be distant from their bases by $\frac{1}{3}p$, we have, supposing the material of the $\triangle$ and trapezium collected in their centres of gravity,

$$x \times (A + B) = A \cdot \frac{p}{3} + B \cdot \frac{2p}{3}$$

$$y \times (A + B) = A \cdot \frac{2p}{3} + B \cdot \frac{p}{3}$$

But $A = \frac{pa}{2}$, $B = \frac{pb}{2}$.

$\therefore\ x = \frac{p}{3} \cdot \frac{a + 2b}{a+b}$, $y = \frac{p}{3} \cdot \frac{b + 2a}{a+b}$, which give the proportion required.

260. Let a be the length of the bar; and A its whole mass; then supposing the bar prismatic, its mass $\propto$ length $\times$ density $\propto x \times x^n \propto x^{n+1}$; $\therefore$ the mass M of the length s is expressed by

$$M = \frac{Ax^{n+1}}{a^{n+1}}$$

$$\therefore dM = \frac{(n+1)A}{a^{n+1}} x^n dx$$

and the distance X of the centre of gravity of the length x from its origin is

$$X = \frac{\int x dM}{M} = \frac{n+1}{n+2} x.$$

Let $x = a$.

Then $X = \frac{n+1}{n+2} a$. Q. E. I.

261. Let $a, b, -c$ be the distances of A, B, C, from the

plane, and x that of their common centre of gravity. Then

$$x \times (A + B + C) = Aa + Bb - Cc$$

$$\text{and } x = \frac{Aa + Bb - Cc}{A + B + C}$$

x being positive 0, or negative, according as Cc is $<$ $=$ or $>$ $Aa + Bb$.

262. Let $QP'q$, (Fig. 80,) be the cutting plane, and bring the generating parabola into such a position Qaq that its plane shall be $\perp$ plane $QP'q$. Then the section of the cutting plane with the surface of the paraboloid will be an ellipse whose semi-axes (a, b) are Qp, and $\sqrt{p \times Pp}$, p being the middle point of Qq, Pp parallel to the axis Am, and p the latus rectum of the parabola. Hence putting the parameter of the diameter $Pp = P$, PN ($\perp$ to Qq) $= x$, and $\angle PpN = \alpha$, the area of the ellipse $QP'q = \pi ab = \pi a \sqrt{p \times Pp} = \pi \sqrt{P . Pp} \times \sqrt{p . Pp}$ $= \pi \sqrt{P . p} \times Pp = \dfrac{\pi \sqrt{P . p}}{\sin. \alpha} \times x$.

Now supposing the segment of the paraboloid QPq generated by the ellipse moving parallel to itself up to P, we have its volume V expressed by

$$V = \frac{\pi \sqrt{P . p}}{\sin. \alpha} . \int x dx$$

$\therefore$ the distance X of its centre of gravity from a line $\perp$ PN and passing through P is

$$X = \frac{\int x dV}{V} = \frac{2x^3}{3x^2} = \frac{2}{3} x$$

$\therefore$ the centre of gravity is in the straight line drawn $\perp$ PN and cutting off a third part from PN, Pp towards Qq. But it is also in Pp ($\because$ Pp bisects every diameter of the generating ellipse). It is $\therefore$ in their intersection, or it is in Pp distant from p by $\dfrac{1}{3} Pp$.

Hence the position of the centre of gravity (G) of the segment of the paraboloid is determined by bisecting Qq in p, drawing pP parallel to the axis Am, and taking $pG = \dfrac{1}{3} Pp$.

263. By the question, we have

$$\frac{\int yxdx}{\int ydx} = \frac{x}{n}$$

$$\therefore (n-1)\,xy = \int ydx$$

and differentiating again, &c.

$$\frac{n-2}{n-1} \cdot \frac{dx}{x} = -\frac{dy}{y}$$

$$\therefore y = c \,.\, x^{\frac{2-n}{n-1}}$$

Now the distance of the centre of gravity of a solid of revolution from the vertex is

$$\frac{\int xy^2dx}{\int y^2dx}$$

$\therefore$ the distance required is

$$D = \frac{\int x^{\frac{4-2n}{n-1}+1}dx}{\int x^{\frac{4-2n}{n-1}}dx} = \frac{\frac{n-1}{2}\,x^{\frac{2}{n-1}}}{\frac{n-1}{3-n}\,x^{\frac{3-n}{n-1}}} = \frac{3-n}{2}\,x$$

Let $n = \frac{5}{3}$. Then $y = cx^{\frac{1}{2}}$; or the curve is the common parabola; and $D = \frac{2}{3}\,x$ for the paraboloid.

264. Let the edges AB, CD, (Fig. 81,) of the pyramid which do not meet be bisected in P, Q. Then P, Q, being joined and bisected in G, G will be the centre of gravity of the pyramid.

For, joining AG and producing it to meet Bn drawn parallel to PQ, and intersecting BQ in g, we have, by similar $\triangle$,

$$Qg : Bg :: QG : Bn :: PG : Bn$$
$$:: AP : AB :: 1 : 2$$

$\therefore g$ is the centre of gravity of the $\triangle$ BCD; and similarly it may be proved (if BG be joined and produced to meet AQ in g') that g' is the centre of gravity of the $\triangle$ ACD.

Now conceiving the pyramid to consist of $\triangle$ parallel to BCD, each $\triangle$ will balance about Ag, because it passes through their

centres of gravity; $\therefore$ the centre of gravity of the pyramid is in Ag; for the like reason it is also in Bg'; it is $\therefore$ in their point of intersection G, or it is in the middle point of the line joining the middle points of any two edges not meeting, of the pyramid.

265. Upon the diameter of the circle which passes through (S) let fall $\perp$ s from A, B, C, &c., and let their distances from the common centre of gravity (G) be denoted by a, b, c, &c.

Then joining AS, AG, we have

$$SA^2 = AG^2 + R^2 + 2Ra$$

$$\text{and } A \times SA^2 = A \,.\, AG^2 + A \,.\, R^2 + 2R \times aA,$$

Similarly,

$$B \times SB^2 = B \,.\, BG^2 + B \,.\, R^2 + 2R \times b \,.\, B$$

$$C \times SC^2 = C \times CG^2 + C \,.\, R^2 + 2R \times c \,.\, C$$

$$\&c. = \&c.$$

$$\therefore A \times SA^2 + B \times SB^2 + C \times SC^2 + \&c., = A \times AG^2 + B \times BG^2 + C \times CG^2 + \&c.$$

$$+ R^2 \times (A + B + C + \&c.)$$

$$+ 2R \times (A \,.\, a + B \,.\, b + C \,.\, c + \&c.)$$

$$\text{But } A \,.\, a + B \,.\, b + C \,.\, c + \&c. = 0.$$

$$\therefore A \times SA^2 + B \times SB^2 + \&c. = A \times AG^2$$

$$+ B \times BG^2 + C \times CG^2 + \&c. +$$

$$R^2 \times (A + B + C + \&c.)$$

which is constant when R is.

266. The equation to a spherical surface, the co-ordinates originating at the centre, is

$$x^2 + y^2 + z^2 = r^2.$$

Hence the distance (Y) of the centre of gravity from the plane of $(y, z,)$ (See Whewell's *Mech.* p. 101,) is

$$Y = \frac{\iint zxdxdy}{\iint zdxdy} = \frac{\int dy \int xdx \sqrt{r^2 - y^2 - x^2}}{\iint zdxdy}.$$

But $\int xdx \sqrt{r^2 - y^2 - x^2} = \dfrac{1}{3}(r^2 - y^2)^{\frac{3}{2}}$ taken between the

limits of $x = 0$, and $x = \sqrt{r^2 - y^2}$, and, by the question, $\iint z\,dx\,dy$ $= \frac{1}{8}$ volume of sphere $= \frac{\pi r^3}{6}$.

$$\therefore Y = \frac{2\int dy\,(r^2 - y^2)^{\frac{3}{2}}}{\pi r^3}.$$

But $\int dy\,(r^2 - y^2)^{\frac{3}{2}} = \left(\frac{\sqrt{r^2 - y^2}}{4} + \frac{3r^2}{8}\right) . y\sqrt{r^2 - y^2} +$ $\frac{3r^4}{8}\sin.^{-1} . \frac{y}{r} = \frac{3r^4\pi}{16}$ when $y = r$.

Hence, for the whole $\left(\frac{1}{8}\text{ sphere}\right)$

$$Y = \frac{3}{8}r.$$

Now the $\left(\frac{1}{8}\text{ sphere}\right)$ being symmetrical with respect to the three co-ordinate planes, it is evident its centre of gravity must be equally distant from each of them.

Hence may be determined the actual position of the centre of gravity of the proposed solid, which is the intersection of the three planes drawn parallel to the co-ordinate planes of (xy), (yz), (xz), and distant from them by the same interval $\frac{3}{8}r$. Its distance from the centre of the sphere is

$$\frac{3\sqrt{3}}{8} . r.$$

On this subject the reader will find much useful and elegant matter in the work above cited. In page 91, lines 7, 8, two errors of calculation having fallen under our observation, we notice them, with the view of facilitating the perusal.

267. The equation to a cycloid being

$$y = \sqrt{2rx - x^2} + \text{vers.}^{-r}\,x$$

we have the distance of the centre of gravity of its arc s from the axis of y originating in the vertex expressed by

$$Y = \frac{\int x ds}{s} = \frac{\int x \times \sqrt{2r} \cdot \frac{dx}{\sqrt{x}}}{\int \sqrt{2r} \cdot \frac{dx}{\sqrt{x}}}$$

$$= \frac{\frac{2\sqrt{2r}}{3} \cdot x^{\frac{3}{2}}}{2\sqrt{2r} \cdot \sqrt{x}} = \frac{x}{3};$$

and its distance from the axis of x by

$$X = \frac{\int y ds}{s} = \frac{\sqrt{2r} \times \int \sqrt{2r - x} \cdot dx + \sqrt{2r} \int \frac{dx}{\sqrt{x}} \text{vers.}^{-r} x}{2\sqrt{2r} \cdot \sqrt{x}}$$

$$= \frac{-\frac{1}{3} \cdot (2r - x)^{\frac{3}{2}} + 2r\sqrt{2r - x} + \sqrt{x} \cdot \text{vers.}^{-r} x - \frac{8}{3} \frac{r^2}{\sqrt{2r}}}{\sqrt{x}}$$

$$= \sqrt{2r - x} \cdot \frac{4r + x}{3\sqrt{x}} - \frac{8r^{\frac{3}{2}}}{3 \cdot \sqrt{2}} \cdot \frac{1}{\sqrt{x}} + \text{vers.}^{-r} x$$

X and Y determining the position of the centre of gravity (G) of an arc (s) of the cycloid measured from the vertex wholly on one side of the axis of x.

If in like manner X′, Y′, be found determining the centre of gravity (G′) of an arc (s') on the other side of the axis, their common centre of gravity, or the centre of gravity of the arc $s + s'$ may be found by dividing the line joining G, G′, in the proportion of g' to g. Hence it appears the centre of gravity of any portion of the arc of a cycloid may be found.

If $s = s'$, the centre of gravity will be in the axis distant from the vertex by $\frac{x}{3}$.

268. This problem is reducible to "If straight lines PQ (Fig. 82,) be made to cut off from two sides AC, BC, of a Δ, segments PA, QB, whose sum = a constant quantity (m) and PQ be divided in G so that

PG : GQ :: BC : AC

then the locus of G is a straight line parallel to the base AB; which admits the following proof:

Let fall Pp, Gg, Qq, $\perp$ AB, and produce QP, BA, to meet in R. Then from similar $\triangle$, we get

$$Qq : Pp :: PQ + PR : PR$$
$$\therefore Qq - Pp : Pp :: PQ : PR, \text{ similarly}$$
$$Pp : Gg - Pp :: PR : PG$$

and componendo

$$Qq - Pp : Gg - Pp :: PQ : PG$$
$$\therefore Gg = \frac{PG}{PQ} \cdot (Qq - Pp) + Pp.$$

But $Qq = QB \,.\, \sin. B$, $Pp = AP \,.\, \sin. A$

$$QB + AP = m$$

and $PG : PQ :: \sin. A : \sin. A + \sin. B$.

Hence, by substituting, we get

$$Gg = \frac{m \sin. A \,.\, \sin. B}{\sin. A + \sin. B}$$

a constant quantity.

$\therefore$ the locus of G is a straight line parallel to AB.

If the question be considered mechanically; then since P and Q are in stable equilibrium in every position, their centre of gravity is the lowest point possible. Hence whatever may be the position of P, Q, their centre of gravity G is at the same distance from the horizontal base AB; i. e., the locus of the centre of gravity is a straight line parallel to the horizon.

269. The spherical sector may be divided into a cone, whose vertex is the centre of the sphere, and a spherical segment, having the same circular base.

Let a, a', be the altitudes of the cone and spherical segment, r the radius of the sphere. Then (p. 199.) the distance (g) of the centre of gravity of the cone from the centre of its base is

$$g = \frac{1}{4} a \; \ldots\ldots\ldots\ldots (1)$$

and the distance (g') of the spherical segment from the same centre is

$$g' = a' - \frac{\int y^2 x dx}{\int y^2 dx} = a' - \frac{a'}{4} \cdot \frac{8r - 3a'}{3r - a'}.$$

But $a' = r - a$; $\therefore$ by substitution and reduction, we have

$$g' = \frac{r-a}{4} \cdot \frac{3r+a}{2r+a} \ldots\ldots\ldots (2)$$

Hence the distance between these centres of gravity is

$$g + g' = \frac{a}{4} + \frac{r-a}{4} \cdot \frac{3r+a}{2r+a} = \frac{3r^2}{4 \cdot (2r+a)}.$$

Again, V, V′ denoting the volumes of the cone and segment, we have

$$V = \frac{1}{3} \text{ base} \times \text{alt.} = \frac{\pi}{3} a \sqrt{r^2 - a^2}, \text{ and } V' = \int \pi y^2 dx$$

$$= \frac{\pi}{3} \cdot (r-a)^2 \cdot (2r+a).$$

$\therefore$ calling G, G′ the distances of their common centre of gravity from g, g', we have

$$G : G' :: V' : V$$

$$:: (r-a)^2 \cdot (2r+a) : a\sqrt{r^2 - a^2}$$

$$\text{But } G + G' = g + g' = \frac{3r^2}{4 \cdot (2r+a.)}$$

$$\therefore G = \frac{(r-a)^2 \cdot (2r+a)}{a\sqrt{r^2-a^2}} \times \left\{\frac{3r^2}{4 \cdot (2r+a)} - G\right\}$$

whence

$$G = \frac{3r^2 \cdot (r-a)^{\frac{3}{2}}}{4\{a\sqrt{r+a} + (r-a)^{\frac{3}{2}}(2r+a)\}}$$

$\therefore$ the distance (D) of the centre of gravity of the spherical sector from the centre of the sphere, $D = \frac{3}{4} a + G$, is known.

Let $a = 0$, or the sector be a hemisphere, then

$$D = \frac{3r}{8}.$$

EQUILIBRIUM IN GENERAL.

270. Let n be the number of equal sides of the prism, and θ the inclination required of the plane to the horizon.

Then since (supposing the prism to be prevented from slipping, by friction or any other cause,) it will roll or not, according as the $\perp$ to the horizon, drawn from its centre of gravity, falls upon the base, or the base produced. Consequently, the limit of equilibrium will correspond to that position of the $\perp$ which passes through an $\angle$ at the base, and the centre of gravity. Now the centre of gravity being equally distant from each of the angles, it easily appears that the $\perp$ bisects the $\angle$ at the base. Hence

$$\theta = R - \frac{1}{2} \angle \text{ at the base,}$$

R being a right $\angle$.

But since the n angles of the prism are equal, each of them $= \frac{2n-4}{n} . R.$

$$\therefore \theta = R - \frac{n-2}{n} R = \frac{2}{n} . R.$$

271. Let the apparent weights be w, w', and the true one x. Then a, b being the corresponding lengths of the arms of the steelyard, we have

$$ax = bw$$

$$bx = aw' = \frac{bww'}{x}$$

$$\therefore x = \sqrt{w\,w'}.$$

Again $\frac{1}{2}(w + w') > \sqrt{w\,w'}$

if $w^2 + 2ww' + w'^2 > 4ww'$ or

if $w^2 - 2ww' + w'^2 > 0$, or

if $(w - w')^2 > 0$, or

if w, w' be unequal, which being the case by hypoth. x is $< \frac{w + w'}{2}$.

272. If θ be the $\angle$ formed by the string at any tack, and W one of the equal weights appended to the string, it is known the pressure (P) on the tack is expressed by

$$P = 2W \cdot \cos. \frac{\theta}{2} \ldots\ldots\ldots (1)$$

Now since in the problem the $\angle$ at the vertex of the isosceles Δ is $= 120°$, those at the base must each $= 30°$, and $\therefore$ the angles at the three tacks, viz., θ, θ', θ'', are each $120°$, or are equal. Consequently, the equat. (1), the pressures P, P,' P'', are equal.

Also $P = 2W \times \cos. 60° = W$.

273. Let the weight of the lever, (which may be any figure of which we can find the centre of gravity) $= w$, a its length, and x the distance of the fulcrum from the end, to which w is attached, its distance from the centre of gravity.

Then, supposing the weight of the lever placed in its centre of gravity, we have

$$w \times y = w \times x$$
$$\therefore y = x$$

But $x + y =$ distance of centre of gravity from the end $= m$, a given quantity.

$$\therefore 2x = m, \text{ and } x = \frac{m}{2}$$

which determines the position of the fulcrum.

274. Since the sphere is supported, the forces which conspire to effect the equilibrium must tend to the same point. Hence

supposing the weight (W) of the sphere placed in its centre of gravity, i. e., in its centre, the force of gravity acting from that point, the other forces, viz., the reactions R, R′, ⊥ planes at the points of their contact with the sphere, must also tend to the centre of the sphere.

Hence, putting the inclinations of the planes to the horizon $= \theta, \theta'$, and forming a $\triangle$ whose sides are parallel to the directions of the forces, we easily get

$$R : W :: \sin. \theta' : \sin. (\theta + \theta')$$
$$R' : W :: \sin. \theta : \sin. (\theta + \theta')$$

and resolving R, R′ into their vertical and horizontal components (V, H), (V′, H′), we have

$$V = R . \cos. \theta, H = R \sin. \theta$$
$$V' = R' \cos. \theta', H' = R' \sin. \theta'$$
$$\left.\begin{array}{l} \therefore V : W . \cos. \theta :: \sin. \theta' : \sin. (\theta + \theta') \\ V' : W . \cos. \theta' :: \sin. \theta : \sin. (\theta + \theta') \end{array}\right\}$$
$$\left.\begin{array}{l} H : W . \sin. \theta :: \sin. \theta' : \sin. (\theta + \theta') \\ -H' : W . \sin. \theta :: \sin. \theta : \sin. (\theta + \theta') \end{array}\right\}$$

In the problem, $\theta = 60^\circ$, and $\theta' = 30^\circ$.

$$\left.\begin{array}{l} \therefore V : W :: 1 : 4 \\ V' : W :: 3 : 4 \end{array}\right\} \text{and } V : V' :: 1 : 3,$$ the analogies required.

275. Let AW be produced to meet BC in (a), and let a prop be placed at (a) instead of the two props at B, C. Then denoting the pressures on the props at A, a, by A, a, we have

$$A : a :: Wa : WA$$
$$\therefore A : A + a :: Wa : Aa$$
$$:: \triangle BWC : \triangle BAC$$
$$\text{But } A + a = W.$$
$$\therefore A : W :: \triangle BWC : \triangle BAC.$$

and in the same manner it may be shewn that

$$B : W :: \triangle WAC : \triangle ABC$$
$$C : W :: \triangle WBA : \triangle ABC$$

and $\therefore$

$$A : B : C :: \triangle BWC : \triangle AWC : \triangle AWB$$

For the general problem, of which this is a particular case, the reader is referred to *Whewell's Mechanics*, p. 52.

276. Since the beam is at rest, the forces which effect its repose, viz., the reactions of the planes at the extremities of the beam, acting in directions $\perp$ planes, and the weight of the beam at its centre of gravity acting $\perp$ horizon, must tend to the same points.

Hence θ, θ' being the inclinations of the planes to the horizon, (a) the length of the beam, m, m' the distances of its centre of gravity from the extremities leaning on the planes, of which θ, θ' are the inclinations, and ϕ the inclination of the beam to the horizon, we easily get (p, p' being the corresponding perpendiculars,) $(m + m')^2 = p^2 + p'^2 - 2pp' \cdot \cos.(\theta + \theta')$.

But from similar $\triangle$ it appears that

$$p = \frac{m \cos. \phi}{\sin. \theta}, \; p' = \frac{m' \cos. \phi}{\sin. \theta'}$$

Hence, substituting, &c., we have

$$\cos.^2 \phi = \frac{(m + m')^2 \sin.^2 \theta \sin.^2 \theta'}{m^2 \sin.^2 \theta' + m'^2 \sin.^2 \theta - 2mm' \sin. \theta \sin. \theta' \cos.(\theta + \theta')}$$

which gives the two positions required.

If the beam be prismatic and of uniform density, or $m = m'$, and moreover if $\theta = \theta'$, then $\cos.^2 \phi = 1$, and $\cos. \phi = \pm 1$, $\phi = 0$, or π, which shew that the beam is then horizontal.

The same principle will also serve to find the position of equilibrium of a beam supported by two given curve surfaces.

277. The forces which produce the equilibrium, viz., the man's weight (W), and the reactions of the beam and swing R, R′, being supposed to act from the man's centre of gravity in the directions vertical, and along the rod and string, if a $\triangle$ be formed by lines drawn parallel to those directions, its sides will represent those forces.

Now, making the inclination of the beam or lever to the vertical $= \alpha$, that of the string in the given position $= \beta$; and the

given lengths of the string and rod $= a, b$, we have the inclination (θ) of the rod to the vertical, expressed by

$$\theta = \sin.^{-1}\left(\frac{a}{b}\sin.\overline{\alpha+\beta}\right) - \alpha \ldots\ldots (1)$$

Hence

$$\begin{aligned} R &: W :: \sin.\beta : \sin.(\theta - \beta) \\ W &: R' :: \sin.(\theta - \beta) : \sin.\theta \\ \text{and } \therefore R &: R' :: \sin.\beta : \sin.\theta, \end{aligned}$$

which, by means of equation (1), gives the values of R, R′, &c.

278. Let a, b, be the lengths of the lever, and α the angle between them; also let θ be the inclination of a to the verticle. Then the weights P and Q hanging from the extremities of a, b, being in equilibrium, we easily get

$$\begin{aligned} P \times a \sin.\theta &= Q \times b \quad \sin.(\alpha - \theta) \\ &= Qb(\sin.\alpha \quad \cos.\theta - \cos.\alpha \sin.\theta) \end{aligned}$$

whence

$$\cot.\theta = \frac{a}{b} \div \frac{P}{Q} \cdot \frac{1}{\sin.\alpha} + \cot.\alpha.$$

279. Since the tension of the string is every where the same, the weight is supported by two *equal* forces, and $\therefore$ the direction in which the weight acts, (viz., the vertical passing through its centre of gravity,) bisects the $\angle$ between the parts of the string. Hence a being the distance of the tacks, b the length of the string, α its inclination to the vertical, and θ the inclination of either part of the string to the vertical, we readily obtain

$$\sin.\theta = \frac{a}{b}.\sin.\alpha.$$

and drawing lines from the tacks making the $\angle\,\theta$, thus determined, with the vertical, their intersection will give the required position of the weigh

280. Since the tension of the string is independent of its

length, the power P is the very same as would be required to support the weight W kept in equilibrium by the string being fixed to a tack in a manner such that the parts of the string may be parallel.

Hence

$$P = \frac{W}{2}.$$

281. Let ABCD (Fig. 81.) be the pyramid, and G its centre of gravity; the forces acting upon G represented in intensity and direction by GA, GB, GC, GD, will counteract each other, or keep a body at G at rest.

For producing AG to meet the plane BCD in g, and resolving the forces GB, GC, GD, into Gg, gB; Gg, gC; Gg, gD, they are equivalent to

$$3Gg + gB + gC + gD.$$

But g being the centre of gravity of the $\triangle$ BCD, it is easily shewn that $gB + gC + gD = 0$, and we know that $GA = 3Gg$. Hence the forces counteract each other, &c.

282. Let w be the weight of any part s $(=\int\sqrt{dx^2 + dy^2})$ of the string measured from its lowest point by the vertical and horizontal co-ordinates (x, y), D the density at the point (x, y), and a the tension of the string at the lowest point.

Then, the equilibrium being produced by the tensions in the directions of tangents at the extremities of s, and its weight acting vertically, we have

$$a : w :: dy : dx.$$

$$\text{Now } D = \frac{dw}{ds}.$$

But $dw = \dfrac{a d^2 x}{dy}$, supposing dy constant.

$$\therefore \ D = \frac{ad^2x}{dy\,ds} = \frac{ad^2x}{dy^2 . \sqrt{1 + \frac{dx^2}{dy^2}}}.$$

which gives the law of density necessary to produce a *catenary* of any required form.

In the problem, it being required to form a cycloid, we have

$$y = \sqrt{2rx - x^2} + \text{vers.}^{-r}x.$$

$$\therefore \ \frac{dx}{dy} = \sqrt{\frac{x}{2r - x}}$$

$$\text{and } \frac{d^2x}{dy} = \frac{rdx}{\sqrt{x}(2r - x)^{\frac{3}{2}}}$$

$$\text{and } \frac{ds}{dx} = \sqrt{\frac{2r}{x}}$$

$$\therefore \ D = \frac{ar}{\sqrt{x}(2r - x)^{\frac{3}{2}}} \times \sqrt{\frac{x}{2r}} = \frac{a\sqrt{r}.}{\sqrt{2}.(2r - x)^{\frac{3}{2}}}$$

$$\therefore \ D \propto \frac{1}{(2r - x)^{\frac{3}{2}}},$$

$$\text{and } w \propto \sqrt{\frac{x}{2r - x}}. \quad \text{Q. E. I.}$$

283. The resistance of each particle of the vertical section AF to fracture or separation from those in contact with it, will be in a direction $\perp$ AF. Hence, supposing AFB a lever whose arms are AF, FCB, and fulcrum F, and P a weight just sufficient to counteract the sum of the resistances of the particles AF, and putting AB $= x$, FA $= y$, the distance of any one of those particles from F $= y'$, and the resistance of each $= m$, (which varies with the nature of the material of which the beam is composed) we have

$$P \times x = \text{sum of all the } m \times y'$$

$$= (\text{sum of the } m) \times \text{dist. of the centre of gravity}$$

of AF from F $= my \cdot \times \dfrac{y}{2} = \dfrac{my^2}{2}$.

$$\therefore y^2 = \frac{2P}{m} \cdot x$$

or the curve required is a common parabola, whose latus rectum is $\frac{2P}{m}$.

284. Let (x', y'), (x, y) be the vertical and horizontal coordinates of a vertical section of the extrados and intrados, originating in their vertices. Then supposing the particles of matter lying between the extrados and intrados in the vertical to be condensed into the intrados, the density of the matter in it $\propto (x' - x)$, and the parts of the bridge are still in equilibrium.

Now let the parts of the arch thus formed be connected by links, or otherwise, and let it be supposed to be inverted; then, in this position, the forces acting upon each point being the same in intensity, but opposite in direction, the arch will not change its form.

Hence (by prob. 182, p. 153)

$$x' - x \propto \text{density at the point } (x, y)$$

$$\propto \frac{d^2x}{dy^2 \cdot \sqrt{1 + \frac{dx^2}{dy^2}}}$$

whence, having given the form of the intrados, that of the extrados may be found, and *vice versâ*.

In the problem the intrados being a cycloid, we have (No. 82,)

$$x' - x \propto D \propto \frac{1}{(2r - x)^{\frac{3}{2}}}.$$

Let $y = 0$, or $x = r$,

and suppose $x' - x$ in this case $= m$.

$$\text{Then } m : x' - x :: \frac{1}{r^{\frac{3}{2}}} : \frac{1}{(2r - x)^{\frac{3}{2}}}$$

$$\therefore x' - x = \frac{mr^{\frac{3}{2}}}{(2r - x)^{\frac{3}{2}}}$$

$$\left.\begin{aligned}\therefore x' &= x + \frac{mr^{\frac{3}{2}}}{(2r-x)^{\frac{3}{2}}}\\ \text{But } y' = y &= \sqrt{2rx - x^2} + \text{vers.}^{-r}x\end{aligned}\right\}$$

whence the relation between x' and y' required.

285. Let θ be the angle required. Then P and 2P being the weights of the arms, we have, placing them in the middle points,

$$P \times a = -2P \times 2a \,.\, \cos.\theta,$$

$2a$ being the length of the shorter arm.

$$\therefore \cos.\theta = -\frac{1}{4}$$

which gives the angle required.

286. The density at any point $(x, y) \propto \dfrac{d^2x}{dy^2\sqrt{1+\dfrac{dx^2}{dy^2}}}$

(p. 211.)

But, by the question

$$y = px^3$$

$$\therefore \frac{dx}{dy} = \frac{1}{3px^2}$$

$$\frac{d^2x}{dy^2} = -\frac{2}{3px^3} \cdot \frac{dx}{dy} = -\frac{2}{9p^2x^5}$$

$$\therefore D \propto \frac{1}{x^5\sqrt{1+\dfrac{1}{9p^2x^4}}} \propto \frac{1}{x^3\sqrt{9p^2x^4+1}}$$

and the weight $\propto \dfrac{dx}{dy} \propto \dfrac{1}{x^2}$.

287. Let m be the cohesion of each of the particles P, P' the weight just sufficient to produce fracture, or to separate the particles in contact with the perpendicular section in the two given positions. Then supposing a, $2b$ the vertical and horizontal

dimensions of the fracture, (which by the question we suppose a parabola whose latus rectum p is given) and l its distance from the end of the beam, we have

area of fracture $\times \frac{4}{3} ab = \frac{4}{3} \cdot \frac{b^3}{p}$, and distance of its centre of gravity from the vertex $= \frac{3}{5} a = \frac{3}{5} \cdot \frac{b^2}{p}$.

Hence (p. 211)

$$P \times l = m \times \frac{4}{3} \cdot \frac{b^3}{p} \times \frac{3}{5} \cdot \frac{b^2}{p} = \frac{4m}{5} \cdot \frac{b^5}{p^2} \text{ and}$$

$$P' \times l = m \times \frac{4}{3} \cdot \frac{b^3}{p} \times \frac{2}{5} \times \frac{b^2}{p} = \frac{8m}{15} \quad \frac{b^5}{p^2}$$

$$\therefore P : P' :: 3 : 2,$$

the relative strength required.

288. Let, generally, θ be the inclination of the plane, and ϕ that of the power, to the horizon. Then since the weight W is supported by the power P, and the reaction of the plane, we have by the $\triangle$ of forces.

$$P : W :: \sin. \theta : \cos. (\theta - \phi)$$

In the problem, $\theta = 30^\circ$, and $\phi = 0$.

$$\therefore P : W :: \sin. 30^\circ : \cos. 30^\circ$$

$$:: 1 : \sqrt{3}.$$

289. Let AC (Fig. 83,) be the rectangle, ADF the $\triangle$ cut off of such a magnitude that the trapezium FDCB may rest with FB horizontal when suspended by F. Now supposing the trapezium divided into the rectangle FC and $\triangle$ DFf, let their matter be collected into their centres of gravity g, g'. Then, the equilibrium being the same whether the matter act by its weight as distributed along the surface of the trapezium, or in portions proportional to the rectangle and $\triangle$, at the points g, g', supposed connected with the fulcrum F by the inflexible lines Fg, Fg', we have (putting AB $= a$, AB $= b$, and AF $= x$)

$$b.(a-x)\times gn=\frac{1}{2}.bx\times g'n'$$

But $gn = \frac{1}{2}(a-x)$

$$g'n'=\frac{2}{3}mf=\frac{2}{3}x$$

$$\therefore \frac{1}{2}.(a-x)^2=\frac{1}{3}.x^2$$

$$\therefore x=\sqrt{\frac{3}{2}}\times(a-x)$$

and $x = \dfrac{a\sqrt{3}}{\sqrt{2}+\sqrt{3}}$

which gives the position of the line required.

290. Let $r_1, r_2 \ldots\ldots r_n$ be the radii of the several axles, $R_1, R_2, R_3, \ldots\ldots R_n$, those of the corresponding wheels; let also $W, p_1, p_2 \ldots\ldots p_{n-1}$, P be the weights applied to the axles sustained by the several powers $p_1, p_2 \ldots\ldots$ P applied to the corresponding wheels. Then, in case of an equilibrium, we have

$$\left.\begin{array}{l} p_1 : W :: r_1 : R_1 \\ p_2 : p_1 :: r_2 : R_2 \\ p_3 : p_2 :: r_3 : R_3 \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ P : p_{n-1} :: r_n : R_n \end{array}\right\}$$

$$\therefore P : W :: r_1\, r_2\, r_3 \ldots r_n : R_1\, R_2 \ldots\ldots R_n$$

the relation between the power and weight generally.

Now by the problem,

$$R_1 = 2r_1,$$
$$R_2 = 2^2\, r_2$$
$$\&c. = \&c.$$
$$R_n = 2^n\, r_n$$

and $P : W :: 1 : p$

$$\therefore 1 : p :: 1 : 2^{1+2+3+\ldots n}$$

$$\therefore 2^{n.\frac{n+1}{2}} = p$$

$$\therefore n . \frac{n+1}{2} = \frac{l.p}{l.2}$$

$$\therefore n^2 + n = 2 \frac{l.p}{l.2}$$

and n may be found by the solution of a quadratic, and by reference to the tables.

291. When the bar is in equilibrium, its centre of gravity and point of suspension are in the same vertical line.

Hence θ the inclination of the line joining the point of suspension and centre of gravity to the bar, is the inclination also of the bar to the vertical; which determines its position.

292. The strength (s) of the beam at any point is measured by the area of its lateral section $\times$ distance of its centre gravity from its under surface,. Hence, x and y being the vertical and horizontal sides of the rectangular section of the beam required, and r the radius of the base of the cylinder, we have

$$s \propto x \times y \times \frac{x}{2} = \text{max.}$$

$$\therefore x^2 y = \text{max.}$$

$$\text{But } \frac{x^2}{4} = r^2 - y^2.$$

$$\therefore y . (r^2 - y^2) = r^2 y - y^3 = \text{max.}$$

$$\therefore r^2 - 3y^2 = 0$$

$$\text{or } y = \frac{r}{\sqrt{3}}, \text{ and } x = 2 \sqrt{\frac{2}{3}} . r.$$

293. When the paraboloid is at rest, the straight line joining its centre of gravity and point of contact with the plane on which it rests, is vertical.

Hence the equation to the generating parabola being

$$y^2 = px$$

a the length of the axis, and θ the inclination of the axis to the

vertical, g the distance of the centre of gravity from the vertex we have,

$$\cot.\theta = \frac{\text{subnormal}}{y}$$

$$= \frac{\frac{p}{2}}{\sqrt{\frac{p}{6}}\cdot\sqrt{4a-3p}} = \sqrt{\frac{3p}{2(4a-3p)}}.$$

By the question $\theta = 60^\circ$

$$\therefore \frac{1}{\sqrt{3}} = \sqrt{\frac{3p}{2(4a-3p)}}$$

whence

$$p : a :: 8 : 15.$$

294. Let the weight (W) of the lever be supposed to be collected in its middle point, its length $= a$, and the distance of the fulcrum from the end to which A is attached $= x$. Then, A being the greater weight, we have

$$A \times x = W.\left(\frac{a}{2} - x\right) + B.(a-x)$$

$$\therefore x = \frac{a}{2}\cdot\frac{W + 2B}{W + A + B}$$

the distance required.

295. The beam being in equilibrium, the directions of the forces, viz., the reactions of its extremes acting $\perp$ planes, and the weight of the beam at its centre of gravity acting vertically, must conspire to the same point.

Hence, supposing θ, θ' the inclinations of the planes to the horizon, ϕ that of the beam of the vertical, and a, a' its parts as divided by the centre of gravity, we readily obtain

$$\frac{a}{a'} = \frac{\sin.\theta}{\sin.\theta'}\cdot\frac{\sin.(\phi - \theta')}{\sin.(\phi + \theta)}$$

$$= \frac{\cot.\theta' - \cot.\phi}{\cot.\theta + \cot.\phi}$$

$$\text{and } \therefore \cot.\varphi = \frac{a'\cot.\theta' - a\cot.\theta}{a + a'}$$

whence it is very easy to determine the actual position of equilibrium of the beam.

296. Let Q = the weight of each pulley, and suppose $p_1, p_2 \ldots\ldots p_n$ the several powers requisite to sustain each pulley, then $p_1 = Q \cdot \frac{1}{2}$, $p_2 = Q \cdot \frac{1}{2^2}$, &c. = &c.

$$\therefore p_1 + p_2 + \ldots\ldots p_n = Q \cdot \left(\frac{1}{2} + \frac{1}{2^2} + \ldots\ldots \frac{1}{2^n}\right)$$

$$= Q \cdot \frac{2^n - 1}{2^n}.$$

Now the weight of the pulleys not being considered, we have

$$P' : W :: 1 : 2^n$$

$$\therefore P = Q \cdot \frac{2^n - 1}{2^n} + P' = \frac{Q \times (2^n - 1) + W}{2^n}$$

$$\therefore 2^n = \frac{W - Q}{P - Q}$$

$$\text{and } n = \frac{1}{l.\,2}\{l.\,(W - Q) - l.\,(P - Q)\}.$$

297. Whatever may be the number of tacks, it is evident, that in the case of an equilibrium the sum of the forces estimated in directions parallel to any given direction must = 0; for otherwise motion would ensue. Hence, in the problem, the vertical tendency of the weights, which is measured by their sum, must be counteracted by the sum of the vertical reactions (in the opposite direction) of the tacks. $\therefore$ sum of the weights = − (sum of reactions) = sum of the vertical pressures. For another proof, see *Whewell's Mech.* p. 48.

298. If the straight line joining the centre of gravity of the $\triangle$ and one of its angles be produced through the middle point

of the opposite side until it is doubled, the whole line will be the diagonal of the parallelogram whose sides are the other two lines joining the centre of gravity, and the remaining angles. Hence the truth of the proposition is manifest.

299. Let w, w'; g, g', be the weights, and distances of the centres of gravity, from the fulcrum of the greater and less arms of the steelyard, respectively. Also let the weights W, W′, W″, &c., attached to the extremity of the less arm (whose length suppose $= m$), be balanced by P, P′, P″, &c., placed on the greater arm at the distances d, d', d'', &c., from the fulcrum. Then

$$\mathrm{W}m + w'g' = \mathrm{P}d + wg$$
$$\mathrm{W}'m + w'g' = \mathrm{P}d' + wg$$
$$\&c. = \&c.$$

$$\text{Hence } d' - d = \frac{\mathrm{W}' - \mathrm{W}}{\mathrm{P}} . m$$

$$d'' - d' = \frac{\mathrm{W}'' - \mathrm{W}'}{\mathrm{P}} . m$$

$$\&c. = \&c.$$

which intervals will evidently be constant, provided the weights W, W, &c., have a common difference, or are in arithmetical progression.

Again, let $d' - d = d'' - d' = \&c. = m$.

Then $\mathrm{P} = \mathrm{W}' - \mathrm{W} = \mathrm{W}'' - \mathrm{W}' = \&c.$ Q. E. D.

300. If a be the tension of the catenary at its lowest point (which being measured along the tangent is then wholly horizontal) x, y the vertical and horizontal co-ordinates originating in that point, and s the length of the corresponding arc; then it is known that

$$s^2 = 2ax + x^2 \ldots\ldots\ldots\ldots\ (1)$$

Hence, putting length of the chain $= 2n$, and the distance between the tacks, which are supposed in the same horizontal line $= 2m$, we have,

$$ds^2 = dx^2 \cdot \frac{(a+x)^2}{2ax+x^2} = (ds^2 - dy^2) \cdot \frac{(a+x)^2}{2ax+x^2}$$

$$\therefore ds = dy \cdot \frac{a+x}{a} = \frac{dy}{a} \cdot \sqrt{a^2+s^2}$$

$$\therefore y = a \times l. \frac{s+\sqrt{a^2+s^2}}{a}$$

and when $s = n$, $y = m$, $\therefore$

$$ae^{\frac{m}{a}} = n + \sqrt{a^2+n^2}$$

which, being transcendental, will give the value of a in terms of the known quantities m, n, e, by approximation only. In practice, it will be sufficient to take from observation the greatest value (α) of x, which will give by means of (eq. 1.)

$$a = \frac{n^2 - \alpha^2}{2n}.$$

301. The solid thus generated will evidently consist of a cone and hemisphere, whose equal circular bases coincide.

Let r be the radius of the sphere, g, g' the distances of the centres of gravity of the cone and hemisphere from the centre of the common base. Then, since $g = \frac{1}{4}$ alt. $= \frac{\sqrt{3}}{4} r$, $g' = \frac{3}{8} r$, the volume of the cone $= \frac{1}{3} \pi \sqrt{3} \cdot r^3$, and that of the $\frac{1}{2}$ sphere $= \frac{2}{3} \pi r^3$, we have, (G, G′ being the distances of the centre of gravity of the whole solid from those of the cone $\frac{1}{2}$ sphere,)

$$G' : G :: \text{cone} : \tfrac{1}{2}\text{sphere} :: \sqrt{3} : 2 :: g : g'$$

$$\therefore G' + G : G :: g' + g : g'.$$

But $g + g' = G + G'$, $\therefore G = g'$, or the centre of gravity of the whole solid coincides with the centre of the sphere or base.

Now the condition of equilibrium of any solid on a plane is that the vertical line passing through the centre of gravity shall fall within the part of contact. But since the line thus drawn in the

present case is $\perp$ tangent plane, and $\therefore$ $\perp$ horizon, it is vertical. Hence the solid will balance in all points of the spherical surface upon the horizontal plane.

302. Let a be the given length of the string to which the body (P) is attached, b the distance between the point of suspension, and that from which the rod acts. Then the equilibrium being produced by the tension (T) of the string, the reaction of the plane, and the weight of the body, we readily get by means of the Δ of forces

$T : P :: \cos.\theta : \sin.(\phi + \theta)$, θ and ϕ being the inclinations of the rod and string to the horizon.

Also $a : b :: \sin.\theta : \sin.(\phi + \theta)$

$\therefore T = \frac{a}{b} P.\cot.\theta \propto \cot.\theta.$ Q. E. D.

303. Let a be the length of the axis of the paraboloid, p its parameter; then by (No. 293, p. 217,) it appears that besides the vertical position of the axis, there are two others in each vertical plane (within the limits we are about to define), determinable by the equation

$$\tan.\theta = \sqrt{\frac{2.(4a-3p)}{3p}} \ldots\ldots\ldots (1)$$

where θ is the inclination of the axis to the vertical passing through the centre of gravity, and point of contact of the paraboloid with the plane.

Now, when $4a - 3p = 0$, or $a = \frac{3p}{4}$, $\theta = 0$, and it is *indifferent* what point of the surface we place in contact with the horizontal plane, as the solid will find but one position of equilibrium; whereas, if a be $> \frac{3p}{4}$, it will require the axis to be placed vertically with great precision, in order that it may remain there, otherwise (the position being then unstable from the altitude of the centre of gravity) it will quit that position, and oscillate in

some vertical plane, until it finally settle in one of the positions defined by eq. 1.

304. Let θ denote the inclination of the rod to the horizon, and P the power necessary to sustain the load W. Then considering the rod an *inclined plane*, the load is supported by the reaction of that plane, and the power. Hence, by means of the $\triangle$ of forces, we have

$$P : W :: \sin.\theta : 1$$

$$\therefore P = W \times \sin.\theta.$$

305. Let a be the length of the beam, and b the distance of its centre of gravity from that extremity which rests against the wall, φ its inclination to the wall. Then, supposing an immoveable vertical obstacle to prevent the other end from sliding, the equilibrium will be maintained by two horizontal forces (the reactions at the extremities,) and two vertical, viz., that of the weight of the beam acting at its centre of gravity, and the vertical reaction of the horizontal plane; which forces may be called

$$F, -F; G, -G.$$

Now the beam being considered a lever whose fulcrum is the obstacle, the weight G will be sustained upon it by the power F. Hence f, g, being drawn from the fulcrum $\perp$ their directions, we have

$$F \times f = G \times g$$

$$\text{or } F = \frac{g}{f}.G = \frac{a-b}{a}.\tan.\varphi$$

which will give the *pressure* required. See also *Cresswell's Translations of Venturoli*, p. 53.

306. Let BA (Fig. 84,) be the position of the beam when sustained by the reaction of the string CB, and that of the vertical wall CA, at A; required to determine this position, and also the limits of the length of BA, so that the equilibrium may be possible.

The equilibrium being maintained by three forces, the reactions of the string and plane acting at the points B, A, in directions along BC and $\perp$ CA, and the weight of the beam acting vertically at its centre of gravity G, those directions must converge to the same point (g).

Hence, putting AG $= a$, BG $= b$, CB $= l$, and AC $= x$, we get

$$x = Cg \,.\, \cos. C = \frac{al}{a+b} \,.\, \frac{l^2 + x^2 - (a+b)^2}{2lx}.$$

$$\text{Hence } x = \frac{\sqrt{a\,(l^2 + \overline{a+b})^2)}}{\sqrt{8}}$$

which determines the position required, the positive value of x only being taken.

The *inferior* limit of l is evidently $a + b$.

If the beam take the position BA′, and its extremity A′ be prevented from falling, (a circumstance which ought to have been stated in the enunciation of the problem), equilibrium, within the same limits of l, will still be maintained.

DYNAMICS.

COLLISION OF BODIES.

307. The plane of the ellipse being supposed horizontal, since the directions of incidence and reflection at the curve will make equal $\angle$ with the tangents, it is easily shewn that the body being projected from an extremity of either axis along the line joining the adjacent extremities of both axes, will go on for ever (friction, &c., are not considered) describing the $\square$ formed by joining the four extremities of the axes.

308. The incident velocity of the ball will be diminished by reflection in the proportion of 1 to n. Hence since

$$s = \frac{1}{2} \cdot \frac{v^2}{g} \propto v^2$$

where s is the space due to the velocity v, and g the measure of the force of gravity, or $32\frac{1}{6}$ feet, the successive altitudes to which the ball will ascend after the reflections of the plane, will be expressed by

$$n^2 s,\ n^4 s,\ n^6 s,\ \&c.,$$

and the whole space described will therefore be

$$S = s + 2n^2 s\,(1 + n^2 + n^4 + \&c.\ \infty)$$

$$= s + 2n^2 s\,\frac{1}{1 - n^2} = s \cdot \frac{1 + n^2}{1 - n^2}.$$

309. Let a, b; α, β, be the velocities of A, B, before and after impact, respectively, and r their relative velocity before impact.

Then since A loses in velocity $\frac{2Br}{A+B}$, and B gains $\frac{2Ar}{A+B}$, (*Wood*, p. 120,) we have

$$\alpha = a - \frac{2Br}{A+B}, \text{ and } \beta = b + \frac{2Ar}{A+B}$$

$\therefore \beta$ is $> \alpha$ if $b.(A+B) + 2Ar$ be $> a.(A+B) - 2Br$ or if $b - a + 2r$ be > 0.

But when the bodies move in the same direction and A overtakes B, $a - b = r$. $\therefore b - a + 2r\ (= r)$ is > 0, and $\therefore \beta$ is $> \alpha$. Q. E. D.

310. Generally, if there be a row of bodies A, Ar, Ar^2.... Ar^{n-1}, at rest, and α_1, α_2 ..... α_n, be their velocities after successive impact; the force of elasticity being to that of compression as $m : 1$, we have

$$\alpha_1 : \alpha_n :: (1+r)^{n-1} : (1+m)^{n-1}$$

For (*Wood*, p. 126)

$$\alpha_1 : \alpha_2 :: A + Ar : (1+m) \times A :: 1+r : 1+m$$

$$\alpha_2 : \alpha_3 :: Ar + Ar^2 : (1+m)rA :: 1+r : 1+m$$

&c. &c.

and $\alpha_{n-1} : \alpha_n :: Ar^{n-2} + Ar^{n-1} : (1+m)\, r^{n-2} A :: 1+r : 1+m$

$$\therefore \alpha_1 : \alpha_n :: (1+r)^{n-1} : (1+m)^{n-1}$$

Now in the problem $n = 5$, $r = 2$ and $m = \frac{2}{3}$.

$$\therefore \alpha_1 : \alpha_5 :: 3^4 : \frac{5^4}{3^4} :: 6561 : 625.$$

311. Let a, b; α, β, be the velocities before and after impact of the bodies A, B; also let the degree of elasticity of these bodies be denoted by m, perfect elasticity being unit, then

$$Aa^2 + Bb^2 > A\alpha^2 + B\beta^2.$$

$$\text{For } \alpha = a - \frac{\overline{1+m}\,.\,B\,.\,\overline{a-b}}{A+B}$$

$$\beta = b + \frac{\overline{1+m}\,.\,A\,.\,\overline{a-b}}{A+B}$$

Hence

$$Aa + Bb = A\alpha + B\beta$$
$$\text{and } \beta - \alpha = m . (a - b)$$
$$\therefore A . (a - \alpha) = B . (\beta - b)$$
$$\text{and } a + \alpha = \beta + b + \overline{1-m} . (a - b)$$
$$\therefore A . (a^2 - \alpha^2) = B (\beta^2 - b^2) + (1-m) B . (a-b)$$
$$\text{or } Aa^2 + Bb^2 = A\alpha^2 + B\beta^2 + \overline{1-m} . B (a - b)$$

Consequently, whether A and B move in the same or opposite directions (for m is $\angle$ 1 and b is $\angle$ a in the former, and $-b$ is positive in the latter case,)

$$Aa^2 + Bb^2 \text{ is} > B . (\alpha^2 + \beta^2)$$

312. Retaining the same notation, by the preceding number, we have

$$Aa + Bb = A\alpha + B\beta$$
$$\text{and } \beta - \alpha = m . (a - b)$$

and by the question

$$Aa^3 + Bb^3 = A\alpha^3 + B\beta^3$$

which easily become

$$A . (a - \alpha) = B (\beta - b) \ldots\ldots\ldots (1)$$
$$\beta - \alpha = m . (a - b) \ldots\ldots\ldots\ldots (2)$$
$$\text{and } a^2 + a\alpha + \alpha^2 = b^2 + b\beta + \beta^2 \ldots . (3)$$

Hence eliminating α, β, after proper reduction, we get

$$m^2 + \frac{b . \overline{1+B} + a . \overline{B + 2A}}{(1 - 2B)(a - b)} . m = \frac{Aa+Bb+a+b}{(1-2B).(a-b)}$$

from which the required degree of elasticity may readily be found.

313. Let PR (Fig. 85,) represent the plane of the billiard table; then A being the ball first struck, draw $A\alpha \perp$ PS and make $ab = a$A; draw also bc parallel to SP, meeting QP produced in r, and make $rc = rb$; join cB cutting PQ in n, and nb, cutting PS in m, and join Am, then AmnB is the path required.

For it is easily seen that $\angle$ Ama = $\angle bma$ = $\angle$ Pmn or the

$\angle$ of incidence at m = the $\angle$ of reflection, and likewise that the $\angle$ Pnm = $\angle$ Pno = $\angle$ QnB.

314. The velocity of A being denoted by (a), that gained by B is (*Wood*, p. 117,)

$$V = \frac{Aa}{A + B}$$

But B = 4A

$$\therefore V = \frac{a}{5}.$$

315. If m, n, be the original velocities of the hard bodies, A, B, then a being added to each of their velocities, let them impinge, and we have the common velocity after impact expressed by (*Wood*, p. 116,)

$$v = \frac{A.(m + a) + B(n + a)}{A + B}$$

$$= a + \frac{Am + Bn}{A + B}$$

$$= a \pm v'$$

according as the second term is positive or negative.

316. Let a, b; α, β, be the velocities before and after impact; then (*Wood*, p. 120,)

$$\alpha = a - \frac{2B.(a + b)}{A + B} = 12 - \frac{2 \times 8 \times 19}{19}$$

$$= -4$$

$$\beta = b + \frac{2A.(a + b)}{A + B} = 7 + \frac{2 \times 11 \times 19}{19}$$

$$= 29.$$

317. The angle of reflection being the same from the curve as from the tangent at the same point of incidence, the $\angle$ of

incidence = $\angle$ of reflection, and the inclinations of *radii vectores* from both foci to the same point being equal, the truth of the proposition is manifest.

318. The same effects being produced whether the falling body be perfectly hard, or the plane, we have, by prob. 308, the space required expressed by

$$n^6 s = \left(\frac{1}{2}\right)^6 a = \frac{a}{64}.$$

319. Let a, b; α, β, be the velocities of A and B before and after impact, γ that gained by B and λ that lost by A during the impact. Then

$$\gamma = \beta - b,\ \lambda = a - \alpha$$

$$\therefore (\mathrm{A}a^2 + \mathrm{B}b^2) - (\mathrm{A}\alpha^2 + \mathrm{B}\beta^2 = 2(\mathrm{A}a\lambda - \mathrm{B}b\gamma) - (\mathrm{A}\lambda^2 + \mathrm{B}\gamma^2).$$

But $\mathrm{A}a\lambda - \mathrm{B}b\gamma = \mathrm{A}\lambda\,.\,(a - b) = \mathrm{A}\lambda\,.\,(\lambda + \gamma) = \mathrm{A}\lambda^2 + \mathrm{A}\lambda\gamma$ $= \mathrm{A}\lambda^2 + \mathrm{B}\gamma^2$. (*Wood*, p. 117.)

$$\therefore \mathrm{A}a^2 + \mathrm{B}b^2 - (\mathrm{A}\alpha^2 + \mathrm{B}\beta^2) = \mathrm{A}\lambda^2 + \mathrm{B}\gamma^2. \qquad \text{Q. E. D.}$$

320. Generally, let $(a, b,)$, $(x, y,)$ be the velocities before and after impact of the spherical bodies A, B, whose degree of elasticity is m: also let (α, β), (θ, φ), denote the inclinations of the directions in which they move before and after impact to the *tangent plane* at their point of concourse; then having given a, b, α, β and m, it is required to determine x, y, θ and φ.

Decomposing (a, b) into their equivalents (a', b') $\perp$ to tangent-plane, and (a_1, b_1) parallel to it, we have

$$\left.\begin{array}{l} a' = a \sin. \alpha,\ b' = b \sin. \beta \\ a_1 = a \cos. \alpha,\ b_1 = b \cos. \beta \end{array}\right\} \ldots\ldots (1)$$

and supposing the bodies A, B, moving $\perp$ plane in opposite directions with velocities (a', b'), their velocities (a'', b'') after impact in the direction of A's motion will be (see *Wood*, p.p. 126 and 133.)

$$\left.\begin{array}{l} a'' = a' - \dfrac{\overline{1+m}\,.\,\mathrm{B}\,.\,(a'+b')}{\mathrm{A}+\mathrm{B}} \\[2ex] b'' = b' + \dfrac{\overline{1+m}\,.\,\mathrm{A}\,.\,(a'+b')}{\mathrm{A}+\mathrm{B}} \end{array}\right\}$$

But it also appears that

$$\left.\begin{array}{l} a'' = a_1 \,.\, \tan.\, \theta = a \cos.\, \alpha \,.\, \tan.\, \theta \\ b'' = b_1 \ \tan.\, \phi = a \cos.\beta \,.\, \tan.\, \phi \end{array}\right\}$$

Hence, and by eq. (1), we get

$$\tan.\, \theta = \frac{A - mB}{A+B} . \tan.\, \alpha - \frac{\overline{1+m}\,.\,B}{A+B} \times \frac{b}{a} \times \frac{\sin.\, \beta}{\cos.\, \alpha}$$

$$\text{and } \tan.\, \phi = \frac{B + \overline{m+2}.A}{A + B} . \tan.\, \beta + \frac{\overline{1+m}\,.\,A}{A+B} \,.\, \frac{a}{b} \,.\, \frac{\sin.\, \alpha}{\cos \beta},$$

which give the values of θ and ϕ.

$$\text{Again, } x = \frac{a_1}{\cos.\theta} = a . \frac{\cos.\, \alpha}{\cos.\, \theta}$$

$$y = \frac{b_1}{\cos.\phi} = b . \frac{\cos.\, \beta}{\cos.\, \phi}$$

which are also known.

In the problem

$$A = 1, B = 2;\ a = 1, b = 2;\ m = \frac{1}{2}, \alpha = 30^\circ \text{ and } \beta = 90^\circ.$$

$$\therefore \tan.\, \theta = -\frac{4}{\sqrt{3}},\ \tan.\, \phi = \infty = \tan.\, 90^\circ.$$

$$x = \frac{\cos.\, 30^\circ}{\cos.\, \theta} = \frac{\sqrt{3}}{2} \times \sqrt{\frac{19}{3}} = \frac{\sqrt{19}}{2}$$

$$y = 2 . \frac{0}{0} = 2.$$

321. Let ABCDE (Fig. 86,) be the billiard table, *abcde* the path required. Then since the angles of incidence and reflection at *a*, *b*, *c*, *d*, *e*, are equal, let them be denoted by $\alpha, \beta, \gamma, \delta, \epsilon$, respectively, and suppose the sides AB, BC, CD, DE, EA, $=$ a, b, c, d, e; also put $aB = u$, $bC = v$, $cD = x$, $dE = y$, $eA = z$.

Now it is evident that

$$\left.\begin{array}{l} \alpha + \beta + B = \pi \\ \beta + \gamma + C = \pi \\ \gamma + \delta + D = \pi \\ \delta + \epsilon + E = \pi \\ \epsilon + \alpha + A = \pi \end{array}\right\} \therefore \left.\begin{array}{l} \alpha + \beta + \gamma + \delta = 2\pi - B - D \\ \alpha + \beta + \delta + \epsilon = 2\pi - B - E \\ \beta + \gamma + \delta + \epsilon = 2\pi - C - E \\ \beta + \gamma + \epsilon + \alpha = 2\pi - C - A \\ \gamma + \delta + \epsilon + \alpha = 2\pi - D - A \end{array}\right\}.$$

$$\text{Also } \alpha + \beta + \gamma + \delta + \epsilon = \frac{5\pi}{2} - \frac{A+B+C+D+E}{2}$$
$$= \frac{5\pi}{2} - \frac{3\pi}{2} = \pi$$

Hence
$$\alpha = C + E - \pi$$
$$\beta = D + A - \pi$$
$$\gamma = B + E - \pi$$
$$\delta = C + A - \pi$$
$$\epsilon = B + D - \pi$$
which are therefore known.

Again,
$$\frac{b-v}{u} = \frac{\sin.\,\alpha}{\sin.\,\beta}$$
$$\frac{c-x}{v} = \frac{\sin.\,\beta}{\sin.\,\gamma}$$
$$\frac{d-y}{x} = \frac{\sin.\,\gamma}{\sin.\,\delta}$$
$$\frac{e-z}{y} = \frac{\sin.\,\delta}{\sin.\,\epsilon}$$
$$\frac{a-u}{z} = \frac{\sin.\,\epsilon}{\sin.\,\alpha}$$
from which eliminating v, x, y, z, we get
$$x = \frac{a \sin.\,\alpha - e \sin.\,\epsilon + d \sin.\,\delta - c \sin.\,\gamma + b \sin.\,\beta}{2 \sin.\,\alpha}$$
$$\text{or} = \frac{1}{2 \,.\, \sin.\,(C+E)} \times \{a \sin.\,(C+E) - e \sin.\,(B+D)$$
$+ d \,.\, \sin.\,(C+A) - c \,.\, \sin.\,(B+E) + b \,.\, \sin.\,(D+A)\}$ which gives the actual position of the point a.

From a, thus determined, draw ab, making with AB the $\angle\,\alpha$, and meeting BC in b, which is the first point of reflection. In like manner the points c, d, e, may be found, and therefore the *perpetual path* required

The same process will apply to find the *perpetual path* on a *table* of any *odd* number $(2n + 1)$ of sides whatever.

In the case of an *even* number ($2n$) of sides, it will conduct to this result,

$$A + C + E + G + \ldots\ldots = B + D + F + \ldots\ldots = \overline{n-1} \cdot \pi$$

which shews that when the problem is possible (in this case) the polygon must be inscriptible in a circle. We leave the further discussion to the student.

322. Let A, Ar, Ar^2 .... Ar^{n-1} be the balls in geometric progression, a_1, a_2 ..... a_{n-1} the distances from one to the other in the same order, and v be the velocity given to A. Then since (*Wood*, p. 117,) the several velocities communicated by A to Ar, Ar to Ar^2, &c., are

$$\frac{Av}{A + Ar} = \frac{v}{1+r}, \ \frac{Ar}{Ar+Ar^2} \times \frac{v}{1+r} = \frac{v}{(1+r)^2}, \text{ \&c.}$$

$$\text{or } \frac{v}{1+r}, \ \frac{v}{(1+r)^2}, \ \frac{v}{(1+r)^3} \ldots\ldots \frac{v}{(1+r)^{n-2}}$$

and in uniform motions

$$S = T \times V, \text{ or } T = \frac{S}{V},$$

the intervals a_1, a_2 ..... a_{n-1} will be described in the corresponding times

$$\frac{1}{v} \cdot a_1, \ \frac{1+r}{v} \cdot a_2, \ \frac{(1+r)^2}{v} \cdot a_3, \ \ldots\ldots \frac{(1+r)^{n-2}}{v} \cdot a_{n-1},$$ and the

whole time required to put Ar^{n-1} in motion, will therefore be

$$t = \frac{1}{v} a_1 + \frac{1+r}{v} a_2 + \frac{(1+r)^2}{v} a_3 + \ldots\ldots \frac{(1+r)^{n-2}}{v} \cdot a_{n-1}$$

Let $a_1 = a_2 = a_3 = \ldots\ldots a_{n-1}$

Then

$$t = \frac{a_1}{v} \times \{1 + (1 + r) + (1 + r)^2 + \ldots\ldots (1 + r)^{n-2}\}$$

$$= \frac{a_1}{v} \times \frac{(1+r)^{n-1} - 1}{r},$$

the time required.

THE GRAVITATION OF BODIES.

323. Since DB = 2AB, the time along BD = time down AB; consequently, the whole time = 2 time down AB = time down (4 . AB).

Hence, if AB be produced to B′ until AB′ = 4AB, and upon AB′ be described a semicircle intersecting BD in D′, and AD′ be joined, AD′ will be the plane required.

For the time down AD′ = time down AB′. (*Wood*, p. 158).

324. Let a be the length of the plane, and θ its inclination to the horizon, and let it be required to find the time down any portions of it (m, n,) measured from the higher and lower ends respectively.

The space (s) fallen through from rest in the time (t) being

$$s = \frac{g \sin. \theta}{2} t^2 \,. \text{ (*Wood* or *Whewell*)}$$

if $x, y,$ denote the terms required, we have

$$x = \sqrt{\frac{2m}{g \sin. \theta}}$$

$$y = \text{time down } a - \text{time down } (a - m)$$

$$= \sqrt{\frac{2a}{g \sin. \theta}} - \sqrt{\frac{2 . (a-m)}{g \sin. \theta}}$$

$$= \sqrt{\frac{2}{g \sin. \theta}} \cdot (\sqrt{a} - \sqrt{a - m}).$$

Let $m = n = \dfrac{a}{2}$

Then $x : y :: 1 : \sqrt{2} - 1.$

325. Let it be required to find the chord of a circle, whose diameter is (a), through which the body falling may acquire a velocity $= \frac{1}{n}$. that acquired through the diameter.

Let x be the chord required.

Then since the times down the chords $\propto$ as their lengths, we have

$$x = \frac{a}{n}, \text{ the chord required.}$$

326. From the given point P draw $\perp$ horizon, a line of such a length that the time through it may $=$ time down the plane, and upon this line describe a semicircle cutting the given plane in the point P′; then P, P′ being joined, PP′ will be the line required.

For the time down the chord PP′ $=$ time down the diameter $=$ time down the given plane.

327. Let x be the plane required. The moving force $= 2B - B = B$,

$$\therefore \text{ the accelerating force } \phi = \frac{B}{3B} \text{ (*Wood*, p. 155)}$$

$$\therefore s = \frac{g}{2} \cdot t^2 \times \phi = \frac{g}{6} \cdot t^2 = 2$$

by the question

$$\therefore t = \frac{2\sqrt{3}}{\sqrt{g}} \text{ seconds.}$$

Hence the space required is

$$x = \frac{g}{2} t^2 = 6 \text{ feet.}$$

328. At the end of the time (t) the heights to which the bodies A, a will have risen, are

$$S = tV - \frac{g}{2} \cdot t^2, s = tv - \frac{g}{2} \cdot t^2$$

$\therefore$ the distance between them is

$$S - s = t \,.\, (V - v)$$

and the distance (D) of the centre of gravity from A is

$$D = \frac{a}{A + a} \,.\, t \,.\, (V - v)$$

$\therefore$ its height (h) is

$$h = t \,.\, \frac{AV + av}{A + a} - \frac{g}{2} \,.\, t^2 = \max.$$

Hence $dh = 0$, gives

$$s = \frac{1}{g} \,.\, \frac{AV + av}{A + a}$$

$$\text{and} \therefore h = \frac{1}{2g} \,.\, \left(\frac{AV + av}{A + a}\right)^2.$$

329. Let x be part required, then by the expression

$$s = \frac{g}{2} \,.\, t^2 \,.\, \sin.\, \theta$$

where s is the distance, descended from rest by the force of gravity (g) in the time t, along a plane inclined to the horizon by the $\angle\, \theta$, we have

$$x = \frac{g}{2} \,.\, t^2 \sin.\, C \text{ (Fig. Cam. Prob.)}$$

$$\text{and } AD = \frac{g}{2}\, t^2 \,.\, \sin.\, D$$

$$\therefore\; x = \frac{\sin.\, C}{\sin.\, D} \,.\, AD = \frac{AD^2}{AC}.$$

330. Let AB (Fig. 87) be the vertical diameter of the $\odot$, and let Mp drawn parallel to AB meet the tangent in M, and the curve in P, p; also let the chords PQ, pq, drawn $\perp$ AB intersect it N, n; then the time down MP + time along PQ with the velocity acquired = time down Mp + time along pq.

For, putting AB $= 2a$, and PM $= x$ we easily get

$$x = \frac{g}{2} \,.\, t^2, pM = 2a - x = \frac{g}{2} \,.\, t'^2$$

$$x = \frac{v^2}{2g},\ 2a - x = \frac{v'^2}{2g}$$

$$\therefore t = \sqrt{\frac{2x}{g}},\ t' = \sqrt{\frac{2}{g}} \cdot \sqrt{2a - x}$$

$$v = \sqrt{2gx},\ v' = \sqrt{2g} \cdot \sqrt{2a - x}$$

But time along $PQ = \dfrac{PQ}{v}$

and time along $pq = \dfrac{pq}{v'} = \dfrac{PQ}{v'}$

Hence, the whole time through MPQ is

$$\sqrt{\frac{2}{g}} \cdot \sqrt{x} + \frac{PQ}{\sqrt{2gx}} = \sqrt{\frac{2}{g}} \cdot (\sqrt{x} + \sqrt{2a - x})$$

and that through Mpq is

$$\sqrt{\frac{2}{g}} \cdot \sqrt{2a - x} + \frac{PQ}{\sqrt{2g} \cdot \sqrt{2a - x}} = \sqrt{\frac{2}{g}} . (\sqrt{2} + \sqrt{2a - x})$$

which being identical, are equal.

331. Let (s) be the space required; then the whole time of descent (t) is expressed by

$$t = \sqrt{\frac{2s}{g}}.$$

Now since $S = \dfrac{g}{2} . T^2$ generally the space described in the 4th second is

$$\frac{g}{2} . (4^2 - 3^2) = \frac{7}{2} g$$

and that described in the last second but 4, is

$$\frac{g}{2} . \{(t - 4)^2 - (t - 5)^2\} = \frac{g}{2} . (2t - 9)$$

Hence, by the equation,

$$7 : 2t - 9 :: 1 : 3$$

$$\therefore t = 15$$

$$\text{and } s = \frac{g}{2} . t^2 = \frac{g}{2} \times 225$$

$$= 3618 \text{ feet } 9 \text{ inches nearly.}$$

332. Let t be the time required. Then by the expression for the circumstances of motion on inclined planes,

$$s = \frac{gt^2 \sin. \theta}{2}$$

we have by the question

$s = 30$ feet, $\theta = 30^\circ$, $g = \frac{3}{4}$ of the force of gravity $= \frac{3}{4} \times$

$(32 \frac{1}{6}$ feet$) = 24 \frac{1}{8}$ feet nearly

$$\therefore t^2 = \frac{60 \times 2}{24\frac{1}{8}} = 4.9740932$$

$$\therefore t = 2.23 \text{ seconds nearly.}$$

333. Let Aa, Bb (Fig. 88) be the perpendiculars to the horizon bT. Then, joining AB let it be produced to meet ba in T, and take TP a mean proportional between TA, TB, and describe the circle APB passing through A, P, B, and join AP, PB, P is the point required.

For since

$$TP^2 = TA \times TB$$

the circle touches the horizon in P.

Hence time down BP $=$ time AP.

The point B will fall within or without ab according as the arc PAB is $>$ or $< \frac{1}{2} \odot$.

334. Let AB $= a$, AC $= b$, and the distance fallen by the latter body before it is overtaken $= x$; then

$$b - a + x$$

is the distance fallen by the former in the same time t with the velocity (v) acquired at B. Hence

$$b - a + x = tv + \frac{g}{2} . t^2$$

$$= t\sqrt{2ga} + \frac{g}{2} . t^2$$

But $x = \frac{g}{2} \times t^2$

$$\therefore b - a = \sqrt{\frac{2x}{g}} \times \sqrt{2ga}$$

$$\text{and } x = \frac{(b-a)^2}{4a}.$$

335. If θ be the inclination of the plane, the moving force with which P descends is

$$P - Q \sin. \theta$$

$\therefore$ the accelerating force (ϕ) is

$$\phi = \frac{P - Q \sin. \theta}{P + Q}$$

Now $V = g\phi t = g \cdot \frac{P - Q \sin. \theta}{P + Q} \times t$ the velocity of P acquired in the time t, the inclination of the plane being any whatever.

In the problem, $\theta = 30°$.

$$\therefore V = g \cdot \frac{2P - Q}{2(P + Q)}.$$

336. Let t be the time in which the bodies meet, a the distance between the points of projection, and from the upper of these let x be the distance of the point of concourse; then

$$x = tv + \frac{g}{2} \cdot t^2$$

$$a - x = tv' - \frac{g}{2} t^2$$

$$\therefore t = \frac{a}{v+v'}$$

$$\text{and } x = \frac{av}{v+v'} + \frac{g}{2} \cdot \frac{a^2}{(v+v')^2}.$$

Let $x = \frac{a}{2}$. Then by reduction

$v'^2 - v^2 = ga = (\sqrt{ga})^2 =$ the square of the velocity acquired down $\frac{a}{2}$, since $s = \frac{V^2}{2g}$.

337. Let α be the inclination of the given plane to the horizon, θ of that sought; then the moving force is

$$P . \sin. \alpha - W \sin. \theta$$

and $\therefore$ the accelerating force (φ) is

$$\varphi = \frac{P \sin. \alpha - W \sin. \theta}{P + W}.$$

Now $t^2 \propto \frac{s}{\varphi} \propto \frac{1}{\sin. \theta} \times \frac{1}{P \sin. \alpha - W \sin. \theta} = \text{min.}$

$\therefore$ $P \sin. \alpha \sin. \theta - W \sin.^2 \theta = \text{max.}$ and putting the differential $= 0$, we get

$$\sin. \theta = \frac{P \sin. \alpha}{2W}$$

giving the inclination required.

338. Here the moving force is evidently 1 oz. and $\therefore$ the accelerating force is $\varphi = \frac{1}{33}$ by *avoirdupoise weight.*

Hence $s = \frac{g}{2} . \varphi t^2 = \frac{g}{66} . t^2$

and $s = \frac{v^2}{2g\varphi} = \frac{33v^2}{2g}$

But s by the question $= 12$ feet.

$$\therefore t = \sqrt{\frac{12 \times 66}{g}} = 6 \sqrt{\frac{33}{g}}$$

$$v = 2 \sqrt{\frac{2g}{11}}.$$

339. Here the accelerating force is $\varphi = \frac{H}{L} = \frac{1}{6}$,

$$\therefore s = \frac{g}{2}\varphi t^2 = \frac{v^2}{2g\varphi}$$

$$\text{or } 20 = \frac{g}{12} \times t^2 = \frac{3v^2}{g}$$

$$\therefore t = 4\sqrt{\frac{15}{g}}, v = 2 . \sqrt{\frac{5g}{3}}.$$

340. Let a be their distance at first, x that after the lapse of the time t; then since the spaces described in that time are

$$tv + \frac{g}{2} . t^2, \text{ and } tv' - \frac{g}{2} . t^2$$

we have

$$a - x = t . (v + v') \ldots\ldots\ldots\ldots (1)$$

Now when the bodies meet $x = 0$, and $\therefore t = \frac{v}{v+v'}$. Hence if for t in equation (1) we put $\frac{a}{v+v'}$, there results, after reduction

$$x = \frac{a}{2}$$

the distance required.

341. The moving force being

$$P - W . \frac{H}{L} = \frac{P}{6} \text{ by the question, we have}$$

$$\varphi = \frac{P}{6 \times 2P} = \frac{1}{12}$$

$$\text{and } s = \frac{g}{2} \times \varphi t^2$$

$$\text{or } 12 = \frac{g}{2} \times \frac{1}{12} t^2 \therefore \text{ gives}$$

$$t = 12 \sqrt{\frac{2}{g}}.$$

342. Here the moving force is 1 oz. and, by the question, we have, therefore,

$$\varphi = \frac{1}{32+32+1} = \frac{1}{65}$$

$$\therefore s = \frac{g}{2}\varphi t^2 = \frac{v^2}{2g\varphi}$$

$$= \frac{g}{130} \times 25 = \frac{65v^2}{2g}, \text{ gives}$$

$$s = \frac{5g}{26}, \text{ and } v = \frac{g}{13}.$$

343. Here $\varphi = \dfrac{H}{L} = \dfrac{1}{6}$.

$\therefore s = \dfrac{g}{2}\varphi t^2 = \dfrac{v^2}{2g\varphi}$ gives, by the question,

$$t^2 = \frac{40}{g} \times 6 = \frac{16 \times 15}{g}$$

$$\text{or } t = 4\sqrt{\frac{15}{g}}, \text{ and } v = 2\sqrt{15g}.$$

344. Let x be the weight required. Then since $P = Q$, the moving force $= x$, and $\varphi = \dfrac{x}{2P + x}$.

Hence $v = g\varphi t$, will give, by the question,

$$48 = g \times \frac{x}{2P+x} \times 6$$

$$\text{and } \therefore x = \frac{16P}{g-8}$$

the weight required.

$$\text{Again, } s = \frac{g}{2}.\varphi t^2 = \frac{g}{2} \times \frac{8}{g} \times 36$$

$$= 144 \text{ feet.}$$

345. Let x denote the required weight. Then the moving force is $P - x$

$$\text{and } \varphi = \frac{P-x}{P+x}.$$

$$\text{Hence } s = \frac{v^2}{2g\phi} = \frac{a^2}{2g} \times \frac{P + x}{P - x}$$

$$\text{and } x = \frac{2gs + a^2}{2gs - a^2} \times \frac{1}{P}.$$

346. Let F, F′ be the accelerating forces, then by the question,

$$\frac{25}{36} = \frac{v^2}{v'^2} = \frac{F}{F'} \cdot \frac{t}{t'} = \frac{F}{F'} \cdot \frac{4}{5}$$

$$\therefore \frac{F}{F'} = \frac{125}{144}.$$

Hence, if Q and Q′ be the quantities of matter, the ratio of the moving forces is

$$\frac{5}{4} = \frac{F \times Q}{F' \times Q'} = \frac{125}{144} \times \frac{Q}{Q'}$$

which gives

$$\frac{Q}{Q'} = \frac{25}{36},$$

the ratio required.

347. Here $\phi = \frac{H}{L}$, and $\therefore s = \frac{1}{2} g\phi t^2$ gives, by the question,

$$t = \sqrt{\frac{2c}{g}}.$$

348. Let m be the degree of elasticity, and x the altitude required; also let a be the given velocity of projection.

Then the velocity with which the body reaches the plane is

$$\sqrt{(a^2 + 2gx)},$$

and that with which it quits the plane is

$$m\sqrt{(a^2 + 2gx)}.$$

But by the question x is the space due to this velocity, and the time is given, $\therefore$

$$x = \frac{v^2}{2g} = \frac{m^2}{2g}(a^2 + 2gx)$$

$$\text{and } x = \frac{g}{2} \cdot t^2$$

$$\begin{aligned} \therefore m^2 &= \frac{gt^2}{a^2 + 2gx} \\ &= \frac{gt^2}{g^2t^2 + g^2t^2} \\ &= \frac{1}{2g} \\ \therefore m &= \sqrt{\frac{1}{2g}}. \end{aligned}$$

349. Let a be the radius of the circle, (x', y'); (x, y) the corresponding co-ordinates of the *circle and locus* measured from the horizon along the vertical diameter; then in any time t, considering the chord $\sqrt{x'^2 + y'^2}$ an inclined plane whose height is x', the form

$$s = \frac{g}{2} \cdot t^2 \times \frac{H}{L}$$

$$\text{gives } \sqrt{x'^2 + y'^2} - \sqrt{x^2 + y^2} = \frac{gt^2}{2} \cdot \frac{x'}{\sqrt{x'^2 + y'^2}}$$

$$\text{also } \frac{x}{x'}, = \frac{y}{y'}$$

$$\text{and } y'^2 = 2ax' - x'^2$$

whence eliminating x', y', &c., we have

$$y^2 = \frac{4a - gt^2}{2} \cdot x - x^2$$

or the *locus* is a circle whose radius is $\frac{4a - gt^2}{4}$.

350. Let a, b, be the altitudes fallen by A, B, and x the space described along the horizon (which they will describe *uniformly with the velocities acquired*, since the plane of reflection pro-

jects then along the horizon). Then, the velocities acquired being

$$\sqrt{2ga}, \text{ and } \sqrt{2gb}$$

if the whole time of motion be denoted by t, we have

$$t = \sqrt{\frac{2}{g}a} + \frac{x}{\sqrt{2ga}} = \sqrt{\frac{2}{g}b} + \frac{x}{\sqrt{2gb}}$$

$$\therefore x = 2\sqrt{ab}.$$

Now, if a circle pass through the points of departure and touch the horizontal line, the distance, as is well known, of the point of contact from the concourse of the tangent and line passing through those points $= \sqrt{ab}$. $\therefore$ if a circle, &c. &c. Q. E. D.

351. Let t, t', be the times down the chords corresponding to ordinates y, y', and abscissæ x, x'. Then considering the chords, planes inclined to the horizon by the $\measuredangle$ 45° and 15°, we have, by the form $s = \frac{g}{2} \cdot \frac{H}{L} t^2$

$$t^2 : t'^2 :: \frac{x^2 + y^2}{x} : \frac{x'^2 + y'^2}{x'}$$

$$\text{But } x = y \tan. 45^\circ = y$$

$$\text{and } x' = y' \tan. 75^\circ = \frac{y'}{\tan. 15^\circ} = y' \sqrt{\frac{2+\sqrt{3}}{2-\sqrt{3}}}$$

$$= y' . (2 + \sqrt{3})$$

$$\therefore t^2 : t'^2 :: 2y : y' . \frac{(2+\sqrt{3})^2 + 1}{2 + \sqrt{3}} = 4y'$$

$$:: y : 2y'$$

$$:: \frac{b}{a}\sqrt{x.(2a-x)} : 2\frac{b}{a} . \sqrt{x'.(2a-x')}$$

$$:: \sqrt{x.(2a-x)} : 2\sqrt{x'.(2a-x')}$$

a being the semi-axis.

*352. Let x be the power required. Then the moving force is

$$x - Q \sin. 30^\circ = x - \frac{Q}{2}$$

$$\text{and } \phi = \frac{x - \frac{Q}{2}}{x + Q}$$

$$\therefore s = \frac{g}{2} \phi t^2 \text{ gives}$$

$$40 = \frac{g}{2} \cdot \frac{x - \frac{Q}{2}}{x + Q} \times 25$$

$$\therefore x = Q \cdot \frac{5g + 32}{10g - 32}.$$

353. Let the distance of the bodies A, B, at first $= a$, and that of the required point C from A $= x$; then α, β, being the velocities with which the bodies impinge,

$$\alpha = \sqrt{2gx};$$

$$\beta = \frac{A}{B} \sqrt{2gx},$$

by Collision of Bodies.

Hence B's velocity of projection is

$$V = \sqrt{\{\beta^2 + 2g(a - x)\}}$$

$$= \sqrt{2g} \cdot \sqrt{\left(a + \frac{A^2 - B^2}{B^2} x\right)}.$$

Now t being the time elapsed before the first impact of the bodies, we have

$$x = \frac{g}{2} t^2$$

$$\text{and } a - x = tV - x$$

$$\therefore a = tV = \sqrt{\frac{2}{g}} \sqrt{x} \times \sqrt{2g} \cdot \sqrt{\left(a + \frac{A^2 - B^2}{B^2} x\right)}$$

which being reduced, &c., gives

$$x = \frac{aB}{2(A + B)}.$$

Hence β is also known.

$$\therefore a = \sqrt{\frac{2}{g}} \cdot \sqrt{x} \times \frac{\sqrt{2g}}{B} (A \sqrt{x} + B \sqrt{a-x})$$

and by reduction and the solution of the resulting quadratic equation, we get

$$x = \frac{aB}{2(A^2 + B^2)} \times (A + B + \sqrt{2AB})$$

the distance required.

354. Let a denote the sum of the velocities; then v being that of A, we have

$$Av^2 + B \cdot (a - v)^2 = \text{min.}$$

$$\therefore 2Avdv - 2B \cdot (a - v)\, dv = 0$$

$$\text{and } Av = B \cdot (a - v)$$

$$\text{or } A : B :: \frac{1}{v} : \frac{1}{a-v}. \qquad \text{Q. E. D.}$$

355. Let $a =$ the length of the plane; then, since $s = \frac{g}{2} \sin. \theta \cdot t^2$, we have

$$a = \frac{g}{2} \sin. \theta\, t^2$$

$$\frac{a}{n} = \frac{g}{n} \cdot \sin. \theta\, t'^2$$

$$\therefore t = \sqrt{\frac{2a}{g \sin. \theta}},\ t' = \sqrt{\frac{2a}{ng \sin. \theta}},$$

$$\therefore t' : t - t' :: \frac{1}{\sqrt{n}} : 1 - \frac{1}{\sqrt{n}} :: 1 : \sqrt{n-1}$$

the relation required.

356. Let t, t' be the times required. Then, by the question,

$$400 = \frac{g}{2} \cdot t^2$$

$$500 = \frac{g}{2} \cdot \varphi t'^2 = \frac{g}{2} \sin. 30^\circ\, t'^2$$

$$= \frac{g}{4}\, t'^2$$

$$\therefore t^2 : t'^2 :: 400 : 1000$$

$$\text{and } t : t' :: \sqrt{2} : \sqrt{5}.$$

357. Let l be the whole length of the plane, a that part of it through which the body falls before it impinges on the horizontal plane, and θ the inclination required; then

$$a = \frac{g}{2} \sin. \theta \,.\, t^2 = \frac{v^2}{2g \sin. \theta}$$

gives

$$v = \sqrt{2ag \sin. \theta}$$

the velocity with which the body moves equally along the horizon; also time of descending the height ($l \sin. \theta$) with this uniform velocity = time through l, by the question,

$$\text{or } \frac{l \sin. \theta}{\sqrt{2ag \sin. \theta}} = \sqrt{\frac{2}{g \sin. \theta}} \,.\, l$$

$$\therefore \sin. \theta = 2 \sqrt{\frac{a}{l}}$$

which gives the inclination θ.

Since $\sin. \theta$ cannot be > 1

a cannot exceed $\frac{l}{4}$.

358. Let a be the length of the vertical line, A, B the bodies, and x the space described by the lower body before impact; then t being the interval elapsed before impact

$$\left.\begin{aligned} x &= tv - \frac{g}{2} \,.\, t^2 \\ a - x &= tv + \frac{g}{2} t^2 \\ \text{and } v &= \sqrt{2ga} \end{aligned}\right\}$$

Hence eliminating t and v, we get

$$x = \frac{7}{16} a \,\ldots\ldots\ldots\ldots\, (1).$$

Again, the velocities α, β, with which the bodies impinge, are

$$\alpha = \sqrt{2g(a-x)+2ga} = \sqrt{2g}\sqrt{(2a-x)}$$

$$\beta = \sqrt{2ga-2gx} = \sqrt{2g}\,.\sqrt{(a-x)}.$$

$\therefore v = \dfrac{\alpha-\beta}{2}$ their common velocity

$$= \sqrt{\frac{g}{2}}\left(\sqrt{2a-x} - \sqrt{a-x}\right)$$

$$= \sqrt{\frac{ga}{2}}\,.\left(\frac{5}{4}-\frac{3}{4}\right) \text{ substituting for } x.$$

$$= \frac{1}{2}\sqrt{\frac{g}{2}} \times \sqrt{a}.$$

Also if T be the required interval,

$$\frac{7}{16}a = \mathrm{T}v + \frac{g}{2}\mathrm{T}^2$$

$$= \frac{1}{2}\sqrt{\frac{g}{2}}\sqrt{a}\,.\,\mathrm{T} + \frac{g}{2}\mathrm{T}^2,$$

$$\therefore \frac{g}{2}\mathrm{T}^2 + \frac{1}{2}\sqrt{a}\sqrt{\frac{g}{2}}\,.\,\mathrm{T} = \frac{7}{16}a$$

$$\therefore \mathrm{T}\sqrt{\frac{g}{2}} = \frac{\sqrt{a}}{4} \pm \sqrt{\left(\frac{a}{16}+\frac{7a}{16}\right)}$$

$$= \frac{\sqrt{a}}{4} \times (1+2\sqrt{2})$$

$$\therefore \mathrm{T} = \sqrt{\frac{a}{g}} \times \frac{1+2\sqrt{2}}{2\sqrt{2}}.$$

359. Let l be the length of the inclined plane, and θ the required elevation; then its height $= l.\sin.\theta$, and the velocity acquired down $l.\sin.\theta$ is

$$v = \sqrt{2g\,.\,l\,.\,\sin.\theta}, \text{ and time down it is}$$

$$t = \sqrt{\frac{2}{g}}\,.\sqrt{l\sin.\theta}.$$

Now $l = tv + \dfrac{g}{2}.\sin.\theta\,.\,t^2$

$\therefore l = 2\,l\sin.\theta + l\sin.^2\theta$, which gives

$\sin.\theta = -1+\sqrt{2}$; whence θ is known.

360. Here $x = ta - mt^2$

and $\therefore 4m . \left(\frac{a^2}{4m} - x\right) = a^2 - 4amt + 4m^2 t^2$

$$= (a - 2mt)^2.$$

361. Let t be the time required; then v being the velocity of projection, and s the space described, we have

$$s = tv - \frac{g}{2} t^2$$

$$\text{or } s = tv + \frac{g}{2} t^2$$

according as the body is projected upwards or downwards. Hence, in the former case

$$t = \frac{v \pm \sqrt{v^2 - 2gs}}{g} \ldots\ldots\ldots (1)$$

and in the latter

$$t = \frac{- v \pm \sqrt{h^2 + 2gs}}{g} \ldots\ldots\ldots (2)$$

Equation (1) denotes that the problem is impossible when v is $<$ $\sqrt{2gs}$, or $<$ then that acquired by falling vertically through s; also when possible, that $t = \frac{v - \sqrt{v^2 - 2gs}}{g}$ is the time in ascending through the space $\frac{v^2}{2g}$ (till all the velocity of projection is destroyed), and afterwards in descending through $\frac{v^2}{2g} - s$.

Equation (2) shews that the problem is always possible in the case of downward projection; but since the time is necessarily positive, the only solution in this case is

$$t = \frac{-v + \sqrt{v^2 + 2gs}}{g}.$$

In the problem, the projection is upwards, $s = 63$ feet, $v = 64$ feet in a second, and we therefore have

$$t = \frac{64 \mp \sqrt{64^2 - 126g}}{g}.$$

362. The force being constant, the space described $\propto$ square of the time. Hence the spaces described in the m, $m-1$, first seconds are as m^2, $(m-1)^2$, and $\therefore$ that described in the m^{th} second is as $m^2-(m-1)^2$, or $2m-1$.

$\therefore$ the spaces described in the m^{th} and x^{th} seconds are as $2m-1$, $2x-1$, or

$$2m-1 : a :: 2x-1 : a\,.\,\frac{2x-1}{2m-1} \text{ the space required.}$$

363. The space described with the velocity of projection continued uniform in four seconds, would be 80 feet; and that due to the retarding force of gravity as modified by the inclined plane,

is $\frac{g}{2}\,.\,\sin.\,\theta\,.\,t^2 = \frac{g}{2}\sin.\,30 \times 4^2 = 4g$,

$\therefore 80-4g = 80-128\frac{2}{3} = -48\frac{2}{3}$ feet nearly, or the

$48\frac{2}{3}$ feet.

364. Let a be the distance between A, B, and x, y, the co-ordinates of the point P, x being measured from A along the line AB. Then considering $AP = \sqrt{y^2+x^2}$, $BP = \sqrt{y^2+(a-x)^2}$ two inclined planes, the times down them from rest will be

$$\sqrt{\frac{2}{gy}}\,.\,\sqrt{y^2+x^2},\quad \sqrt{\frac{2}{gy}}\,.\,\sqrt{y^2+(a-x)^2}$$

$\therefore$ by the question

$$\sqrt{\frac{2}{gy}} \times (\sqrt{y^2+x^2} + \sqrt{y^2+(a-x)^2} = \text{const.}$$

$$\text{or } \sqrt{y^2+x^2}+\sqrt{y^2+(a-x)^2} = c\sqrt{cy}$$

which being rationalized becomes

$$4cy^3 - c^2y^2 + (4cx^2 - 4acx + 2ca^2)\,y - 4a^2x^2 + 4a^3x - a^4 = 0$$

the equation to the locus of P.

365. Since on the inclined plane

$$s = \frac{g}{2} \cdot \sin. \theta \cdot t^2$$

and here $s = 400$ feet, $\theta = 30°$, we readily obtain

$$t = \frac{40}{\sqrt{g}}$$

366. Let l be the common length of the planes, t, t' the times, and v, v' the velocities required; then the velocity acquired down vertical line (l) being $= \sqrt{2gl}$, with which the bodies are projected, we have

$$l = t\sqrt{2gl} + \frac{g}{2} \cdot \sin. 45° \times t^2$$

$$\text{and } l = t'\sqrt{2gl} - \frac{g}{2} \cdot \sin. 30° \times t'^2$$

Hence (See 361, vol. II.)

$$t = 2\sqrt{\frac{l}{g}} \cdot \left(\sqrt{1 + \frac{1}{\sqrt{2}}} - 1\right)$$

$$t' = 2\sqrt{\frac{l}{g}} \cdot (\sqrt{2} - 1)$$

$$\therefore t : t' :: \sqrt{1 + \frac{1}{\sqrt{2}}} - 1 : \sqrt{2} - 1.$$

Again, V, V′ being the velocities acquired and lost by the descent and ascent in the times t, t', we have

$$V = gt \sin. 45° = \sqrt{2gl}\left(\sqrt{1 + \frac{1}{\sqrt{2}}} - 1\right)$$

$$V = gt' \sin. 30° = \sqrt{gl}\ (\sqrt{2}. \mp 1)$$

whenever the velocities at the end of the motion are respectively

$$v = V^2 + 2gl \cdot \sin. 45°$$

$$v' = V'^2 - 2gl \cdot \sin. 30°$$

and substituting these, also may be compared.

367. Let the origin of the co-ordinates $(x, y, z,)$ be taken in the intersection of the given line with the plane of the horizon, and the plane of $(x, z,)$ be the vertical in which that line is posited; that of $(x, y,)$ being the horizon. Also let $(a, b, c,)$ be the co-ordinates of the given point, and α the inclination of the given line to the horizon.

Then drawing any line from the given point to meet the given line in a point whose co-ordinates are $(x, y,)$ it may be considered an inclined plane, whose length and height are easily found to be

$$L = \sqrt{b^2 + (a - x)^2 + (c - x \tan. \alpha)^2}$$

$$\text{and } H = c - x \tan. \alpha.$$

$$\text{But } T^2 = \frac{L^2}{H} \times \frac{2}{g} = \text{minimum}.$$

$$\therefore \frac{b^2 + (a - x)^2}{c - x \tan. \alpha} + c - x \tan. \alpha = \text{min},$$

and putting the differential $= 0$, we get, after proper deductions, and the solution of a quadratic,

$$x = \frac{c \pm \cos. \alpha \sqrt{c^2 - 2ac \tan. \alpha + (a^2 + b^2) \tan.^2 \alpha}}{\tan. \alpha}$$

which determines the position of the line required.

Let $b = 0$, or the line and point be in the same vertical. Then

$$x = \frac{c \pm \cos. \alpha \, (c - a \tan. \alpha)}{\tan. \alpha}.$$

A geometrical determination may be made by drawing a line from the given point (P) parallel to the horizon, and meeting the given line in (Q), and taking along the given line (downwards or upwards, according to the position of P) QR = QP, or by drawing a circle touching the lines QP, QR in P, R. This is either obtainable from the construction of the above equation, or directly from the consideration (which may easily be proved) that the times down the chords, originating in the highest or lowest point of a circle *any how inclined* to the horizon, are equal.

368. The section of the sphere made by a plane passing *vertically* through the lowest point of the sphere, and another taken any where on its surface, is known to be a circle, whose

diameter is the vertical axis of the sphere. Hence, since the times down the chords terminating in the lowest point of a vertical circle are equal to that down its vertical diameter, the time down the line joining the abovementioned points of the sphere is equal to the time down its vertical axis.

369. Let P be the weight of the body, l the length of the plane, θ its inclination to the horizon. Then since the pressure $\perp$ to the plane $=$ P cos. θ, and $\therefore$ by the question friction $= \frac{P\cos.\theta}{n}$ and the moving force down the plane (not reckoning friction) is P sin. θ, the moving force down it when the retarding influence of friction is considered, will be

$$P\sin.\theta - \frac{P\cos.\theta}{n}.$$

Hence the force which accelerates the body down the plane is expressed by

$$\varphi = \frac{n\sin.\theta - \cos.\theta}{n},$$

and $\therefore$ since $s = \frac{g}{2}\varphi t^2$, we have

$$t^2 = \frac{2l}{g}\cdot\frac{1}{n\sin.\theta - \cos.\theta}$$

the positive root of which will give the time required.

PROJECTILES IN A VACUUM.

370. Let the body be projected in a direction inclined to the horizon at the $\angle\ \theta$, with the velocity v, and suppose (y, x) the vertical and horizontal co-ordinates, originating in the point of projection, which determine the position of the body at the end of the time t; then since the velocity estimated vertically and horizontally is $v \sin. \theta$, $v \cos. \theta$, we have

$$x = v \cos. \theta \times t$$

$$y = v \sin. \theta \times t - \frac{g}{2} t^2$$

$$\text{Hence } y = x \tan. \theta - \frac{gx^2}{2v^2 \cos.^2 \theta} \ldots\ldots\ (1)$$

the equation to a parabola, whose greatest ordinate, corresponding abscissa, and parameter, are respectively

$$\frac{v^2}{2g} . \sin.^2 \theta, \frac{v^2}{2g} \sin. 2\theta, \frac{2v^2}{g} . \cos.^2 \theta \ldots\ldots (2)$$

Consequently the equation to the trajectory referred to its vertex, by the horizontal and vertical co-ordinates (w, z) is

$$w^2 = \frac{2v^2}{g} \cos.^2 \theta \times z \ldots\ldots\ldots (3).$$

To resolve the problem, we have by (1) (referring the given co-ordinates a, b, of the plane, to the point of projection)

$$-b = a \tan. \theta - \frac{ga^2}{2v^2 \cos.^2 \theta}$$

which gives

$$v^2 = \frac{ga^2}{2 \cos.^2 \theta\, (b + a \tan. \theta)} = \text{min.}$$

$$\therefore \cos.^2 \theta\, (b + a \tan. \theta) = \text{max.} = u$$

Hence, putting $\frac{du}{d\theta} = 0$, we have

$$\tan. 2\theta = \frac{a}{b}$$

$$\text{and } \theta = \frac{1}{2} . \tan.^{-1} \frac{a}{b}$$

which gives the direction required; whence also the velocity.

371. Let the equations to the plane (considered a straight line), and parabola described (370), referred to the point of projection, be

$$y' = ax' + b$$

$$y = x \tan. \theta - \frac{gx^2}{2v^2 \cos.^2 \theta};$$

then, since when the body reaches the plane . $y = y'$, and $x = x'$, we have

$$ax + b = x \tan. \theta - \frac{gx^2}{2v^2 \cos.^2 \theta}$$

whence

$$x = \frac{v \cos. \theta}{g}\left\{v \cos. \theta\,(a - \tan. \theta) \pm \sqrt{v^2 \cos.^2 (a - \tan \theta)^2 - 2bg}\right\}$$

and $\therefore y = ax + b$, and finally

$$\sqrt{x^2 + y^2}$$

or the range required.

The greatest altitude is (370)

$$\frac{v^2}{2g} \sin.^2 \theta.$$

372. Retaining the notation of (370) we have

$$x = v \cos. \theta \times t$$

$$y = v \sin. \theta \times t - \frac{g}{2} t^2$$

whence, eliminating (v)

$$y = x \tan. \theta - \frac{g}{2} t^2.$$

Hence, making the altitude of the tower $= a$, since by the question

$$\theta = 45^\circ, \text{ and } t = 5''$$

we have

$$-a = a \tan. 45 - \frac{g}{2} \times 25$$

$$\therefore a = 33 \frac{1}{24} \text{ feet nearly,}$$

373. The space due to a velocity of 744 feet in a second is

$$s = \frac{v^2}{2g} = \frac{(144)^2}{32\frac{1}{6}} = 644 : 58 \text{ feet nearly.}$$

Also the space described in two seconds by a body projected upwards with that velocity is

$$S = tv - \frac{g}{2} t^2$$

$$= 288 - 2g = 223 \frac{2}{3} \text{ feet nearly.}$$

374. At the end of the time (t) the positions of the bodies are determined by (370)

$$x = a \cos. \theta \times t, \; y = a \sin. \theta \times t - \frac{g}{2} t^2$$

$$x' = b \cos. \theta \times t, \; y' = b \sin. \theta \times t - \frac{g}{2} t^2$$

which give

$$\left.\begin{aligned} y &= x \tan. \theta - \frac{gx^2}{2a^2 \cos.^2 \theta} \\ y' &= x' \tan. \theta - \frac{gx'^2}{2b^2 \cos.^2 \theta} \end{aligned}\right\} \dots\dots\dots (1)$$

Hence, if (y'', x'') denote the co-ordinates of the centre of gravity, since

$$y'' - y : x'' - x :: y' - y : x' - x$$

we have

$$y'' = \frac{y' - y}{x' - x} x'' + \frac{yx' - xy'}{x' - x}$$

But $x' = \frac{b}{a} x$, and (b being $> a$)

it readily appears that

$$x'' - x : x' - x :: \mathrm{B} : \mathrm{A} + \mathrm{B}$$

and $\therefore x = \frac{a}{\mathrm{B}b + \mathrm{A}a} \times (\mathrm{A} + \mathrm{B})\, x''$; also

y and y' are given by (eq. 1)

$\therefore$ substituting, we finally obtain

$$y'' = x'' \tan.\theta - \frac{g\,(\mathrm{A}+\mathrm{B})^2}{2\,(b\mathrm{B} + a\mathrm{A})^2 \cos.^2\theta} \cdot x''^2 \ldots\ldots (2)$$

whose greatest ordinate, corresponding abscissa, and paramete (found by putting $dy'' = 0$, &c.) are respectively

$$\frac{(b\mathrm{B} + a\mathrm{A})^2 \sin.^2\theta}{2g\,(\mathrm{A}+\mathrm{B})^2},\ \frac{(b\mathrm{B} + a\mathrm{A})^2 \sin.2\theta}{2g\,(\mathrm{A}+\mathrm{B})^2},$$

and $\frac{2\,(b\mathrm{B} + a\mathrm{A})^2 \cos.^2\theta}{g\,(\mathrm{A}+\mathrm{B})^2}$.

375. Let a be the altitude of the tower. Then (370)

$$-a = x \tan.\theta - \frac{gx^2}{2v^2 \cos.^2\theta},$$

whence

$$x = \frac{v^2 \sin.2\theta \pm \sqrt{v^4 \sin.^2 2\theta + 4gav^2\,(1 + \cos.2\theta)}}{2g}$$

$$\therefore v \sin.2\theta + \sqrt{v^2 \sin.^2 2\theta + 4ga\,(1 + \cos.2\theta)} = \max.$$

by the question; which gives by the rule, after proper reductions,

$$\cos.2\theta = \frac{-v^2 + \sqrt{v^4 + 4ag(v^2 + ag)}}{v^2 + ag} = \frac{2ag}{v^2 + ag}$$

determining the direction required.

Let the point of projection be situated in the horizon; then $a = 0$ and $\cos.2\theta = 0 = \cos.90^\circ$.

$$\therefore \theta = 45^\circ.$$

376. Making the diameter ($2r$) of the circle horizontal, we have (370) and by the question,

$$y = x \tan. 45 - \frac{gx^2}{2v^2 \cos.^2 45}$$

$$= x - \frac{gx^2}{v^2}$$

But when $x = r$ (by question) $y = 0$, $\therefore$

$$0 = r - \frac{gr^2}{v^2}$$

which gives

$$v^2 = rg = 2 \cdot g \cdot \frac{r}{2}$$

or the velocity of projection is that acquired down $\frac{1}{2} r$.

Also (by 370) the greatest ordinate corresponding abscissa, and parameter of the parabola, are respectively

$$\frac{v^2}{4g} = \frac{r}{4}, \frac{v^2}{2g} = \frac{r}{2}, \text{ and } \frac{v^2}{g} = r.$$

377. The greatest ordinates and corresponding abscissæ of the parabolas described are (370,) if 2θ and θ are the angles of projections,

$$\left.\begin{aligned} y &= \frac{v^2}{2g} \sin.^2 2\theta \\ y' &= \frac{v^2}{2g} \sin.^2 \theta \end{aligned}\right\} \text{and} \left\{\begin{aligned} x &= \frac{v^2}{2g} \sin. 4\theta \\ x' &= \frac{v^2}{2g} \times \sin. 2\theta \end{aligned}\right\}$$

But by the question $x' = x$, $\therefore \cos. 2\theta = \frac{1}{2}$,

$\therefore \theta = 30°$. Hence, the areas A, A′, are

$$A = \frac{4}{3} y \times x = \frac{v^4 \sqrt{3}}{8}, A' = \frac{4}{3} y' \times x' = \frac{v^4 \sqrt{3}}{8}$$

$\therefore A : A' :: 3 : 1$.

378. Here $\theta = 60°$, $x =$ side of an equilateral $\Delta = a$, when $y = 0$. Consequently (370)

$$0 = a \tan. 60° - \frac{a^2 g}{2v^2 \cos.^2 60°}$$

$$= a\sqrt{3} - \frac{2a^2 g}{v^2}$$

which gives

$$v = \sqrt{\frac{2ag}{\sqrt{3}}}$$

the required velocity.

Also $a = v \cos. 60^\circ \times t = \frac{v}{2} t$

$$\therefore t = \sqrt{\frac{2a\sqrt{3}}{g}}.$$

379. The velocity with which the bodies quit the plane is given by

$$v^2 = V^2 - 2gl. \sin. 30^\circ$$
$$= 80^2 - 50g.$$

Hence (370) the equation to the tragectory is

$$y = x \tan 30^\circ - \frac{x^2 g}{2v^2 \cos.^2 30^\circ}$$
$$= \frac{x}{\sqrt{3}} - \frac{2gx^2}{3v^2},$$

which is a parabola whose greatest ordinate, corresponding abscissa and parameter are (370)

$$\frac{v^2}{3g}, \frac{v^2\sqrt{3}}{4g} \text{ and } \frac{3v^2}{2g}.$$

380. Let r be the radius of the ⊙ ; then the velocity (v) acquired down it is

$$v = \sqrt{2gr}$$

and the time

$$t = \sqrt{\frac{2}{gr}}$$

Now θ being the inclination of the plane to the vertical, we have

$$s = tv \pm \frac{g \cos. \theta}{2} t^2$$

according as the body is projected downwards or upwards.

$$\therefore\ s = 2r \pm r \cos.\ \theta$$
$$= r + r\,(1 \pm \cos.\ \theta).$$

Let the negative sign be taken, then

$$s = r + r \text{ vers. } \theta \qquad \text{Q. E. D.}$$

381. Let v, v' be the required velocities of projection: then, generally, s, s', being the spaces described in the time t along the planes whose inclinations to the horizon are θ, θ', with these velocities we have

$$s = tv \pm \frac{g}{2} \sin.\ \theta \times t^2$$

$$s' = tv' \pm \frac{g}{2} \sin.\ \theta' \times t^2$$

according as the bodies are projected downwards or upwards.

Hence we easily get

$$t = \frac{s' \sin.\ \theta - s \sin.\ \theta'}{v' \sin.\ \theta - v \sin.\ \theta'} = \frac{\mp v + \sqrt{v^2 \pm 2sg \sin.\ \theta}}{g \sin.\ \theta}$$

which, by reduction, gives

$$2v' = v \times \left(\frac{s'}{s} + \frac{\sin.\theta'}{\sin.\theta}\right) + \left(\frac{s'}{s} - \frac{\sin.\theta'}{\sin.\theta}\right)\sqrt{v^2 \pm 2sg \sin.\ \theta}.$$

From this result it appears that the ratio $\frac{v'}{v}$ may be obtained, when $\frac{s'}{s}$, $\frac{\sin.\ \theta'}{\sin.\ \theta}$ are given, only in the case of $\frac{s'}{s} = \frac{\sin.\ \theta'}{\sin.\ \theta}$.

Hence, since $\frac{\sin.30^\circ}{\sin.45^\circ} = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$ does not equal $\frac{\sqrt{3}}{\sqrt{2}}$, and $\frac{\sin.60^\circ}{\sin.45^\circ}$ does $= \frac{\sqrt{3}}{\sqrt{2}}$, we conclude that Mr. Woodhouse made a corresponding oversight in the enunciation. When $\frac{s'}{s} = \frac{\sin.\ \theta'}{\sin.\ \theta}$, we have

$$v' = v\,.\,\frac{s'}{s} \text{ or } = v\,.\,\frac{\sin.\ \theta'}{\sin.\ \theta}.$$

$$\therefore\ \frac{v}{v'} = \frac{\sin.45^\circ}{\sin.60^\circ} = \frac{\sqrt{2}}{\sqrt{3}} \text{ the ratio required.}$$

382. Let the inclination of the plane, passing through the centre of the circle, and $\perp$ tangent at its extremity, be θ; then s being the distance of the point of projection from the tangent, v the velocity of projection, v' that with which the body is reflected by the tangent plane, t the time from the hand to the tangent, and t' that from the tangent to the hand, we easily obtain

$$s = tv - \frac{g}{2} \sin.\theta \times t^2$$

$$s = t'v' + \frac{g}{2} \sin.\theta \times t'^2 \text{ which give}$$

$$t = \frac{v - \sqrt{v^2 - 2gs \sin.\theta}}{g \sin.\theta} \text{ or } = \frac{2s}{v + \sqrt{v^2 - 2gs \sin.\theta}}$$

$$t' = \frac{-v' + \sqrt{v'^2 + 2gs \sin.\theta}}{g \sin.\theta} \text{ or } = \frac{2s}{v' + \sqrt{v'^2 + 2gs \sin.\theta}}$$

Hence,

$$t + t' = \frac{v - \sqrt{v - 2gs \sin.\theta} - v' + \sqrt{v'^2 + 2gs \sin.\theta}}{g \sin.\theta}$$

But v' ($=$ velocity with which the body strikes the tangent plane) $= \sqrt{v^2 - 2gs \sin.\theta}$.

Hence $t + t' = \dfrac{2(v - \sqrt{v^2 - 2gs \sin.\theta}}{g \sin.\theta}$

$$= \frac{2r}{v} \text{ by the question.}$$

Hence $r - s = \dfrac{gr}{2v^2} . \sin.\theta$;

and making $r - s$ the radius vector (since it is the man's distance from the centre of the circle), the polar equation to his locus, is

$$\rho = \frac{gr}{2v^2} . \sin.\theta.$$

If it were required to determine the curve by *rectangular coordinatès*, we have

$$\rho = \sqrt{x^2 + y^2}, \sin.\theta = \frac{x}{\sqrt{x^2 + y^2}}$$

x being vertical; which give

$$y^2 = \frac{gr}{2v^2} x - x^2$$

the equation to a circle whose radius is $\dfrac{gr}{4v^2}$.

If it were required to determine the curve by *rectangular co-ordinates*, we have $s = r + \sqrt{x^2 + y^2}$, $\sin.\ \theta = \frac{y}{\sqrt{x^2 + y^2}}$, which being substituted would give the equation.

In both cases the equation will be very complex. It would be rather laborious than difficult to rationalize these results.

383. Let a be the altitude through which the body falls to acquire the velocity of projection v, and suppose that after describing with this velocity the arc s of the trajectory, it is reflected by a hard normal plane, at the point whose vertical and horizontal co-ordinates, originating in the point of projection, are (y, x); then it being again projected by the intervention of this plane with a certain velocity v', that plane (by the question) must be so situated as to reflect the body precisely to a point whose co-ordinates $(-y', x')$ measured from the point of reflection are $a - y$, x.

Now, by (370), we have

$$- y = x \tan.\ 0 - \frac{gx^2}{2v^2 \cos.^2 0} = - \frac{gx^2}{2v^2}$$

$$\therefore y = \frac{gx^2}{2v^2} = \frac{x^2}{4a} \text{ ....(1)}$$

the equation to the first *parabola*.

Again, since the horizontal velocity of a projectile is uniform, we have (t being the time of describing x with the velocity v).

$$x = tv, \text{ or } t = \frac{x}{v}.$$

Hence, and by equation (1), since the body is reflected with *half* the velocity of compact;

$$2v' = \frac{ds}{dt} = \frac{dx}{dt}\sqrt{1 + \frac{dy^2}{dx^2}} = v\sqrt{1 + \frac{x^2}{4a^2}}.$$

Also, since the body, after reflection, is projected in the direction of the tangent at the point (x, y), we have the inclination (θ) of that direction to the horizon expressed by

$$\tan.\ \theta = \frac{dy}{dx} = \frac{x}{2a}.$$

Hence, we get

$$-y' = x' \tan.\theta - \frac{gx'^2}{2v'^2 \cos.^2\theta}.$$

or $y - a = \frac{x^2}{2a} - \frac{x^2}{a}$, which gives by means of (1),

$$x = \frac{2a}{\sqrt{3}}, \text{ and } y = \frac{a}{3} \ldots\ldots(2)$$

$$\tan.\theta = \frac{1}{\sqrt{3}} = \tan. 30^\circ, \ldots\ldots(3)$$

$$\text{and } v' = \frac{v}{2}\sqrt{1+\frac{1}{3}} = \frac{v}{\sqrt{3}} = \sqrt{\frac{2ga}{3}} \ldots\ldots(4)$$

and ∴ the equation to the second parabola

$$y' = \frac{x'}{\sqrt{3}} - \frac{x'^2}{a} \ldots\ldots(5)$$

Hence, since θ is also the inclination of the normal plane to the vertical, we readily obtain its equation, referred to the point of projection by y'', x'', viz.,

$$y'' = \frac{7a}{3} - x'' \sqrt{3} \ldots\ldots(6)$$

determining the position required.

Again, t, t_1, t_2 being the times of describing a from rest, and x with the uniform horizontal velocities v, v', we have

$$t = \sqrt{\frac{2a}{g}},\ t_1 = \frac{x}{v} = \sqrt{\frac{2a}{3g}},\ t_2 = \frac{x}{v'} = \sqrt{\frac{2a}{g}}$$

∴ the whole time of motion is

$$t + t_1 + t_2 = \sqrt{\frac{2a}{g}} \times \left(2 + \frac{1}{\sqrt{3}}\right).$$

384. By (374) the equation to the locus of the centre of gravity is

$$y'' = x'' \tan. 45^\circ - \frac{g \times 49}{2 \times 49 \cos.^2 45^\circ \times 16} \times x''^2$$

$$= x'' - \frac{g}{16} \times x''^2.$$

the equation to a parabola whose parameter and greatest ordinate are $\frac{16}{g}$, and $\frac{4}{g}$ respectively.

385. By (370) the equation to the trajectory is

$$y = x \tan. 30^\circ - \frac{gx^2}{2 \times (193)^2 \times \cos.^2 30^\circ}$$

$$= \frac{x}{\sqrt{3}} - \frac{2gx^2}{3(193)^2}.$$

And that of the plane referred also to the point of projection, is

$$y' = x' \sqrt{3} - a.$$

where a is the distance of the point of projection from the plane's intersection with the vertical.

Now, when the body strikes the plane, we have, $x = x'$, $y = y'$, which give

$$x^2 + \frac{(193)^2 \sqrt{3}}{g} x = \frac{3a\,(193)^2}{2g}.$$

And since $x = tv = 193t$, we have

$$t^2 + \frac{\sqrt{3}}{g} \times 193 \times t = \frac{3a}{2g}$$

by the resolution of which the time required may be found.

386. Let θ be the $\angle$ required, and a the given velocity; then (370) the greatest ordinate and corresponding abscissa being

$$y = \frac{a^2}{2g} \sin.^2 \theta, \text{ and } x = \frac{a^2}{2g} \times \sin. 2\theta,$$

we have

$$\frac{4}{3} yx = \text{area of parabola} = \frac{a^4}{3g^2} \sin.^3 \theta.$$

Hence, by the question,

$$\sin.^2 \theta \times \sin. 2\theta = \text{max.} = u$$

which, by putting $\frac{du}{dx} = 0$, gives

$$\cos. \theta = \frac{1}{2}, \text{ or } \theta = 60^\circ$$

387. Let x be the distance of the point of concourse from the bottom of the vertical line; then

$$L - x = \frac{g}{2} t^2$$

$$x = tv - \frac{g}{2} t^2$$

whence $t = \sqrt{\frac{L}{3g}}$, and $x = \frac{5L}{6}$

which determine the point required.

388. By (370) we have generally

$$y = x \tan.\theta - \frac{gx^2}{2v^2 \cos.^2\theta}$$

But here y is negative, $\theta = 0$, and $v^2 = 2ga$ (a being the altitude of the tower).

$$\therefore\ y = \frac{x^2}{4a},$$

the equation to the trajectory.

Let $y = a$, then $x = 2a$.

389. Generally, if A, B; a, b; θ, ϕ denote the weights of the bodies, the velocities of projection, and directions respectively, then we have for the time (t), (x, y; x', y'; x'', y'') denoting the horizontal and vertical co-ordinates of the trajectories described by the projectiles and their centre of gravity,

$$x = ta \times \cos.\theta,\ x = tb \times \cos.\phi$$

$$y = ta \times \sin.\theta - \frac{g}{2}.\ t^2,\ y' = tb \times \sin.\phi - \frac{g}{2}.\ t^2$$

$$\text{Hence } x' = \frac{b}{a}\,\frac{\cos.\phi}{\cos.\theta} \ldots\ldots(1)$$

$$\left.\begin{aligned} y &= x \tan.\theta - \frac{gx^2}{2a^2 \cos.^2\theta} \\ y' &= x' \tan.\phi - \frac{gx'^2}{2b^2 \cos.^2\phi} \end{aligned}\right\} \ldots\ldots(2)$$

Also, since (b being $> a$)

$y'' - y : y' - y :: B : A + B$

and $x'' - x : x' - x :: B : A + B$

we have

$$\left.\begin{array}{l} y''(A + B) = y'B + yA \\ x''(A + B) = x'B + xA \end{array}\right\} \ldots\ldots(3)$$

And eliminating x, x', y', y' from the above five equations (1), (2), (3), we finally obtain

$$y'' = \frac{aA \sin.\theta + bB \sin.\phi}{aA \cos.\theta + bB \cos.\phi} . x'' - \frac{g(A + B)^2 x''^2}{2(aA \cos.\theta + bB \cos.\phi)^2} \ldots(4)$$

the equation to a parabola, whose greatest ordinate, corresponding abscissa, and parameter, are easily found to be

$$\frac{(aA \sin.\theta + bB \sin.\phi)^2}{2g(A + B)^2}, (aA \sin.\theta + bB \sin.\phi)$$

$$\times (aA \cos.\theta + bB \cos.\phi) \times \frac{1}{g(A + B)^2} \text{ and}$$

$$\frac{2(aA \cos.\theta + bB \cos.\phi)^2}{g(A + B)^2} \text{ respectively.}$$

No. (374) is but a particular case of this.

In the problem, $A = 2, a = 20, \theta = 60°$; $B = 3, b = 25$, and $\phi = 30°$, whence it is easy to find the required quantities.

390. Since the horizontal velocities of both *sound* and the *projectile* are constant, and describe (by supposition) the same space in the same time, these velocities must be equal ($s = tv$). Hence, a being that of sound, we have, by the question, and by (370)

$$y = 0 = x \tan.\theta - \frac{gx^2}{2a^2 \cos.^2\theta}$$

$$\therefore x = 0, \text{ or } = \frac{a^2}{g} \times \sin. 2\theta \text{ the range required.}$$

391. I e a be the height of the tow . by (370) we get

$$-a = \frac{a}{2} \tan. \theta - \frac{ga^2}{8v^2 \cos.^2\theta}$$

$$\therefore v^2 = \frac{ga}{8}$$

and $v = \sqrt{\frac{ga}{8}}$ the velocity required.

392. By (370) we have

$$y = x \tan.\theta - \frac{gx^2}{2v^2 \cos.^2\theta}$$

where, by the question, $y = 0, \theta = 15, x = 100$ feet,

Hence, substituting, &c., we get

$$\frac{v^2}{2g} = 100 \text{ feet, height required.}$$

393. Generally, we have

$$s = tv - \frac{g}{2}\frac{\text{H}}{\text{L}} \times t^2$$

since the space described with the velocity v continued uniform would be tv, and that through which the body would descend down the plane $\frac{g}{2}.\frac{\text{H}}{\text{L}}.t^2$. But by the question, $v = 30$ feet, $\frac{\text{H}}{\text{L}} = \frac{1}{6}, t = 12''$.

$\therefore s = 360 - 12g = -26$ feet nearly, or the body will be lower down the plane by 26 feet than the point of projection.

Hence the velocity (u) of the body after $12''$ will by $30 +$ that acquired by falling through 26 feet of the plane.

$$\therefore u = 30 + g \times 12 \times \frac{1}{6} = 94\frac{1}{3} \text{ feet.}$$

394. Here

$$u = gt \times \frac{\text{H}}{\text{L}} = \frac{1}{10}gt = 30$$

$$\therefore t = \frac{300}{g} = \frac{1800}{193} = 9''\frac{63}{193} \text{nearly.}$$

395. By (370) we have

$$y = x \tan. \theta - \frac{gx^2}{2v^2 \cos.^2 \theta}$$

where, by the question,

$\theta = 45°, v = 50$ feet;

$$\therefore \text{ when } y = 0, x = \frac{2500}{g} = \frac{15000}{193} \text{ feet nearly.}$$

396. By (370) and the question, we have

$$0 = y = 150 \tan. 30° - \frac{g(150)^2}{2v^2 \cos.^2 30°},$$

which gives

$$v = \frac{1}{10} \sqrt{\frac{\sqrt{3}}{g}}$$

397. By (370) and the question,

$$y = x \tan. 60° - \frac{gx^2}{2(100)^2 \cos.^2 60°}$$

$$= x\sqrt{3} - \frac{gx^2}{5000}.$$

Hence, if b be the distance of the foot of the inclined plane from the point of projection, (since the equation to the section of that plane made by the plane of motion is $y' = (x' - b) \tan. 30° = (x' - b)\frac{1}{\sqrt{3}}$), by making $y = y'$, $x = x'$, we have

$$x\sqrt{3} - \frac{gx^2}{5000} = (x - b)\frac{1}{\sqrt{3}}$$

$$\therefore x = 50\sqrt{\frac{2b}{g\sqrt{3}}}$$

the horizontal range.

Hence, the time required is

$$t = \frac{x}{u} = \frac{x}{100 \cos. 60°} = \sqrt{\frac{2b}{g\sqrt{3}}}.$$

Again, the greatest altitude above the horizon is (370)

$y = \frac{v^2}{2g} \sin.^2 \theta = \frac{(100) \times 3}{8g} = \frac{3750}{g}$ feet, whence it is easy to find that above the plane.

398. By (389) the greatest altitude of the centre of gravity of A, B, projected with the velocities a, b, is generally,

$$\frac{(aA \sin. \theta + bB \sin. \varphi)^2}{2g\,(A + B)^2}.$$

Here $\theta = \varphi = 90^\circ$; $\therefore$ the greatest altitude required is

$$\frac{(aA + bB)^2}{2g(A + B)^2}.$$

399. By (370) the greatest altitude is

$$y = \frac{v^2}{2g} \sin.^2 \theta$$

Also

$$x = \frac{v^2}{2g} \sin. 2\theta$$

Consequently, if

$$y' = ax' + b$$

be the equation, referred to the same origin, of the section of the inclined plane, made by the plane of motion, we have

$$y - y' = \frac{v^2}{2g} (\sin.^2 \theta - a \sin. 2\theta) - b$$

for the greatest altitude required.

400. Making the intersection of the horizon with the bank, the line of abscissæ (x) (originating in the point of projection), and taking the ordinates (y) in the plane of the bank, it is evident that the motion parallel to (x) will be uniform, and that parallel to (y) the same as down the *common inclined plane*, whose inclination (α) is the same as that of the bank. Hence, (v) being the velocity of projection, and θ the $\angle$ which its direction makes with the line of intersection, we have,

$$x = v \cos. \theta \times t$$

$$y = v \sin. \theta \times t - \frac{g}{2} \sin. \alpha \times t^2$$

$$= x \tan. \theta - \frac{g \sin. \alpha \times x^2}{2v^2 \cos.^2 \theta}$$

the equation to a parabola.

The greatest ordinate, corresponding abscissa, and parameter, are easily found to be

$$y = \frac{v^2}{2g} \cdot \frac{\sin.^2 \theta}{\sin. \alpha}, \; x = \frac{v^2}{2g} \cdot \frac{\sin. 2\theta}{\sin. \alpha},$$

$$p = \frac{2v^2}{g} \cdot \frac{\cos.^2 \theta}{\sin. \alpha}.$$

The place where the body meets the horizon is determined by putting $y = 0$, which gives

$$y = \frac{v^2}{g} \cdot \frac{\sin. 2\theta}{\sin. \alpha}.$$

401. Let AB = BC (Fig. 89.) represent the equal times of descent and ascent; Aa, Cc the initial velocities, and Bb, BD the ultimate velocities; then the spaces described (*Wood*, p. 189,) are denoted by

ABba, BCcD.

But, by the question, the spaces described are equal, and aDb = DEc (being the spaces due to gravity). Hence

ABDa = BCED; and since

AB = BC, ∴ Aa = BD = CE.

∴ $Cc - Aa = Ec$ = velocity acquired by falling vertically in the time AB.

402. Let a, b, be the altitudes fallen through by A, B, to acquire the velocities u, v; then

$$u = \sqrt{2ga}, \quad v = \sqrt{2gb}$$

Hence, x being the horizontal distance moved, we have

$$t = \sqrt{\frac{2a}{g}} + \frac{x}{\sqrt{2ga}} = \sqrt{\frac{2b}{g}} + \frac{x}{\sqrt{2gb}}$$

$$\therefore\ x = 2\sqrt{ab}$$

$\therefore$ the time of B's uniform motion is

$$\frac{x}{v} = \frac{2\sqrt{ab}}{\sqrt{2gb}} = \sqrt{\frac{2a}{g}} = \text{the time of A's descent.}$$

403. Let θ be the inclination of the line to the horizon, then the time down it is

$$a = \frac{g}{2} t^2 \sin.\ \theta$$

$$\therefore\ t^2 = \frac{2a}{g \sin.\ \theta} = 2 \times \frac{2a}{g}$$

by the question. Hence

$$\sin.\ \theta = \frac{1}{2} = \sin.\ 30^\circ.$$

$$\therefore\ \theta = 30^\circ.$$

404. By 370, we have

$$y = x \tan.\ 60^\circ - \frac{9x^2}{2v^2 \cos.^2 60^\circ}$$

$$= x\sqrt{3} - \frac{2gx^2}{v^2}$$

But by the question,

$$x = 300 \times \cos.\ 30^\circ = 150\sqrt{3}$$

$$\text{and } y = 300 \times \sin.\ 30^\circ = 150$$

$$\text{whence } v = 15\sqrt{2g}.$$

Hence the greatest altitude above the plane is easily found, by the rules of max. and min., to be 75 feet; and the time of flight

$$= \frac{150\sqrt{3}}{v \cos.\ 60} = 10\sqrt{\frac{6}{g}}.$$

405. Let α be the inclination of the given plane to the horizon, a the vertical distance between the given point and the plane; also suppose the co-ordinates (x, y) of the required locus

to originate in the intersection of this vertical with the plane, x' being measured $\perp$ and y parallel to the intersection of the plane with the horizon; then (x', y') being the horizontal and vertical co-ordinates of the path, described by the body projected parallel to the horizon, we easily obtain

$$y' = a - x \sin. \alpha$$
$$x'^2 = y^2 + x^2 \cos.^2 \alpha$$

And by 370,

$$y' = \frac{gx'^2}{2v^2}.$$

Hence

$$y^2 = \frac{2av^2}{g} - \frac{2v^2 \sin. \alpha}{g} x - x^2 \cos.^2 \alpha$$

the equation to an ellipse, whose transverse and conjugate axes are $\frac{2v}{g \cos. \alpha} \sqrt{2ag + v^2 \tan.^2 \alpha}$, and $\frac{2v}{g} \sqrt{2ag + v^2 \tan.^2 \alpha}$ respectively.

406. Let a be the greatest range; then (370)

$$\frac{a}{2} = \frac{v^2}{2g} \sin.^2 \theta$$

But the range is greatest when $\theta = 45°$.

$$\therefore v^2 = ga.$$

Hence, by the question,

$$d = x - \frac{x^2}{a}$$

$$\therefore x = \frac{a \pm \sqrt{a^2 - 4ad}}{2}$$

407. By (370.)

$$x = tv \cos. \theta$$
$$y = tv \sin. \theta - \frac{g}{2} t^2$$

Whence

$$\sin. \theta = \frac{\sqrt{t^2 v^2 - x^2}}{tv}$$

$$\therefore y^2 + gt^2 \times y = \frac{t^2}{4} (4v^2 - g^2 t^2) - x^2$$

the equation to a circle, whose radius is tv, and the co-ordinates of whose centre are

$$0 \text{ and } tv - \frac{gt^2}{2}.$$

408. Let p be the parameter of the given parabola, whose equation is

$$x^2 = py$$

x, and y being the horizontal and vertical co-ordinates, originating in the vertex; then, supposing (a, b), (a', b') the co-ordinates of the given points of projection, the $\angle\ (\alpha, \alpha')$ of projection, or the inclinations of the tangents to the parabola, at those points are $\left(\frac{dy}{dx} = \tan.\ \theta\right)$ determined by

$$\tan.\ \alpha = \frac{2a}{p}, \tan.\ \alpha' = \frac{2a'}{p}.$$

Hence (370)

$$b = \frac{v^2}{2g} \sin.^2\alpha,\ b' = \frac{v'^2}{2g} \sin.^2\alpha'$$

$$\text{And } \therefore\ v^2 = 2gb\,(1 + \cos.^2\alpha) = 2gb\left(1 + \frac{p^2}{4a^2}\right)$$

$$v'^2 = 2gb'\,(1 + \cos.^2\alpha') = 2gb'\left(1 + \frac{p^2}{4a'^2}\right).$$

Also, at the point of concourse,

$$a - x = tv \cos.\ \alpha,\ a + x = tv' \cos.\ \alpha'$$

$$b - y = tv \sin.\ \alpha - \frac{g}{2}t^2,\ b' - y = tv' \sin.\ \alpha' - \frac{g}{2}t^2$$

$$\therefore\ t = \frac{b - b'}{v \sin.\ \alpha - v' \sin.\ \alpha'} = \frac{\sqrt{b} + \sqrt{b'}}{\sqrt{2g}}$$

$$\therefore\ x = \frac{t}{\text{vers.}\ \alpha} = \frac{\sqrt{p}}{g}(\sqrt{v} + \sqrt{v'})$$

$$\text{And } y = \frac{x^2}{p} = \frac{1}{g^2}(\sqrt{v} + \sqrt{v'})^2$$

the co-ordinates of the point required.

409. Let m denote the degree of elasticity; then, θ, ϕ, χ, being the $\angle$ of reflection made with the radii, we have, (*Wood*, p. 131.)

$$\tan.\,\theta = m \tan.\,\phi$$
$$\tan.\,\phi = m \tan.\,\chi$$
$$\text{and } \theta + \phi + \chi = \frac{\pi}{2} \text{ (radius of } \odot = 1)$$

$$\text{Hence } \tan.^2\theta = \frac{m^3}{m^2 + m + 1}$$

which gives the $\angle$ required.

410. Let θ be the inclination of the plane AB to the horizon, a, b the horizontal and vertical co-ordinates of the point A referred to the point B; then, v denoting the velocity at B, or that with which the body is reflected by the vertical plane in a direction inclined to the horizon by the $\angle\theta$, the trajectory described, referred by (x, y) to B, will be (370).

$$y = x \tan.\,\theta - \frac{gx^2}{2v^2 \cos.^2\theta}.$$

But, by the question, the body strikes the plane again in A; $\therefore$

$$-b = a \tan.\,\theta - \frac{ga^2}{2v^2 \cos.^2\theta}$$

$$\therefore v = \frac{a}{\cos.\,\theta}\sqrt{\frac{g}{2}} \times \frac{1}{\sqrt{a \tan.\theta + b}}$$

But $b = a \tan.\,\theta$, $a = \sqrt{a^2 + b^2} \times \cos.\,\theta$.

$$\therefore v = \frac{\sqrt{g}}{2} \times \sqrt{\frac{a^2 + b^2}{b}} \text{ or } = \sqrt{\frac{ag}{2\sin.2\theta}}.$$

411. Let

v, v_1, v_2.....denote the velocities of projection after each rebound.
θ, θ_1, θ_2......the elevations of projection.
t, t_1, t_2.......the times of flight.
r, r_1, r_2.......the horizontal ranges.
h, h_1, h_2......the greatest altitudes.
a, a_1, a_2......the areas of the parabolas.

Then, since the velocity of incidence on the horizontal plane is = that of the reflection immediately preceding, and (*Wood*, p. 181.)

$$v_1 = \frac{\cos.\theta}{\cos.\theta_1} v,\ v_2 = \frac{\cos.\theta_1}{\cos.\theta_2} v_1, \&c. = \&c.$$

and $\tan.\theta_1 = n \tan.\theta$, $\tan.\theta_2 = n \tan.\theta_1$, &c. = &c.

$\therefore$ (370)

$r = \frac{v^2}{2g}\sin.2\theta$, $r_1 = \frac{v_1^{\ 2}}{2g}\sin.\ 2\theta$, &c., become by substitution, and reduction.

$$\left.\begin{aligned} r_1 &= nr \\ r_2 &= nr_1 = n^2 r \\ r_3 &= nr_2 = n^3 r \\ \&c. &= \&c. \end{aligned}\right\} \therefore r + r_1 + \ldots\infty = \frac{r}{1-n}$$

Hence also $t = \frac{r}{v\cos.\theta}$, $t_1 = \frac{r_1}{v_1\cos.\theta}$, &c., become

$$\left.\begin{aligned} t_1 &= nt, \\ t_2 &= n^2 t \\ t_3 &= n^3 t \\ \&c. &= \&c. \end{aligned}\right\} \text{and } \therefore t + t_1 + \ldots\infty = \frac{t}{1-n}$$

Moreover,

$$h = \frac{v^2}{2g}\sin.^2\theta,\ h_1 = \frac{v_1^{\ 2}}{2g}\sin.^2\theta_1, \text{ \&c., become}$$

$$\left.\begin{aligned} h_1 &= n^2 h \\ h_2 &= n^4 h \\ \&c. &= \&c. \end{aligned}\right\} \text{and } \therefore h + h_1 + \ldots\infty = \frac{h}{1-n^2}$$

Hence, $a = \frac{2}{3} rh$, $a_1 = \frac{2}{3} r_1 h_1$, &c., give

$$\left.\begin{aligned} a_1 &= n^3 a \\ a_2 &= n^6 a \\ a_3 &= n^9 a \\ \&c. &= \&c. \end{aligned}\right\} \text{and } \therefore a + a_1 + \ldots\infty = \frac{a}{1-n^3}$$

Also, since the parameter $p_m = \frac{r_m^2}{4h_m}, = \frac{n^{2m}r^2}{4n^{2m}h} = \frac{r^2}{4h} = p$,

the parabolas have all the same parameter.

412. Let v be the velocity of projection, v' that after the body has been in motion t'', and u the velocity due to its altitude above the horizontal plane after such interval; then, s being the arc described, since

$$v' = \frac{ds}{dt} = \frac{dx}{dt}\sqrt{1+\frac{dy^2}{dx^2}}$$

$$y = x\tan.\theta - \frac{gx^2}{v^2\cos.^2\theta}\ (370).$$

and $x = tv\cos.\theta$

we obtain

$$\frac{dx}{dt} = v\cos.\theta, \frac{dy}{dx} = \tan.\theta - \frac{gx}{v^2\cos.^2\theta}$$

$$\text{and} \therefore v'^2 = v^2 + \frac{g^2x^2}{v^2\cos.^2\theta} - 2gx\tan.\theta$$

$$= v^2 - 2gy$$

$$= v^2 - u^2. \quad \text{Q. E. D.}$$

413. By 370, and the question the equation to the trajectory, is

$$y = \frac{x}{\sqrt{3}} - \frac{2gx^2}{3v^2};$$

$\therefore$ putting $y = 0$, the horizontal range is

$\frac{v^2\sqrt{3}}{2g}$, and consequently the co-ordinates of the point where the projectile strikes the sonorous body are,

$\frac{v^2}{g\sqrt{3}}$, and $\frac{v^2}{9g}$, and its distance from the point of projection is therefore

$$\sqrt{\frac{v^4}{3g^2} + \frac{v^4}{81g^2}} = \frac{2v^2}{9g}\sqrt{7}.$$

Now, let a denote the velocity of sound as determined by experiment; then the time of its moving uniformly through $\frac{2v^2}{9g}\sqrt{7}$

is $\frac{2v^2}{9ag}\sqrt{7}$

Also the time in which the projectile reaches the sonorous body is that of describing $\frac{v^2}{g\sqrt{3}}$ with the *uniform* velocity v cos. 30°, or

$\frac{2v}{3g}$

Hence, m being the whole time given, we have

$$m = \frac{2v}{3g} + \frac{2v^2}{9ag}\sqrt{7}.$$

which gives

$$v = -\frac{3a}{2\sqrt{7}} + \frac{3\sqrt{a^2+2agm\sqrt{7}}}{2\sqrt{7}}$$

the velocity required.

414. By (370)

$$y = x \tan. \theta - \frac{gx^2}{2v^2 \cos.^2 \theta}$$

and by the question, $y = 80$ feet, $x = 120$ feet.

$v^2 = 2g \times 80 = 160\,g$,

$\therefore$ by substitution and reduction

$$1 = 4 \tan. \theta - \frac{3}{2 \cos.^2\theta}$$

$$\therefore \tan.^2\theta - \frac{8}{3} \tan. \theta = -\frac{5}{3}$$

$$\therefore \tan. \theta = 1 \text{ or } \frac{5}{3}$$

$$\therefore \theta = 45^\circ \text{ or } = \tan.^{-1} \frac{5}{3}.$$

OSCILLATIONS.

415. Generally, if t be the time of an oscillation, and l the length of the pendulum, we have (see *Translat. of Venturoli*, p. 103, or *Wood*, Prop. 74.)

$$t = \pi \sqrt{\frac{l}{g}} = 1 \text{ by the question.}$$

$$\therefore g = l\pi^2$$

$$\text{And } s = \frac{g}{2} t^2 = \frac{\pi^2 lt^2}{2}$$

the space described in t''.

416. Let l be the length of the cycloidal pendulum; then the time of an oscillation is

$$t = \pi \sqrt{\frac{l}{g}}$$

and the time down the length of the pendulum

$$t_{,} = \sqrt{2 . \frac{l}{g}}$$

$$\therefore t : t_{,} :: \pi : \sqrt{2}$$

417. By 715, and the question

$$\frac{60}{180} = \pi \sqrt{\frac{l}{g}}$$

$$\therefore l = \frac{g}{9\pi^2} \text{ the required length.}$$

418. Since the moving force here is $A - B = 5 - 3 = 2$, the accelerating force $\varphi = \frac{A - B}{A + B} = \frac{1}{4}$

$$\therefore s = \frac{g}{2} \varphi t^2 = \frac{g}{8} t^2 = 1$$

by the question.

$$\text{Hence } t = 2\sqrt{\frac{2}{g}} = \pi\sqrt{\frac{l}{g}}$$

by the question.

$$\therefore l = \frac{8}{\pi^2} \text{ the length required.}$$

419. Since the arc of a cycloid = 2 × corresponding chord of the generating circle, and the tangent at any point is parallel to that chord, it easily appears that (Fig. in *Enunciation*)

$$\angle V' = 30^\circ$$

$\therefore$, l being the length of the semi-cycloid ABV, or of ABV′, we have the time down A′BV′ (t) expressed by

$$l = \frac{g}{2} \sin. V' \times t^2 = \frac{g}{4} t^2$$

$$\therefore t = 2\sqrt{\frac{l}{g}}$$

Now $t_{,}$, the time of $\frac{1}{2}$ an oscillation, is

$$t_{,} = \frac{\pi}{2}\sqrt{\frac{l}{g}}$$

$$\therefore t : t_{,} :: 4 : \pi$$

420. The inclination of the chord (= rad. of ⊙ and originating in lowest point) is evidently 30°, and the time down it is $\therefore$

$$t = \sqrt{\frac{2}{g} \cdot \frac{r}{\sin. 30^\circ}} = 2\sqrt{\frac{r}{g}}$$

Also the time of oscillation of a pendulum (r) is

$$t_{,} = \pi\sqrt{\frac{r}{g}}$$

$$\therefore t : t_{,} :: 2 : \pi$$

421. Let x be the length of the plane, l that of the $\frac{1}{2}$ cycloid; then, the elevation of the plane being 30° (see 719),

the time down it is $\sqrt{\frac{2x}{g \sin. 30^{\circ}}} = 2\sqrt{\frac{x}{g}}$; also the time of $\frac{1}{2}$ an oscillation is $\frac{\pi}{2}\sqrt{\frac{l}{g}}$.

Hence, by the question,

$$\frac{\pi}{2}\sqrt{\frac{l}{g}} = 2\sqrt{\frac{x}{g}}$$

And $x = \frac{\pi^2 l}{16}$ the length required.

422. The time (t) of an oscillation is $\pi\sqrt{\frac{l}{g}}$, and the ($t_{,}$) down the axis $\left(\frac{l}{2}\right)$ is $\sqrt{\frac{2}{g} \cdot \frac{l}{2}} = \sqrt{\frac{l}{g}}$. $\therefore$

$$t : t_{,} :: \pi : 1. \qquad \text{Q. E. D.}$$

Also, if l be the length of the common seconds' pendulum, we have

$$1 = \pi\sqrt{\frac{l}{g}}, \therefore \frac{g}{2} = \frac{\pi^2 l}{2}$$

the space through which a body would fall in 1″ by the force of gravity.

423. Let l be the length of the pendulum which loses or gains n'' in 24 hours, and $\therefore$ vibrates $24 \times 60 \times 60 \mp n$ times in a day; $l \mp x$ that of the common pendulum, which vibrates $24 \times 60 \times 60$ times a day. Then, since the number of vibrations for a given time $\propto$ inversely as the square roots of the lengths, we have $24 \times 60 \times 60 \mp n : 24 \times 60 \times 60 :: \sqrt{l \mp x} : \sqrt{l}$

$$\therefore \mp x = \frac{ln}{24^2 \times 60^2 \times 60^2} \times (n \mp 48 \times 60 \times 60)$$

$$= \frac{ln}{(24 \, . \, 60 \, . \, 60)^2} \times (n \mp 172800)$$

the quantity by which the pendulum l ought to be shortened or lengthened, in order to beat true time.

424. Let $\frac{l}{2}$ be the axis of the cycloid, then the velocity of projection is

$$v = \sqrt{2g \cdot \frac{l}{2}} = \sqrt{lg}$$

and since (370)

$$y = x \tan. 45^\circ - \frac{gx^2}{2lg \cos.^2 45^\circ} = x - \frac{x^2}{l}$$

the horizontal range is $x = l$. Hence the time of flight is

$$t = \frac{l}{\sqrt{lg} \cos. 45^\circ} = \sqrt{\frac{2l}{g}}.$$

Again, the time of an oscillation is

$$t_{,} = \pi \sqrt{\frac{l}{g}}$$

$$\therefore t : t_{,} :: \sqrt{2} : \pi$$

425. Since, generally, the time of an oscillation is

$$t = \pi \sqrt{\frac{l}{g}}$$

$$l = g \frac{t^2}{\pi^2} = g \cdot \frac{60^2}{61^2 \pi^2} \text{ by the question.}$$

And $l' = \frac{g}{\pi^2}$

$$\therefore l' - l = \frac{g}{\pi^2 61^2} (61^2 - 60^2) = g \cdot \left(\frac{11}{61\pi}\right)^2.$$

426. Let l be the length of the pendulum; then $2l =$ length of the whole cycloid, and, since the time of an oscillation is

$$t = \pi \sqrt{\frac{l}{g}},$$

and that down $2l$ is

$$t_{,} = 2 \sqrt{\frac{l}{g}}$$

we have

$$t : t_{,} :: \pi : 2.$$

427. Let the body descend from rest through the arc FA (Fig. 90); then the greatest velocity will be at the lowest point A;

and, since $s \propto = v^2$, if BR $= \frac{1}{4}$ BA, the velocity of a body descending from rest through BA at R will $= \frac{1}{2}$ velocity at A. Hence, through R drawing RM $\perp$ AB, and meeting the curve in M, the point M is determined in which the velocity of the body descending down the cycloid $= \frac{1}{2}$ the greatest velocity.

428. Since generally, $v^2 = 2gs$, $\frac{1}{2}$ the velocity due to $2r$ (r being the radius of the generating $\odot$ of the cycloid), is

$$\frac{1}{2}\sqrt{4gr} = \sqrt{gr}.$$

Now the velocity V of the moving point in the cycloid : velocity of the moving point in the circle :: $ds : dz$, s and z denoting the cycloidal and circular arcs respectively.

But $s = 4r - 2$ chord of supp. of z

$$= 4r - \frac{4 \cos. \frac{z}{2}}{r}$$

$$\therefore \frac{V}{\sqrt{gr}} = \frac{ds}{dz} = \frac{2 \sin. \frac{z}{2}}{r} = \frac{2}{r}\sqrt{\frac{r^2 - r \cos. z}{2}}$$

$$= \sqrt{\frac{2(r - \cos. z)}{r}} = \sqrt{\frac{2(2r - x)}{r}}$$

x being the abscissa corresponding to s, measured from the vertex along the axis of the cycloid.

Hence $V = \sqrt{2g(2r - x)}$ = velocity due to $2r - x$ = that of a body oscillating in the cycloid. (*Venturoli*, p. 102.)

429. In the generating $\odot$ AB (Fig. 90,) inflect AP = its radius, and draw R'PM' $\perp$ AB; then since AM' = 2AP = AB $= \frac{1}{2}$ AF, AF is bisected in M'.

Now (*Translat. of Venturoli*, p. 103.) the time through any arc of the cycloid, whose abscissa, measured from the lowest point is x, is

$$t = \sqrt{\frac{d}{2g}} \cos.^{-1} \frac{2x - d}{d}$$

$$\text{Let } \therefore x = \text{AR} = \sqrt{\frac{d^2}{4} - \text{PR}^2} = \frac{d}{4}.$$

$$\text{Then } t = \sqrt{\frac{d}{2g}} \cos.^{-1} \left(-\frac{1}{2}\right) = \sqrt{\frac{d}{2g}} \times 120^\circ$$

But when $x = 0$, the time ($t_{,}$) down FA is

$$t_{,} = \sqrt{\frac{d}{2g}} \cos.^{-1} (-1) = \sqrt{\frac{d}{2g}} \times 180^\circ,$$

$$\therefore t : 2t_{,} :: 1 : 3.$$

430. In *Newton's Construction* (Prop. 50, Princep.), we have

$$\text{CA} : \text{CO} :: \text{CO} : \text{CR}$$

$$\text{or AO} + \text{CO} : \text{CO} :: \text{CO} : \text{CO} - \text{OR}$$

$$\therefore \text{AO} : \text{CO} :: \text{OR} : \text{CO} - \text{OR}.$$

But in the common cycloid, SOQ is a straight line, and $\therefore$ CO $= \infty$.

$$\therefore \text{CO} = \text{CO} - \text{OR}, \text{ and AO} = \text{OR}.$$

Also BW is parallel to RA, and BV = AO = OR = VW

$\therefore$ PT = 2PV = PS, &c. &c.

431. Let R be the radius of the base, r that of the wheel; then (by Prop. 49, Princip. Newt.) it easily appears that

$$\text{AP} = \frac{4r\,(\text{R}-r)}{\text{R}} \cdot \text{vers.} \frac{\text{AB}}{2}$$

$$= \text{vers.} \frac{\text{AB}}{2} \text{ to rad.} \frac{4r\,(\text{R}-r)}{\text{R}}$$

Also by Prop. 52 of that work, we learn that the time through AP $\propto$ arc vers, AP to rad. $\frac{\text{AL}}{2} \left(= \frac{4r\,(\text{R}-r)}{\text{R}}\right) \propto \frac{\text{AB}}{2} \propto \text{AB}$.

Hence $T_{AP} : T_{AL} :: AB : AL$

And $\therefore T_{AP} : T_{BL} :: AB : BL$.

432. Since generally,

$$t = \pi \sqrt{\frac{l}{g}}$$

by the question we have

$$t = \pi \sqrt{\frac{20}{g}} = 2\pi \sqrt{\frac{5}{g}}$$

$\therefore$ the whole time is $T = 20 \cdot \pi \sqrt{\frac{5}{g}}$

Hence $s = \frac{g}{2} T^2 = 1000 \pi^2$, the space required.

433. Let l be the length of either pendulum; then the distance descended vertically from the highest to the lowest points of the $\frac{1}{2}$ ⊙ and cycloid, will be

$$l \text{ and } \frac{l}{2}$$

and the velocities in the curves (v, v') will be those due to such distances.

$$\therefore v : v' :: \sqrt{2gl} : \sqrt{2g\frac{l}{2}} :: \sqrt{2} : 1.$$

Now the chord (C) of the quadrant $= l\sqrt{2}$ the arc (A) of the cycloid $= 2 . \frac{l}{2} = l$.

$$\therefore C : A :: \sqrt{2} : 1$$

$$\therefore v : v' :: C : A$$

434. The semi-cycloidal arc FA (Fig. 90) is bisected by inflecting $AP = \frac{AB}{2}$, &c. &c. (429.)

Also

$$T_{FM}' = \sqrt{\frac{d}{2g}} \cos.^{-1} \frac{2x - d}{d} \quad \text{(\textit{Venturoli}, p. 108.)}$$

$$= \sqrt{\frac{d}{2g}} \cos.^{-1} \left(-\frac{1}{2}\right) \left(\text{for } x = \frac{d}{4}\right)$$

$$= \sqrt{\frac{d}{2g}}.\ 120^\circ$$

And $T_{FA} = \sqrt{\frac{d}{2g}}\ 180^\circ$, $\therefore T_{M'A} = \sqrt{\frac{d}{2g}}\ 60^\circ$

And $\therefore T_{FM}' : T_{M'A} :: 2 : 1$.

CENTRAL FORCES.

435. Generally let the force $\propto \frac{1}{\rho^n}$.

Then if μ be the absolute force tending to a centre, we have

$$v\,dv = -\mathrm{F}d\rho = \frac{\mu d\rho}{\rho^n}$$

$$\therefore \frac{v^2}{2} = \frac{\mu}{n-1} \times \left(\frac{1}{\rho^{n-1}} - \frac{1}{a^{n-1}}\right)$$

a being the value of ρ when $v = 0$.

Hence

$$v = \sqrt{\frac{2\mu}{a^{n-1}(n-1)}} \cdot \sqrt{\frac{a^{n-1} - \rho^{n-1}}{\rho^{n-1}}}.$$

Hence for any other centre where the absolute force is μ' and a' the distance at which $v' = 0$, we have

$$v' = \sqrt{\frac{2\mu'}{a'^{n-1}(n-1)}} \quad \sqrt{\frac{a'^{n-1} - \rho'^{n-1}}{\rho'^{n-1}}}$$

$$\text{and } v : v' :: \sqrt{\frac{\mu}{\mu'} \cdot \frac{a'^{n-1}}{a^{n-1}}} : \sqrt{\frac{\rho'^{n-1}}{\rho^{n-1}}} \sqrt{\frac{a^{n-1} - \rho^{n-1}}{a'^{n-1} - \rho'^{n-1}}}$$

If the velocities be taken when at the centres, or when $\rho = 0$

$$v : v' :: \sqrt{\frac{\mu}{\mu'} : \frac{a^{n-1}}{a'^{n-1}}} \ \ldots\ldots(1)$$

If $a = a'$. Then

$$v : v' :: \sqrt{\mu} : \sqrt{\mu'} \ldots\ldots(2).$$

436. The nature of the reciprocal spiral is such that the angle θ traced out by the *radius-vector* ρ, is always reciprocally proportional to ρ. Hence then the equation required is

$$\rho = \frac{a}{\theta}$$

a being a constant.

This equation being similar to that of the *hyperbola*, referred to its asymptotes, which is

$$x = \frac{ab}{y}$$

causes it also to be named the *Hyperbolical Spiral.*

Once for all we shall investigate the expression for the force in any curve, first in terms of its Polar Co-ordinates (ϱ), (θ), and secondly, in terms of (ϱ) and the perpendicular upon the tangent (p).

If s be the arc described, in the time t and v the velocity at the end of that time, and F the accelerating force, then we have

$$v = \frac{ds}{dt},\ \mathrm{F} = \frac{dv}{dt}$$

$$\therefore \mathrm{F} = \frac{d^2s}{dt^2} \text{ and } vdv = \mathrm{F}ds \ldots\ldots\ldots\ldots(a)$$

Hence, if the force be decomposed into two others, X and Y, parallel to the axes of x and y we have

$$\left.\begin{aligned} \mathrm{X} &= \frac{d^2x}{dt^2} \\ \mathrm{Y} &= \frac{d^2y}{dt^2} \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots(b)$$

$$\text{and } y\mathrm{X} - x\mathrm{Y} = \frac{1}{dt}.\,d.\,\frac{ydx - xdy}{dt}$$

But since

$$\mathrm{X} : \mathrm{F} :: x : \varrho$$
$$\mathrm{Y} : \mathrm{F} :: y : \varrho$$

$\therefore y\mathrm{X} - x\mathrm{Y} = 0$, and we have, after integrating,

$$ydx - xdy = cdt \ldots\ldots\ldots\ldots\ldots\ldots(c)$$

But $y = \varrho \sin.\theta$, $x = \varrho \cos.\theta$, and

$$x^2 + y^2 = \varrho^2$$

$$\therefore ydx - xdy = \varrho^2 d\theta$$

$$\therefore \varrho^2 d\theta = cdt \ldots\ldots\ldots\ldots\ldots\ldots(d)$$

Again from equation (b)

$$2\int\{\mathrm{X}dx + \mathrm{Y}dy\} = \frac{dx^2 + dy^2}{dt^2}$$

$$= c^2 \times \frac{\varrho^2 d\theta^2 + d\varrho^2}{\varrho^4 d\theta^2}$$

But $Xdx + Ydy = Fd\varrho$, since

$X = F\cos.\theta$, $Y = F.\sin.\theta$, and $F = (X^2 + Y^2)$

$$\therefore 2\int F d\varrho = c^2 \frac{\varrho^2 d\theta^2 + d\varrho^2}{\varrho^4 d\theta^2}$$

Hence

$$F = \frac{c^2}{\varrho^3} - \frac{c^2}{2} \cdot \frac{d\left(\dfrac{d\varrho^2}{\varrho^4 d\theta^2}\right)}{d\varrho} \ldots\ldots\ldots\ldots (e)$$

from which expression having given the equation of the curve, we can find the force F.

We also have

$$d\theta = \frac{c d\varrho}{\varrho\sqrt{(-c^2 - 2\varrho^2\int F d\varrho)}} \ldots\ldots\ldots\ldots\ldots (f)$$

by which, having given the force, we obtain the Equation to the Trajectory.

Again, when the equation is in p and ϱ, since generally

$$p : \sqrt{(\varrho^2 - p^2)} :: \varrho d\theta : d\varrho$$

we have

$$\frac{d\varrho^2}{\varrho^4 d\theta^2} = \frac{1}{p^2} - \frac{1}{\varrho^2}$$

and $$\frac{d.\left(\dfrac{d^2\varrho}{\varrho^4 d\theta^2}\right)}{d\varrho} = \frac{2}{\varrho^3} - \frac{dp}{d\varrho} \cdot \frac{2}{p^3}$$

and substituting in (e)

$$F = \frac{c^2}{p^3} \cdot \frac{dp}{d\varrho} \ldots\ldots\ldots\ldots\ldots\ldots (g)$$

Applying expression (e) to the reciprocal spiral, since

$$\varrho = \frac{a}{\theta}$$

we get

$$\frac{d\varrho^2}{\varrho^4 d\theta^2} = \frac{a^2}{c^2}$$

and $\therefore F = \dfrac{c^2}{\rho^3}.$

437. Generally, *required the space through which a body must fall from rest towards a centre of force either externally or in-*

ternally, with respect to the trajectory, in order to acquire the velocity in the trajectory at the point corresponding to the radius-vector along which the body falls.

By 436

$$v = \frac{ds}{dt} = \frac{c\sqrt{(\varrho^2 d\theta^2 + d\varrho^2)}}{\varrho^2 d\theta}$$

$$= \frac{c\sqrt{\left(\varrho^2 + \frac{d\varrho^2}{d\theta^2}\right)}}{\varrho^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots (a)$$

$$\text{or} = \frac{c}{p} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (b)$$

Hence if $r + z$ be the distance from the centre at which the body begins to fall in the direction of the given *radius-vector* r, when the fall is exterior to the trajectory; then since

$$vdv = -\mathrm{F}d\varrho$$
$$\text{and } c^2 = 2\int -\mathrm{F}d\varrho$$

and integrating between $\varrho = r + z$ and $= r$ and equating the value of v hence derived with its value obtained from (a) or (b) we get z in terms of r.

Ex. 1. Let $\mathrm{F} = \frac{\mu}{\varrho^n}$; then

$$v^2 = 2\mu\int - \frac{d\varrho}{\varrho^n}$$

$$= \frac{2\mu}{(n-1)(r+z^{n-1})} \cdot \frac{(r+z)^{n-1} - \varrho^{n-1}}{r^{n-1}}$$

$$= \frac{2\mu}{(n-1)r^{n-1}} - \frac{2\mu}{n-1} \cdot \frac{1}{(r+z)^{n-1}}.$$

Hence by equation (a) $\frac{1}{(r+z)^{n-1}} = \frac{1}{r^{n-1}} - \frac{n-1}{2\mu} \times \frac{c^2}{r^4} \times$

$\left(r^2 + \frac{dr^2}{d\theta^2}\right) \ldots\ldots (c)$, $\frac{dr}{d\theta}$ being obtained from the equation to the curve.

And by Eq. (b)

$$\frac{1}{(r+z)^{n-1}} = \frac{1}{r^{n-1}} - \frac{n-1}{2\mu} \cdot \frac{c^2}{p^2} \ldots\ldots (d)$$

Ex. 2. *In the Ellipse.*

$$\varrho = \frac{a.(1-e^2)}{1+e\cos.\theta}$$

$$\text{and } p^2 = \frac{b^2\varrho}{2a-\varrho}$$

Hence by 487,

$$F = \frac{ac^2}{b^2\varrho^2}$$

$$\therefore n = 2 \text{ and } \mu = \frac{ac^2}{b^2} = \frac{c^2}{a}\cdot\frac{1}{1-e^2}$$

and after proper reductions, we get

$$\frac{1}{r+2} = \frac{1}{2a}$$

$$\therefore z = 2a - r$$

the external fall to acquire the velocity in an ellipse.

Ex. 3. *In the Parabola.*

$$\left.\begin{aligned} \varrho &= \frac{2a}{1+\cos.\theta} \\ \text{and } p^2 &= a\varrho \end{aligned}\right\}$$

and applying either of the expressions (c) or (*d*) it is found that

$$z = \infty.$$

Ex. 4. *In the Hyperbola.*

$$\left.\begin{aligned} \varrho &= \frac{b^2}{a(1+e\cos.\theta)} \\ \text{and } p^2 &= \frac{b^2\varrho}{2a+\varrho} \end{aligned}\right\}$$

Whence F is shewn to be repulsive and ascends through

$$z = -(2a + r).$$

If internal falls are required then $\int F d\varrho$ must be integrated between $\varrho = r$, and $\varrho = r - z$.

and we shall get, when

$$F = \frac{\mu}{\varrho^n}$$

$$v^2 = \frac{2\mu}{n-1} \cdot \left\{ \frac{1}{(r-z)^{n-1}} - \frac{1}{r^{n-1}} \right\}$$

$$= \frac{c^2}{r^4} \sqrt{} \left(\varrho^2 + \frac{dr^2}{d\theta^2} \right)$$

$$\text{or} = \frac{c^2}{p^2}$$

by the question. Hence

$$\frac{1}{(r-z)^{n-1}} = \frac{1}{r^{n-1}} + \frac{c^2(n-1)}{2\mu r^4} \left(r^2 + \frac{dr^2}{d\theta^2} \right) \ldots\ldots(e)$$

$$\text{or } \frac{1}{(r-z)^{n-1}} = \frac{1}{r^{n-1}} + \frac{c^2(n-1)}{2\mu} \cdot \frac{1}{p^2} \ldots\ldots\ldots\ldots(f)$$

from whence, by means of the equation to the given curve, z may be found.

Ex. 1. *In the Problem it is required to find z for the common Parabola.*

Hence

$$\varrho = \frac{2a}{1 + \cos.\theta}$$

$$\text{or } p^2 = a\varrho$$

By the former

$$\frac{d\varrho^2}{d\theta^2} = \frac{4a^2 \sin.^2\theta}{(1 + \cos.\theta)^4}$$

$$\therefore \varrho^2 + \frac{d\varrho^2}{d\theta^2} = \frac{\varrho^3}{a}.$$

Also by 436,

$$F = \frac{c^2}{2a \times \varrho^2}$$

$$\therefore \mu = \frac{c^2}{2a}$$

and by substitution in (e) we get

$$\frac{1}{r-z} = \frac{1}{r} + \frac{a}{r^4} \cdot \frac{r^3}{a}$$

$$= \frac{2}{r}$$

$$\therefore z = \frac{r}{2}.$$

By the expressions (*e*), (*f*) we have also

(1). *For the Ellipse referred to its focus.*

$$z = r - \frac{2ar}{4a-r} = \frac{2ar - r^2}{4a - r}.$$

(2). *For the circle when the force* $\propto \frac{1}{\varrho^{n+1}}$, we have generally

$$z = r - r\sqrt{\frac{2}{2+n}}.$$

(3). *Required the space fallen through to acquire m times the velocity in a curve, when*

$$F \propto \frac{1}{\varrho^{n+1}}.$$

If we put $\frac{2pd\varrho}{dp} = \gamma$ which is the chord of curvature, then it is easily shewn from expressions (*d*) and (*f*) that

$$z = r\left\{\left(\frac{2r + \frac{2+n}{n+1}\cdot\frac{\gamma}{2}m^2}{2r}\right)^{\frac{n+1}{n+2}} - 1\right\}$$

$$\text{or } z' = r\left\{1 - \left(\frac{2r - \frac{n+2}{n+1}\cdot\frac{\gamma}{2}m^2}{2r}\right)^{\frac{n+1}{n+2}}\right\}$$

according as the body falls externally or internally.

438. It is well known (see *Vince's Fluxions*, *Simpson's*, &c., that the velocity of a body at any point of its orbit, is the same as would be acquired through $\frac{1}{4}$ the chord of curvature to that point, with the force constant.

But this chord is (see *Vince*).

$$\frac{2pd\varrho}{dp}$$

Hence,

$$v^2 = 2F \times \frac{1}{4} \frac{2pd\varrho}{dp}$$

generally; and in the circle

$$\frac{2pd\varrho}{dp} = 2\varrho$$

if ϱ be its radius.

$$\therefore v'^2 = 2F \times \frac{1}{4} \times 2\varrho$$

$$\text{and } v^2 : v'^2 :: \frac{d\varrho}{dp} : \frac{\varrho}{p} \ldots\ldots\ldots (a)$$

which gives the general relation required.

To apply this to the conic sections, we have

$$p^2 = a\varrho$$

$$p^2 = \frac{b^2\varrho}{2a - \varrho}$$

$$p^2 = \frac{b^2\varrho}{2a + \varrho}$$

for the Parabola, Ellipse, and Hyperbola respectively. Hence

$$\frac{d\varrho}{dp} = \frac{2p}{a}, \text{ or } = \frac{b^2\varrho}{ap^2} \times \frac{\varrho}{p}$$

$$\therefore v^2 : v'^2 :: \frac{2p}{a} : \frac{\varrho}{p} :: 2 : 1$$

$$v^2 : v'^2 :: b^2\varrho : \frac{a.b^2\varrho}{2a - \varrho} :: 2a - \varrho : a$$

$$v^2 : v' :: b^2\varrho : \frac{ab^2\varrho}{2a + \varrho} :: 2a + \varrho : a$$

439. Suppose a body moving in a curve PQ Fig. 91.; its motion at P is in the direction of the tangent PR, and if PR described in a given time be taken to represent the velocity v, and be resolved into Pr, Rr, then Pr is the velocity with which it is approaching the centre S. Again QR$=$Tp is due to the force, and we therefore have, when there is no angular velocity, the whole approach to the centre $=$ PT. But with this angular velocity the approach is only

$SQ = Sp$. Hence Tp must be the recess from the centre caused by the centrifugal force.

We have therefore

$$RQ : Tp :: \frac{PQ^2}{PV} : \frac{QT^2}{2Sp} \text{ ultimately.}$$

$$:: \frac{SP^2}{PV} : \frac{Sy^2}{2SP} \text{ ultimately.}$$

and $F : \varphi :: 2SP^3 : Sy^2 . PV$

$$:: 2\varrho^3 : p^2 \frac{2pd\varrho}{dp}$$

$$:: \frac{\varrho^3}{p^3} : \frac{d\varrho}{dp} \text{.......}(b)$$

φ being the centrifugal force; which is the relation required.

To apply it to the hyperbolic spiral, we have

$$p = \frac{a\varrho}{\sqrt{(a^2 + \varrho^2)}}$$

$$\therefore \frac{dp}{d\varrho} = \frac{a}{\sqrt{(a^2 + \varrho^2)}} - \frac{a\varrho^2}{(a^2 + \varrho^2)^{\frac{3}{2}}}$$

$$= \frac{a^3}{(^2 + \varrho^2)^{\frac{3}{2}}} = \frac{p^3}{\varrho^3}$$

$\therefore F : \varphi :: 1 : 1$

and $F = \varphi$.

440. The velocity in a circle is that which is due to $\frac{1}{4}$ chord of curvature or to $\frac{1}{2}$ its radius.

Hence

$$v^2 = 2F \times \frac{\varrho}{2} = F\varrho \text{..............}(a)$$

F being considered constant.

Hence the periodic time is

$$P = \frac{2\pi\varrho}{v} = \frac{2\pi\varrho}{\sqrt{(F\varrho)}}$$

$$= \frac{2\pi\sqrt{\varrho}}{\sqrt{F}} \text{............}(b)$$

Now at the surface of the earth $F = g = 32\frac{1}{6}$ feet nearly.

Hence at the distance of n times the earth's radius R from its centre, we have

$$F : g :: \frac{1}{R^2} : \frac{1}{(nR)^2}$$

$$:: 1 : \frac{1}{n^2}$$

$$\therefore F = \frac{g}{n^2}$$

$$\text{and } v^2 = \frac{g}{n^2} \times nR = \frac{gR}{n} \ldots\ldots\ldots (a')$$

$$P = \frac{2\pi n\sqrt{(nR)}}{\sqrt{g}} \ldots\ldots\ldots\ldots (b')$$

Let $n = 2$, and change $R = 4000$ miles (it $= 3958$ miles) into feet, or g into miles, and the problem will be resolved.

Suppose it were required to find the velocity and periodic time of the moon; then we have $n = 60$ and

$$v^2 = \frac{g \times 4000}{60} = \frac{200g}{3}.$$

$$\text{and } P = \frac{2\pi 60 \times \sqrt{(60 \times 4000)}}{\sqrt{g}} = 24000\,\pi\sqrt{\frac{6}{g}}$$

$$\text{But } g = \frac{32\frac{1}{6}}{3 \times 1760} = \frac{1}{3.5.11} \text{ miles nearly.}$$

$$\therefore v = \sqrt{\frac{40}{99}}$$

$= .6356$ miles in a second nearly.

and $P = 237285$ seconds nearly.

$= 27$ days, 10 hours, 59 seconds, nearly.

which is sufficiently near for a rough calculation, the *sidereal* revolution requiring $27^d\ 7^h\ 43'\ 11''.5$.

441. Generally let the comet be supposed to change the velocity of the earth from v to $m \times v$.

Now let first be investigated *the nature of the orbit, from having*

given the velocity of projection V, *the perpendicular* P *and radius vector* R, *at the point of projection*, when $F \propto \frac{1}{\varrho^2}$.

By (436)

$$F = \frac{c^2}{p^3} \cdot \frac{dp}{d\varrho} = \frac{\mu}{\varrho^2}$$

$$\therefore \frac{c^2 dp}{p^3} = \frac{\mu d\varrho}{\varrho^2}$$

$$\therefore \frac{c^2}{2p^2} = \frac{\mu}{\varrho} + C$$

But when $\varrho = R, p = P$

$$\therefore \frac{c^2}{2} \times \left(\frac{1}{p^2} - \frac{1}{P^2}\right) = \mu \times \left(\frac{1}{\varrho} - \frac{1}{R}\right)$$

Whence

$$p^2 = \frac{c^2R^2}{2\mu P^2 - c^2R} \times \frac{\varrho}{\frac{2\mu P^2R}{2\mu P^2 - c^2R} - \varrho}$$

and putting $\frac{\mu P^2 R}{2\mu P^2 - c^2R} = a$,

$$p^2 = \text{ and } \frac{c^2RP^2}{2\mu P^2 - c^2R} = b^2$$

we have

$$p^2 = \frac{b^2\varrho}{2a \mp \varrho} \ldots\ldots\ldots\ldots (a)$$

according as $2\mu P^2$ is $>$ or $<$ c^2R.

But this is the equation to a conic section, and the trajectory is an Ellipse, Hyperbola, or Parabola, according as

$$2\mu P^2 - c^2R$$

is positive, negative, or 0, that is according as $2\mu P^2 > < \text{ or } = c^2R$.

Again the velocity in the curve is obtained from (437)

$$v = \frac{c}{p}$$

$$\therefore V = \frac{c}{P}$$

and the trajectory is an Ellipse, Hyperbola or Parabola, according as

$$2\mu P^2 \text{ is } >, < \text{ or } = P^2V^2R$$

or according as

$$2\mu \text{ is } >, < \text{ or } = V^2 R.$$

This being premised, it is evident that the orbit of the Earth after impact with the Comet is an Ellipse, Hyperbola, or Parabola, according as

$$2\mu \text{ is } >, < \text{ or } = m^2 v^2 \times R$$

or as

$$2 \text{ is } >, <, \text{ or } = m^2 \ldots\ldots\ldots (e)$$

since

$$v^2 = FR = \frac{\mu}{R}.$$

$$\text{Hence } a = \frac{\mu P^2 R}{2\mu P^2 - c^2 R} = \frac{v^2 R^2 P^2}{2 R P^2 v^2 - m^2 v^2 P^2 R}$$

$$= \frac{R}{2 - m^2} \ldots\ldots\ldots (f)$$

$$\text{or} = \frac{R}{m^2 - 2} \ldots\ldots\ldots (g)$$

according as the new orbit is an Ellipse or Hyperbola.

Also

$$b^2 = \frac{c^2 R P^2}{2\mu P^2 - c^2 R} = \frac{m^2 v^2 R P^3}{2 R v^2 P^2 - m^2 v^2 P^2 R}$$

$$= \frac{m^2 P}{2 - m^2} \ldots\ldots\ldots (h)$$

$$\text{or} = \frac{m^2 P}{m^2 - 2} \ldots\ldots\ldots (k)$$

according as the new orbit is an Ellipse or Parabola.

Hence then equation (a) becomes

$$\left.\begin{aligned} p^2 &= \frac{m^2 P}{2 - m^2} \times \frac{\rho}{2 \times \frac{R}{2 - m^2} - \rho} \\ \text{or} &= \frac{m^2 P}{m^2 - 2} \times \frac{\rho}{2 \times \frac{R}{m^2 - 2} + \rho} \end{aligned}\right\} (l)$$

which are the equations to the new Ellipse or Hyperbola.

If $m^2 = 2$; then

$$p^2 = m^2 P\rho \times \frac{\infty}{2\,\infty} = P \cdot \rho \ldots\ldots (m).$$

the equation to the Parabola.

Thus are the new orbits determined.

Again, since in the Circle, and Ellipse, the peredic times Y Y′ are in the sesquiplicate ratio of the major axes; therefore

$$Y : Y' :: R^{\frac{3}{2}} : \left(\frac{R}{2-m^2}\right)^{\frac{3}{2}}$$

$$:: (2-m^2)^{\frac{3}{2}} : 1$$

$$\therefore Y' = \frac{Y}{(2-m^2)^{\frac{3}{2}}} = \frac{365\frac{1}{4}\text{ days}}{(2-m^2)^{\frac{3}{2}}}$$

the length of the year in the new orbit when it is an Ellipse.

In the problem

$$m^2 = \frac{3}{2}$$

$$\therefore Y' = 365\,\frac{1}{6}\,. \times 2^{\frac{3}{2}}$$

$$\text{But } 2^{\frac{3}{2}} = 2\sqrt{2}$$

$$= 2\left\{1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{2}{\&c.}}}\right\}.$$

$$\therefore Y' = \frac{14}{5}\,365\frac{1}{4}\text{ days nearly}$$

$$= 1022\,\frac{7}{10}\text{ days nearly.}$$

442. In the Ellipse referred to its centre we have

$$\rho^2 = \frac{b^2}{1-e^2\cos.^2\theta}$$

Whence

$$F = \frac{c^2}{a^2 b^2}\cdot\rho$$

$\therefore$ the absolute force is

$$\mu = \frac{c^2}{a^2 b^2} \ldots\ldots\ldots\ldots (1)$$

Again

$$dt = \frac{\rho^2\, d\theta}{c} = \frac{\rho^2\, d\theta}{ab\sqrt{\mu}}$$

$$= \frac{b}{\sqrt{a\mu}} \cdot \frac{d\theta}{1 - e^2 \cos.^2\theta}$$

$$= \frac{\sqrt{(1-e^2)}}{\sqrt{\mu}} \cdot \frac{d\theta}{1-e^2\cos.^2\theta}$$

$$\therefore 2t = \sqrt{\frac{1-e^2}{\mu}} \times \left\{ \int \frac{d\theta}{1-e\cos.\theta} + \int \frac{d\theta}{1+e\cos.\theta} \right\}$$

$$= \frac{1}{\sqrt{\mu}} \left\{ \cos.^{-1}\left(\frac{e-\cos.\theta}{e\cos.\theta - 1}\right) + \cos.^{-1}\left(\frac{e+\cos.\theta}{e\cos.\theta+1}\right) \right\} + C.$$

See the *Integral Tables of Hirsch*, (Baynes and Son, Paternoster-Row.) And if the Integral be taken between $\theta = 0$ (when $t = 0$) and $\theta = \pi$ we get

$$2t = \frac{2\pi}{\sqrt{\mu}},$$

the whole Periodic Time required.

Hence

$$2t \propto \frac{1}{\sqrt{\mu}}. \qquad \text{Q. E. D.}$$

The above general expression for $2t$ may be somewhat simplified by putting

$$\frac{e-\cos.\theta}{e\cos.\theta - 1} = \cos.\phi, \quad \frac{e+\cos.\theta}{e\cos.\theta+1} = \cos.\chi$$

which give

$$\text{Sin.}\,\phi = \sqrt{(1-e^2)} \cdot \frac{\sin.\theta}{e\cos.\theta - 1}$$

$$\text{Sin.}\,\chi = \sqrt{(1-e^2)} \cdot \frac{\sin.\theta}{e\cos.\theta + 1}$$

Hence, substituting in

$$\cos.(\phi + \chi) = \cos.\phi \cdot \cos.\chi - \sin.\phi \cdot \sin.\chi$$

and, thence deriving $\phi + \chi$ and substituting in the above expression, it will be less complex.

443. Let SY, Sy (fig. 92,) be perpendiculars upon the tangents PY, Qy, and draw PO, QO, at right angles to the tangents at the points of contact P, Q, and meeting in O.

Then $\angle\, YSy = d\varphi = \angle\, O = 2\,\angle V$

$$\therefore d\theta : d\varphi :: \angle QSP : 2\angle V$$

$$:: \frac{QT}{\rho} : \frac{2QT}{PV}$$

$$:: PV : 2\rho$$

But $PV = \dfrac{2p\,d\rho}{dp}$

$$\therefore d\theta : d\varphi :: \frac{d\rho}{dp} : \frac{\rho}{p} \ldots\ldots\ldots\ldots\ldots (a)$$

the same as the expression for the squares of the Linear Velocities in a curve and circle at the same distance. See 438.

In the Logarithmic Spiral

$$p = a\rho$$

$$\therefore dp = a\,d\rho$$

and $d\theta : d\varphi :: 1 : 1.$ Q. E. D.

444. If V be the velocity of projection, R the distance, and P the perpendicular upon the direction of projection; then (441) we have

$$p^2 = \frac{b^2\rho}{2a \mp \rho}$$

for the equation to the Trajectory,

where

$$a = \frac{\mu P^2 R}{2\mu P2 - c^2 R}, \quad b^2 = \frac{c^2 RP^2}{2\mu P^2 - c^2 R},$$

and $c = VP$

whence the curve will be an *Ellipse*, or *Hyperbola*, or *Parabola*, according as

$$2\mu \text{ is } > < \text{or} = RV^2.$$

But since $v\,dv = F\,d\rho$

$$= \frac{\mu\, d\rho}{\rho^2}$$

$$\therefore v^2 = \mu . \left(\frac{1}{\rho} - \frac{1}{2R}\right) \text{ by the question;}$$

$$\text{and } V^2 = \frac{\mu}{2R}.$$

Hence the orbit will be an Ellipse, Hyperbola, or Parabola, according as

$$2\mu \text{ is } >, <, \text{ or } = \frac{\mu}{2}.$$

It is consequently an Ellipse, and the equation becomes, since

$$a = \frac{\mu P^2 R}{2\mu P^2 - \frac{P^2\mu}{2}} = \frac{2R}{3},$$

$$\text{and } b^2 = \frac{P^2 \mu P^2}{2\left(2\mu P^2 - \frac{P^2\mu}{2}\right)} = \frac{P^2}{3},$$

$$p^2 = \frac{P^2\rho}{3\left(\frac{4}{3}R - \rho\right)} \quad \ldots\ldots\ldots (1)$$

Hence also, if T, T' denote the Periodic Times in this Ellipse, and in the circle whose radius is $2R$ the given altitude, then we get

$$T : T' :: a^{\frac{3}{2}} : (2R)^{\frac{3}{2}}$$

$$:: \frac{(2R)^{\frac{3}{2}}}{3^{\frac{3}{2}}} :: (2R)^{\frac{3}{2}}$$

$$:: 1 : 3\sqrt{3}.$$

445. Let APV (Fig. 93,) be the given cycloid, AB its base, and VB its axis. Then the figure being completed in Newton's manner, as in the diagram, we have

$$RP^2 : QT^2 :: ZP^2 : ZT^2 :: VF^2 : EF^2$$

$$:: VB : BE$$

and since chord of curvature $PV = 4PM$, therefore

$$RP^2 = 4PM . RQ$$

$$\therefore 4PM \times RQ : QT^2 :: VB : PM$$

$$\therefore \frac{QR}{QT^2} = \frac{VB}{4PM^2}$$

$$\therefore F \propto \frac{1}{PM^2}.$$

446. By (440 . *b*)

$$P = \frac{2\pi\sqrt{\rho}}{\sqrt{F}}$$

and by the question

$$F \times P^2 = q \times \rho$$

$$\therefore q \cdot \rho = 4\pi^2\rho$$

$$\therefore q = 4\pi^2.$$

447. Since the velocity in the same curve $\propto \frac{1}{p}$; let $P_{,} P'$ be the perpendiculars upon the tangent corresponding to the greatest and least velocities; then by the question, we have

$$\frac{1}{P} : \frac{1}{p} :: \frac{1}{p} : \frac{1}{P'}$$

$$\therefore p^2 = PP' = (a - \sqrt{a^2 - b^2}) \cdot (a + \sqrt{a^2 - b^2})$$

$$= b^2$$

$$\therefore p = b$$

or the point required is at the extremity of the *minor axis*.

448. Since $F = \frac{\mu}{\rho^2}$, we have

$$v^2 = \mu \left\{ \frac{1}{\rho} - \frac{1}{r} \right\}$$

r being the whole distance fallen through.

$$\text{and } dt = \frac{d\rho}{v} = \sqrt{\mu r} \cdot \frac{\sqrt{\rho d\rho}}{\sqrt{r - \rho}}$$

$$= \sqrt{\frac{r}{\mu}} \cdot \frac{\rho d\rho}{\sqrt{(r\rho - \rho^2}}$$

$$\therefore t = \sqrt{\frac{r}{\mu}} - \sqrt{(r\rho-\rho^2)} + \frac{r}{2}\int\frac{d\rho}{\sqrt{(r\rho-\rho^2)}}$$

But $\int\frac{d\rho}{\sqrt{(r\rho-\rho^2)}} = \text{vers.}^{-1}\frac{2\rho}{r}$

see Hirsch's Integral Tables.

$$\therefore t = \sqrt{\frac{r}{\mu}}\left\{\frac{r}{2}\text{vers.}^{-1}.\frac{2\rho}{r} - \sqrt{(r\rho-\rho^2)}\right\} + C.$$

But when $\rho = r$, $t = 0$

$$\text{and } C = \sqrt{\frac{r}{\mu}}\left\{\frac{r}{2}\text{vers.}^{-1}.2\right\} = \sqrt{\frac{r}{\mu}}.\frac{r}{2}\pi$$

$$\therefore t = \sqrt{\frac{r}{\mu}}\left\{\frac{r}{2}(\text{vers.}^{-1}.\frac{2\rho}{r} + \pi) - \sqrt{(r\rho-\rho^2)}\right\}$$

Hence, if T, T′ denote the times down the first last halves of the r, we have

$$T = \sqrt{\frac{r}{\mu}}\left\{\frac{r}{2}\frac{\pi}{2} + \frac{r}{2}\right\}$$

$$\text{and } T + T' = \sqrt{\frac{r}{\mu}}.\frac{r}{2}\pi.$$

$$\therefore T : T + T' :: \frac{\pi}{2} + 1 : \pi$$

$$\text{and } T : T' :: \frac{\pi}{2} + 1 : \frac{\pi}{2} - 1$$

$$:: Q + R : Q - R.$$

449. Since $F \propto \frac{1}{\varrho^2}$, the curve is a conic section with force in the focus; and since moreover, by the question, it turns into itself again, it must be either an ellipse or a circle. When the angle of reflection or angle of incidence α is 45°, and QA (Fig. 94,) $= \frac{SQ}{2}$, then the orbit is a circle whose radius is SQ, because then SQ is at right angles to the tangent Tt, and the velocity in a circle is that which is acquired down $\frac{1}{2}$ its radius. In all other cases the orbit is an Ellipse, of which S is one focus.

Let QA $= r$, SQ $=$ R; then the perpendicular upon the tangent Tt, is

$$P = R \cdot \sin.(\pi - 2\alpha) = R \cdot \sin. 2\alpha$$

and by 441 the equation to the Ellipse is

$$p^2 = \frac{b^2 \varrho}{2a - \varrho} \ldots\ldots\ldots\ldots (a)$$

where

$$a = \frac{\mu P^2 R}{2\mu P^2 - c^2 R}, \; b^2 = \frac{c^2 R P^2}{2\mu P^2 - c^2 R}$$

But if V be the velocity acquired down r, then

$$V^2 = \mu\left(\frac{1}{r} - \frac{1}{R+r}\right) = \frac{\mu R}{r(R+r)}$$

$$\text{and } V = \frac{c}{P}, \; \therefore c^2 = V^2 P^2$$

$$\therefore a = \frac{\mu R}{2\mu - V^2 R} = \frac{rR(R+r)}{2r(R+r) - R^2}$$

$$\text{and } b^2 = \frac{V^2 P^2}{2\mu - V^2 R} = \frac{R^3 \sin.^2 (2\alpha)}{2r(R+r) - R^2}$$

whence the Ellipse may be constructed.

Again, by 448 the whole time of descent to the centre is

$$\sqrt{\frac{(R+r)}{\mu}} \cdot \frac{R+r}{2} \pi$$

and by 440, the Periodic Time in the circle, whose radius is SA, is

$$\frac{2\pi V(R+r)}{\sqrt{\frac{\mu}{(R+r)^2}}}$$

$$\therefore T : T' :: \frac{\pi}{2} : 2\pi :: 1 : 4.$$

450. In the Logarithmic Spiral

$$\varrho = m a^\theta$$

$$\therefore \frac{d\varrho}{d\theta} = mla \cdot a^\theta = \varrho la$$

$$\therefore \frac{d\varrho^2}{\varrho^4 d\theta^2} = \frac{la}{\varrho^2}$$

$$\text{and } F = c^2 \frac{1 + la}{\varrho^3}$$

$$\therefore c = \sqrt{\frac{\mu}{1 + la}}.$$

$$\text{Hence } dt = \frac{\varrho^2 d\theta}{c}$$

$$= \sqrt{\frac{1 + la}{\mu (la)^2}} \times -\varrho d\varrho$$

$$\therefore t = \sqrt{\frac{1 + la}{\mu (la)^2}} \cdot \left(\frac{r^2}{2} - \frac{\varrho^2}{2}\right)$$

r being that value of ϱ which corresponds to $t = 0$.

Hence the whole time to the centre is

$$T = \sqrt{\frac{1 + la}{\mu (la)^2}} \times \frac{r^2}{2}$$

and the Periodic Time in a circle whose radius is r and force $=$ $\frac{\mu}{\varrho^3}$ is by 440

$$P = \frac{2\pi \sqrt{r}}{\sqrt{\frac{\mu}{r^3}}} = \frac{2\pi r^2}{\sqrt{\mu}}$$

$$\therefore T : P :: \sqrt{\frac{1 + la}{(la)^2}} : 4\pi$$

451. Let $T = m a^{\frac{3}{2}}$ be the Periodic Time. Then dt is to T as area described in dt is to the whole Ellipse, or we have

$$dt : T :: \frac{\varrho^2 d\theta}{2} : \pi ab$$

$$\text{and } dt = \frac{m}{2\pi} \cdot \frac{\sqrt{a}}{b} \times \varrho^2 d\theta$$

Again, it is easily shewn that

$$\frac{QR}{QT^2} = \frac{AC}{2BC^2}$$

$$\text{or } \frac{d^2\varrho}{2\varrho^2 d\theta^2} = \frac{a}{2b^2}$$

$$\therefore dt^2 = \frac{m^2}{4\pi^2} \cdot \varrho^2 d\varrho^2$$

$$\text{and } \frac{d^2\varrho}{dt^2} = \frac{4\pi^2}{m^2} \cdot \frac{1}{\varrho^2}$$

$$\text{or } F \propto \frac{1}{\varrho^2}.$$

This is in substance the proof usually given of the Inverse of Kepler's Third Law, see Thorpe's Newton, p. 114; but it is merely *argumentum in circulo*. In getting the equation,

$$\frac{QR}{QT^2} = \frac{AC}{2BC^2}$$

it is assumed that the focus of the ellipse is the centre of force. If this be taken for granted, there is no occasion to have given the law of the Periodic Times; for since the nature of the orbit, *viz.*, an ellipse and centre of force are given, the law of force may be found at once from either of the formulæ in 436.

We have repeated this vulgar error for the purpose of more clearly exposing it. A legitimate demonstration can hardly be attained prior to the discovery of something like an Inverse Method of Definite Integrals.

452. *To investigate the Paracentric Velocity in any curve,* we have

$$\begin{aligned} \text{V in orbit} : \text{Paracentric Vel.} &:: ds : d\varrho \\ &:: \varrho : \sqrt{(\varrho^2 - p^2)} \end{aligned}$$

$$\text{But } v. \text{ in orbit} = \frac{c}{p}$$

$$\begin{aligned} \therefore \text{Paracentric Vel.} &= \frac{c\sqrt{\varrho^2 - p^2}}{p\varrho} \\ &= c\sqrt{\left(\frac{1}{p^2} - \frac{1}{\varrho^2}\right)} \ldots\ldots (a) \end{aligned}$$

In the reciprocal spiral

$$p^2 = \frac{a^2\varrho^2}{a^2 + \varrho^2}$$

$$\therefore \text{P. Vel.} = c\sqrt{\left(\frac{a^2+\varrho^2}{a^2\varrho^2} - \frac{1}{\varrho^2}\right)}$$
$$= \frac{c}{a}$$

which is therefore constant. Q. E. D.

453. Since $dt = \dfrac{\varrho^2 d\theta}{c}$

and when ellipse is referred to its focus

$$\varrho = \frac{b^2}{a} \cdot \frac{1}{1+e\cos\theta}$$

$$\therefore dt = \frac{b^4}{a^2 c} \cdot \frac{d\theta}{(1+e\cos\theta)^2}$$

$$\therefore t = \frac{b^4}{a^2 c\,.(1-e^2)} \times \left\{\frac{1}{\sqrt{(1-e^2)}} \cos^{-1}.\frac{e+\cos\theta}{1+e\cos\theta} - \frac{e\sin\theta}{1+e\cos\theta}\right\} + C$$

See Hirsch's Integral Tables, p. 274.

$$= \frac{a^2.(1-e^2)}{c}.\left\{\frac{1}{\sqrt{(1-e^2)}} . \cos^{-1}.\frac{e+\cos\theta}{1+e\cos\theta} - \frac{e\sin\theta}{1+e\cos\theta}\right\};$$

since when $t = 0$, $C = 0$, and $b^2 = a^2 - a^2 e^2$;

Let $\theta = \pi$. Then

$$t = \frac{a^2(1-e^2}{c} . \frac{\pi}{\sqrt{(1-e^2)}} = \frac{a^2\sqrt{(1-e^2)}}{c} . \pi$$

But $c = b\sqrt{\dfrac{\mu}{a}} = a\sqrt{(1-e^2)} . \sqrt{\dfrac{\mu}{a}}$ (437)

$$\therefore T = 2t = \frac{2a^{\frac{3}{2}}\pi}{\sqrt{\mu}} \quad \ldots\ldots\ldots\ldots (a)$$

But when the Ellipse is referred to its centre, we have (442)

$$T' = \frac{2\pi}{\sqrt{\mu}}$$

$$\therefore T : T' :: a^{\frac{3}{2}} : 1.$$

454. By 436 we have

$$F = \frac{d^2y}{dt^2}$$

$$\text{and } \therefore q = \frac{1}{dt^2}$$

dt being constant.

455. When the lines drawn from the moon to the sun and the earth are at right angles to one another, it is evident the sun will not disturb the moon's attraction to the earth. The exact point in the orbit may be found by describing a semicircle upon the line joining the sun and earth, it being the intersection of this semicircle and the moon's orbit.

The moon will be *nearly* in quadratures when this takes place.

456. By (439)

$$F : \varphi :: 2CP^3 : Cy^2 \times PV$$

$$\therefore \text{ when } F = \varphi$$

$$2\,CP^3 = Cy^2 \times PV$$

But in the Ellipse

$$Cy \times CD = AC \,.\, BC$$

$$\text{and } PV = \frac{2CD^2}{CP}$$

as we learn from many geometrical Treatises on Conics.

Henc

$$2CP^3 = \frac{AC^2 . BC^2}{CD^2} \times \frac{2CD^2}{CP}$$

$$\therefore CP^4 = AC^2 . BC^2$$

$$\text{and } CP^2 = AC \,.\, BC$$

$$\text{and } \therefore CP = \pm\sqrt{AC \,.\, BC}$$

which gives the points required.

457. By (448) we have

$$T = 2\pi \frac{r^{\frac{3}{2}}}{\sqrt{\mu}} \propto r^{\frac{3}{2}}$$

458. Since, if $\theta = \angle SPy$

$$p = \varrho \sin \theta$$

and when θ is a minimum, $\sin \theta$ is also;

therefore, then

$$d \sin \theta = d \cdot \frac{p}{\varrho} = 0$$

$$\text{or } \varrho dp - p d\varrho = 0$$

$$\therefore \frac{d\varrho}{dp} = \frac{\varrho}{p}$$

when θ is a minimum

But by 443

$$\angle^r. \text{vel. of } \varrho : \angle^r. \text{vel. of } . p :: \frac{d\varrho}{dp} : \frac{\varrho}{p}$$

$$\therefore \angle^r. \text{vel. of } \varrho = \angle^r. \text{of } p$$

when θ is a minimum. Q. E. D.

459. Let PAp (Fig. 95,) be the ellipse described with the centre of the earth S as a centre of force; this centre will be the focus. Then since SP is $\perp$ to the tangent to the surface RPH, and TP $= \frac{1}{2}$ right angle, we have

$$\angle \text{TPH} = \angle \text{RPQ} = \angle \text{QPS}$$

and TQ touches the ellipse;

$\therefore$ the other focus lies somewhere in PH. But it also lies in SA which bisects the arc Pp, or it is the point of intersection of SA and PH. H then is the other focus.

Now since $\angle S = 30^\circ$. we have

$$\text{PH} = \text{PS} . \tan. 30^\circ = \text{R} . \frac{1}{\sqrt{3}}$$

$$\therefore 2a = \text{R} . \left(1 + \frac{1}{\sqrt{3}}\right) \; . \; . \; . \; . \; . \; . \; . \; (m)$$

the axis major.

$$\text{Also SH} = \sqrt{\left(R^2 + \frac{R^2}{3}\right)}$$

$$= \frac{2R}{\sqrt{3}}.$$

$$\therefore b^2 = a^2 - \frac{SH^2}{4} = \frac{R^2}{4} \cdot \left(1 + \frac{1}{\sqrt{3}}\right)^2 - \frac{R^2}{3}$$

$$= R^2 \left(\frac{1}{4} - \frac{1}{3} + \frac{1}{12} + \frac{2}{\sqrt{3}}\right) = R^2 \cdot \frac{2}{\sqrt{3}}$$

$$\therefore b = R \sqrt{\frac{2}{\sqrt{3}}}.$$

Now the velocity of projection $= \sqrt{g \frac{PV}{4}}$

and $PV = \dfrac{2CD^2}{AC} = \dfrac{2a'^2}{a}$

But $a'^2 + b'^2 = a^2 + b^2$

and $a'b' = ab \sin. (30 + 45)^\circ$

$= ab \sin. 75^\circ$

whence we find a' and substituting, &c., we get the required velocity of projection.

460. This is easily derived from 441. In that Problem, the velocity in the circle is supposed from v to become mv by the impulse; which impulse also takes place, so that the $\perp$ upon its direction $=$ P. On these conditions we find

$$a = \frac{R}{2 - m^2}, \; b^2 = \frac{m^2 P}{2 - m^2}.$$

Consequently since

$$b^2 = a^2 (1 - e^2)$$

ae being the eccentricity, we have

$$\frac{m^2 P}{2 - m^2} = \frac{R^2}{(2 - m^2)^2} \times (1 - e^2)$$

$$\text{or } m^2 (2 - m^2) = \frac{R^2}{P} (1 - e^2)$$

and solving the Equation

$$m^2 = 1 \pm \sqrt{(1 - \frac{R^2}{P} . \overline{1 - e^2})}$$

$$\text{and } m = \pm \sqrt{\{1 \pm \sqrt{(1 - \frac{R^2}{P} . \overline{1 - e^2})}\}}.$$

whence

$$(m - 1) v$$

which is the velocity required, when the direction of the impulse is any whatever.

When the impulse is made in the direction of the body's motion, we have

$$P = R$$

$$\text{and } m = \pm \sqrt{\{1 \pm \sqrt{(1 - R . \overline{1 - e^2})}\}}.$$

461. This is true for all the planets. For generally

$$v \propto \frac{1}{p}$$

and p decreases continually from the mean to the perihelion distance.

462. Let a be the given distance from the plane; then we have

$$\frac{d^2y}{dt^2} = F = \frac{\mu}{y^2}$$

$$\therefore \frac{2dyd^2y}{dt^2} = \frac{2\mu dy}{y^2}$$

$$\therefore \frac{dy^2}{dt^2} = 2\mu . \left(\frac{1}{y} - \frac{1}{a}\right)$$

$$\therefore \frac{dy}{dt} = \frac{\sqrt{2\mu}}{a} . \sqrt{\frac{a-y}{y}}$$

Also let V be the velocity of projection, α the inclination of its direction to the horizon; then V . cos. α is the velocity parallel to the horizon

$$\text{and } dx = \text{V cos. } \alpha \times dt$$

$$\therefore \frac{dy}{\sqrt{\frac{a-y}{y}}} = \sqrt{\frac{2\mu}{a \text{V cos.} \alpha}} \cdot dx$$

$$\text{or} \frac{ydy}{\sqrt{(ay - y^2)}} = \frac{1}{\text{Vcos.} \alpha} \cdot \sqrt{\frac{2\mu}{a}} \cdot dx.$$

$$\text{But} \int \frac{ydy}{\sqrt{(ay - y^2)}} = -\sqrt{(ay-y^2)} + \frac{a}{2}\int \frac{dy}{\sqrt{ay-y^2}}$$

$$= -\sqrt{(ay-y^2)} + \frac{a}{2} \times \text{vers.}^{-1} \frac{2}{a} y + \text{C}$$

Let $y = a$, then $x = 0$, and $\text{C} = -\frac{a}{2} \cdot \pi$ and we get

$$\frac{1}{\sqrt{\cos. \alpha}} \sqrt{\frac{2\mu}{a}} \cdot x + \frac{a}{2} \pi = \frac{a}{2} \text{ vers.}^{-1} \frac{2}{a} y - \sqrt{(ay - y^2)}$$

which is the equation to a *quasi-cycloid*, the diameter of whose generating circle is a.

463. If ψ denote the angular velocity of the radius rector, we have

$$\psi = \frac{d\theta}{dt} = \frac{cd\theta}{\varrho^2 d\theta} = \frac{c}{\varrho^2}$$

But $c = b \sqrt{\frac{\mu}{a}}$ (See 453,) $= \sqrt{a.(1-e^2)\mu}$.

$$\text{and } \varrho = \frac{a.(1-e^2)}{1 + e \cos. \theta}$$

$$\therefore \psi = \frac{\sqrt{\mu}}{a^{\frac{3}{2}}.(1 - e^2)^{\frac{3}{2}}} \times (1 + e \cos. \theta)^2$$

$$= \frac{\sqrt{\mu}}{a^{\frac{3}{2}}.(1 - e^2)^{\frac{3}{2}}} \times (1 + 2e \cos. \theta) \text{ nearly,}$$

and the variation of ψ is nearly as cos. θ.

464. The equation to the circle referred to a point in the circumference is

$$\rho = 2r \sin.\theta$$

when θ is measured from the tangent at that point.

$$\therefore \frac{d\rho}{d\theta} = 2r \cos.\theta$$

$$\text{and } \therefore \frac{d\rho^2}{\rho^4 d\theta^2} = \frac{4r^2\left(1 - \frac{\rho^2}{4r^2}\right)}{\rho^4}$$

$$= \frac{4r^2}{\rho^4} - \frac{1}{\rho^2}$$

Hence (436)

$$F = \frac{c^2}{\rho^3} - \frac{c^2}{2}\left(\frac{2}{\rho^3} - \frac{16r^2}{\rho^5}\right)$$

$$= 8c^2 r^2 \times \frac{1}{\rho^5} = \frac{\mu}{\rho^5}$$

$$\therefore c^2 = \frac{\mu}{8r^2}$$

μ being the absolute force.

Again,

$$dt = \frac{\rho^2 d\theta}{c}$$

$$= 4r^2 d\theta \sin.^2\theta$$

$$\therefore t = \frac{2r^2}{c}(\theta - \sin.\theta.\cos.\theta) + C$$

Let $t = 0$, when $\theta = 0$; then $C = 0$.

Let $\theta = \pi$, then

$$T = \frac{2r^2}{c}\pi = \frac{4\sqrt{2}\times r^3}{\sqrt{\mu}}\pi \text{, the Periodic Time.}$$

Again, when force is in the centre, we have (440)

$$T' = 2\pi\sqrt{\frac{r}{F}}$$

$\therefore$ By the question

$$\frac{4\sqrt{2}\,.\,r^3\pi}{\sqrt{\mu}} = \frac{2\pi\sqrt{r}}{\sqrt{F}}$$

$$\text{or } \frac{2\sqrt{2}\,r^{\frac{5}{2}}}{\sqrt{\mu}} = \frac{1}{\sqrt{F}} = \frac{r^{\frac{5}{2}}}{\sqrt{\mu'}}$$

when $F = \frac{\mu'}{r^5}$ or $\propto \frac{1}{r^5}$ for different circles.

Hence $\mu : \mu' :: 8 : 1$.

465. Since the body is as much retarded in its whole ascent as it is accelerated in its descent, the time of revolution is the same as that of a body moving in a circle with the uniform velocity

$$\sqrt{2gh'}$$

$$\text{and } \therefore T' = \frac{2\pi l}{v'} = \frac{2\pi l}{\sqrt{2gh'}} = \frac{2\pi l m}{\sqrt{4gl}}$$

$$= \pi m \sqrt{\frac{l}{g}}.$$

Again, the time of an oscillation *in a very small circular arc*, whose radius is l, is

$$T = \pi \sqrt{\frac{l}{g}}$$

$$\therefore T' : T :: m : 1.$$

See 570.

466. Since the force is repulsive, the trajectory will be convex to the plane, and its plane passing through the line of projection will be $\perp$ to the given plane, there being no reason why it should be inclined on one side rather than another.

Hence y, x denoting the co-ordinates $\perp$ and parallel to the plane, x being 0 when $y = a$ the given distance from the plane, we have

$$F = \frac{d^2y}{dt^2} = \frac{\mu}{y^3}.$$

$$\therefore \frac{dy^2}{dt^2} = \int \frac{2 \gamma dy}{y^3} = C - \frac{\mu}{y^2}$$
$$= \mu \cdot \left(\frac{1}{a^2} - \frac{1}{y^2} \right)$$

Again, if β be the given velocity,

$$dx = \beta \, dt$$
$$\therefore \frac{a y dy}{\sqrt{(y^2 - a^2)}} = \frac{dx}{\beta}$$
$$\therefore a \sqrt{(y^2 - a^2)} = \frac{x}{\beta} + C$$

Let $y = a$, then $x = 0$ and $C = 0$

$$\therefore y^2 = \frac{x^2}{a^2\beta^2} + a^2$$
$$= \frac{1}{a^2\beta^2}(x^2 + a^4\beta^2)$$
$$= \frac{a^2}{a^4\beta^2}(x^2 + a^4\beta^2)$$

and consequently the required Trajectory is an Hyperbola whose semiaxes are

$a^2\beta$ and a

respectively.

467. Let PQ (Fig. 96,) be the orbit of the Earth round the Sun, S, and pq that of the Moon round the Earth P; then if P, p be their positions when in conjunction, and Q, q those of the next instant; their orbit in fixed space, *viz.*, the Epicycloid, will then be convex or concave to the Sun, according as

$$QR - qr$$

is positive or negative.

But if P, p; R, r are the respective known Periods and Distances of the Earth and Moon, we have (Princip. prop. IV.)

$$QR : qr :: \frac{R}{P^2} : \frac{r}{p^2}$$

$$\text{and QR} - qr = qr \cdot \left(\frac{\text{R}}{\text{P}^2} - \frac{r}{p^2}\right) \times \frac{p^2}{r}$$

and the orbit is convex or concave according as

$$\frac{\text{R}}{\text{P}^2} - \frac{r}{p^2} \text{ or } \text{R}p^2 - r\text{P}^2$$

is positive or negative.

$$\begin{aligned} \text{But R} &= 93{,}000{,}000 \text{ miles} \\ r &= 237{,}000 \text{ miles} \\ \text{P} &= 365 \text{ days} \\ p &= 29 \text{ days} \end{aligned}$$

is round numbers

$$\begin{aligned} \therefore r\text{P}^2 &= 78213000000 \\ \text{R}p^2 &= 31574325000 \end{aligned}$$

consequently the orbit is concave at conjunction.

468. *Generally, let it be required to find the change of force or quantity of matter, in order that the eccentricity of the new elliptical orbit may be the $\frac{1}{n}$ of the radius of the circular orbit.*

Let r be the radius of the circular orbit.

Then since at the point where the change in the force takes place, the velocity and direction of motion are the same as in the circular orbit, and the elliptical orbit touches the circular in that point, we have

$$\text{F} \propto \frac{\text{V}^2}{\text{PV}} \propto \frac{1}{\text{PV}}$$

$$\text{and F} : \text{F}' :: \frac{1}{2r} : \frac{1}{\frac{2b^2}{a}}, \text{ at the vertex}$$

$$\therefore \text{F}' = \frac{2ra}{2b^2} \cdot \text{F} = \frac{ra}{b^2} \cdot \text{F}$$

But $b^2 = a^2 - (\text{eccent.})^2$

and $a = r + \text{eccentricity} = r \cdot \frac{n+1}{n}$ by the question,

$$\therefore b^2 = \left(r + \frac{r}{n}\right)^2 - \frac{r^2}{n^2} = r^2 + \frac{2r^2}{n}$$

and substituting, we get

$$F' = \frac{n+1}{n+2} . F \ldots\ldots\ldots\ldots\ldots\ldots (a)$$

In the problem $n = 2$

$$\therefore F' = \frac{3}{4} F$$

or the quantity of matter in the Earth must be diminished by $\frac{1}{4}$.

Hence, also,

If the quantity of matter be suddenly increased or diminished m *. fold, required the eccentricity of the new orbit, and the change in the Periodic Time.*

$$\text{If } m = \frac{n+1}{n+2}; \text{ then the eccentricity}$$

$$= \frac{r}{n} = \frac{1-m}{2m-1} . r \ldots\ldots\ldots\ldots (b)$$

$$\text{and } a = r + \frac{1-m}{2m-1} . r = \frac{m}{2m-1} . r \ldots\ldots\ldots (c)$$

Consequently if T be the Periodic Time in the circle, and T' in the new orbit, we have, since $T \propto \frac{(\text{axis maj.})^{\frac{3}{2}}}{\sqrt{\mu}}$

$$T : T' :: \frac{r^{\frac{3}{2}}}{\sqrt{\mu}} : \frac{m^{\frac{3}{2}}}{(2m-1)^{\frac{3}{2}}} . \frac{r^{\frac{3}{2}}}{(m\mu)^{\frac{1}{2}}}$$

$$:: 1 : \frac{m}{(2m-1^{\frac{3}{2}}}$$

$$\text{or } T' = \frac{m}{(2m-1)^{\frac{3}{2}}} T \ldots\ldots\ldots\ldots (d$$

469. Here,

$$F = \frac{\mu}{\varrho^3}$$

$$\therefore vdv = -Fd\varrho = -\mu \frac{d\varrho}{\varrho^3}$$

and if a be the whole distance to the centre, we have

$$v^2 = \frac{a^2 - \varrho^2}{a^2\varrho^2} \cdot \mu$$

$$\therefore dt = \frac{d\varrho}{v} = \frac{a}{\sqrt{\mu}} \cdot \frac{\varrho d\varrho}{\sqrt{(a^2 - \varrho^2)}}$$

$$\text{and } t = \frac{a}{\sqrt{\mu}} \cdot \sqrt{(a^2 - \rho^2)}$$

Now put $a - \rho =$ vers. θ the space fallen through. Then

$$\sin.\ \theta = \sqrt{(2a \cdot \overline{a-\rho} - \overline{a-\rho})^2}$$
$$= \sqrt{(a^2 - \rho^2)} = \frac{\sqrt{\mu}}{a} \cdot t$$

$$\text{and } \tan.\ \theta = \frac{a \cdot \sqrt{(a^2 - \rho^2)}}{\rho} = \frac{a^2}{\sqrt{\mu}} \cdot v.$$

$$\therefore t : v :: \frac{a}{\sqrt{\mu}} \sin.\ \theta : \frac{\sqrt{\mu}}{a^2} \cdot \tan.\ \theta$$
$$:: a^3 \sin.\ \theta : \mu \tan.\ \theta.$$

which shews the enunciation to be inaccurate.

470. The mean angular velocity being V, and those round the foci S and H being denoted by S and H, we have easily

$$V : S :: \frac{1}{AC \cdot BC} : \frac{1}{SP^2}$$

$$\text{and } S : H :: HP \times SP : AC \times BC$$

$$\therefore V : H :: HP \times SP^3 : AC^2 \times BC^2$$

But when the eccentricity is very small, $HP = SP = AC = BC$ nearly,

$\therefore V = H$ nearly. Q. E. D.

This is named *Seth Ward's Hypothesis*. By means of it he approximated to the Solution of Kepler's Problem.

471. Let Q, q; D, d, be the quantities of matter and densities of the Earth and Moon, respectively. Then supposing

them to be perfect spheres whose magnitudes or volumes are S, s, we have

$$Q = SD \text{ and } q = sd$$

But if R and r be the radii of these globes, we also have

$$S = \frac{4}{3}\pi R^3, \text{ and } s = \frac{4}{3}\pi r^3$$

$$\therefore Q = \frac{4}{3}\pi R^3 \times D, \text{ and } q = \frac{4}{3}\pi r^3 \times d$$

$$\therefore R = \left(\frac{3Q}{4\pi D}\right)^{\frac{1}{3}}, \text{ and } r = \left(\frac{3q}{4\pi d}\right)^{\frac{1}{3}}$$

which are therefore known.

Hence, if a be the distance from the centres of the Earth and Moon, we have that between their surfaces, *viz.*,

$$a - \overline{R+r}.$$

Now it is evident that the body must be projected with such a velocity as shall just convey it to the point of equal attraction of the Earth and Moon, in order to place it in equilibrium between them, and therefore that the least additional velocity will cause it to proceed towards the Earth. But if x be the distance of the body at any point of its progress from the centre of the Moon, then $a - x$ is its distance from the centre of the Earth, and in order to the point of equal attraction, we have

$$\frac{q}{x^2} = \frac{Q}{(a-x)^2}$$

which gives

$$x = \frac{a\sqrt{q}}{\sqrt{Q} + \sqrt{q}}.$$

Hence, the velocity requisite to carry the body from the moon's surface to this point of equal attraction, is the same as would be acquired through

$$r - \frac{a\sqrt{q}}{\sqrt{Q}+\sqrt{q}}$$

by falling from rest by the force

$$\frac{q}{x^2} - \frac{Q}{(a-x)^2}$$

and $\therefore vdv = -\text{F}dx = \frac{\text{Q}dx}{(a-x)^2} - \frac{qdx}{x^2}$

$$\therefore v^2 = \frac{2\text{Q}}{a-x} - \frac{2q}{x} + \text{C}$$

But $v = 0$, when $x = \frac{a\sqrt{q}}{\sqrt{\text{Q}}+\sqrt{q}}$, and $\therefore a-x = \frac{a\sqrt{\text{Q}}}{\sqrt{\text{Q}}+\sqrt{q}}$

$$\text{and C} = -\frac{2}{a}(\text{Q} - q)$$

$$\therefore v^2 = \frac{2\text{Q}}{a-x} - \frac{2q}{r} - \frac{2}{a}(\text{Q} - q)$$

$$\text{Let } x = r = \left(\frac{3q}{4\pi d}\right)^{\frac{1}{3}}$$

Then

$$v^2 = \frac{2\text{Q}}{a-r} - \frac{2q}{r} - \frac{2}{a}(\text{Q} - q)$$

which gives the velocity required. Hence is deducible a refutation of that theory of meteoric stones which supposes them to be projected from volcanoes in the moon.

472. Let 2R be the diameter of the primary planet, g the force of gravity upon its surface, and p and a the periodic time and distance of its secondary. Then, since generally (F $\propto \frac{1}{\varrho^2}$ we have (by 453)

$$\text{Period} = \frac{2a^{\frac{3}{2}}\pi}{\sqrt{\mu}}$$

where $\mu = \text{M} + m$, M, m being the masses of the primary and secondary respectively.

But since the attraction at the surface of a sphere whose radius is R is (see *Vince's* Fluxions, 144.)

$$\frac{4\text{R}}{3} \times \text{density}.$$

supposing density to be uniform;

$$\therefore g = \frac{4\text{R}}{3} \times \text{density}.$$

$\therefore$ M $= \frac{4\pi}{3}$ R^3 $\times$ density $= g$R$^2\pi$, which is therefore known.

Hence

$$p = \frac{2a^{\frac{3}{2}}\pi}{\sqrt{(M+m)}}$$

$$\text{and } M + m = \frac{4a^3\pi^2}{p^2}$$

$$\therefore m = \frac{4a^3\pi^2}{p^2} - M$$

$$= \frac{4a^3\pi^2}{p^2} - gR^2\pi$$

the mass required.

473. Generally when the force acts in parallel lines

$$F = \frac{d^2y}{dx^2}$$

y being parallel and $x \perp$ to the direction of the force. But in the Ellipse

$$y^2 = \frac{b^2}{a^2}(a^2 - x^2)$$

$$\therefore 2ydy = \frac{2b^2}{a^2} . xdx$$

$$\text{and } \frac{dy}{dx} = \frac{b^2}{a^2} . \frac{x}{y}$$

$$\therefore \frac{d^2y}{dx} = \frac{b^2}{a^2} . \left(\frac{1}{y} - \frac{x}{y^2} \times \frac{dy}{dx}\right)$$

$$\text{or } F = \frac{b^4}{a^2} \times \frac{1}{y^3}$$

But in the circle it will be likewise found that

$$F = \frac{a^2}{y^3}$$

Hence if the absolute force be changed from $\mu = a^2$ to $\mu' = \frac{b^4}{a^2}$, an ellipse having the same diameter $2a$ as the circle, and any

axis-minor whatever may be changed; and generally if it be required to ascertain the change in the absolute force requisite to make the body describe an ellipse of given axes $2a'$, $2b'$, we have

$$\mu' : \mu :: a^2 : \frac{b'^4}{a'^2}$$

$$\text{and } \therefore \mu' = \frac{a^2 b'^4}{a'^2}.$$

474. The velocity $\propto \frac{1}{p}$. Therefore since

$$p^2 = a\varrho$$

and ϱ at the vertex and extremity of latus rectum is respectively

$$\frac{a}{4} \text{ and } \frac{a}{2}.$$

$$\therefore v : v' :: \frac{2}{\sqrt{a}} : \sqrt{\frac{2}{a}} :: \sqrt{2} : 1.$$

475. Since the body revolves in an hyperbola with the force in the centre, we know from *Newton*, *Prop.* X., or very readily from

$$(436) \text{ since } p^2 = \frac{a^2b^2}{\varrho^2 - a^2 + b^2} \text{ that}$$

$$F \propto \varrho = \mu\varrho.$$

But, by 439,

$$F : \varphi :: \frac{\varrho^3}{p^3} : \frac{d\varrho}{dp}$$

$$\text{and } \frac{\varrho^3}{p^3} = \frac{\varrho^4}{a^2b^2} \cdot \frac{d\varrho}{dp}$$

$$\therefore F : \varphi :: \varrho^4 : a^2b^2.$$

476. $v \propto \frac{1}{p}$, and $p^2 = \frac{b^2\varrho}{2a - \varrho}$.

Hence, since at the focus and mean distance

$\varrho = a - \sqrt{(a^2 - b^2)}$ and a

we have

$$v^2 : v'^2 :: 1 : \frac{a - \sqrt{(a^2 - b^2)}}{a + \sqrt{(a^2 - b^2)}}$$

$$:: a + \sqrt{(a^2 - b^2)} : a - \sqrt{(a^2 - b^2)}$$

$$:: 1 + \sqrt{(1 - e^2)} : 1 - \sqrt{(1 - e^2)}$$

$$:: e^2 : e^2 - 2\sqrt{(1 - e^2)}.$$

477. The angular velocity $= \frac{c}{\varrho^2}$, see 463.

Now the mean angular velocity is that by which a body would uniformly describe a circle in the periodic time of the earth, and $\therefore$

$$V = \frac{2\pi}{T} = \frac{2\pi . \sqrt{\mu}}{2\pi a^{\frac{3}{2}}} \ (453) = \frac{\sqrt{\mu}}{a^{\frac{3}{2}}}$$

and at the mean distance a, the angular velocity is

$$V' = \frac{c}{a^2} = b\sqrt{\frac{\mu}{a}} . \frac{1}{a^2}$$

$$= \frac{b\sqrt{\mu}}{a^{\frac{5}{2}}}$$

$$\therefore V : V' :: 1 : \frac{b}{a} :: a : b$$

$\therefore$ V' is $<$ V in the proportion of b to a. Q. E. D.

478. Generally, let R be the given distance or radius-vector, P the corresponding perpendicular upon the tangent, then since by the question and (436)

$$F = \frac{c^2 dp}{p^3 dp} = \mu\varrho$$

$$\therefore \frac{c^2 dp}{p^3} = \mu\varrho d\varrho$$

$$\text{and } -\frac{c^2}{p^2} = \mu\varrho^2 + C.$$

But when $\varrho = R$, $p = P$

$$\text{and } \therefore C = -\frac{c^2}{P^2} - \mu R$$

$$\therefore \frac{c^2}{p^2} = -\frac{c^2}{P^2} + \mu R^2 - \mu\varrho^2$$

$$\text{and } p^2 = \frac{c^2}{\frac{c^2}{P^2} + \mu R^2 - \mu\varrho^2} \ldots\ldots (a)$$

which is the equation to a conic section, and therefore the orbit has one apse at least. But the problem must be solved without considering the nature of the curve; for since at an apse

$$p = \varrho$$

$\therefore$ at an apse

$$\varrho^2 = \frac{c^2}{\frac{c^2}{P^2} + \mu R^2 - \mu\varrho^2}$$

$$\text{and } -\mu\varrho^4 + \left(\frac{c^2}{P^2} + \mu R^2\right)\varrho^2 = c^2$$

$$\text{and } \therefore \varrho^4 - \left(\frac{c^2}{P^2} + \mu R^2\right).\frac{\varrho^2}{\mu} = \frac{c^2}{\mu}$$

$$\therefore \varrho^2 = \frac{c^2}{2P^2} + \frac{\mu R^2}{2} \pm \sqrt{\left\{\left(\frac{c^2}{2P^2} + \frac{\mu R^2}{2}\right)^2 + \frac{c^2}{\mu}\right\}}$$

and ϱ will therefore have at least two real values, which indicate two apses.

If the velocity of projection be given and equal to β; then

$$\beta = \frac{c}{P} \text{ (see 436).}$$

and $c = \beta P$,

and $\therefore$ equation (a) becomes

$$p^2 = \frac{\beta^2 P^2}{\beta^2 + \mu R^2 - \mu\varrho^2} \ldots\ldots (b)$$

which may be useful hereafter.

479. By 438.

Vel. in parab. : vel. in circle same dist. :: $\sqrt{2} : 1$

and vel. in $\odot$ s. dist. : vel. in ellipse :: $\sqrt{a} : \sqrt{2a - \varrho}$

$\therefore$ vel. in parab. : vel. in ellipse :: $\sqrt{2a} : \sqrt{2a - \varrho}$

But at the perihelion $\rho = a - ae$;

$$\therefore v : v' :: \sqrt{2} : \sqrt{1 + e}$$

OTHERWISE.

Since $v = \dfrac{c}{p}$.

and p in this case is given.

$\therefore v \propto c$.

Now in the ellipse

$$c' = b\sqrt{\frac{\mu'}{a}} \quad \text{(See 458.)}$$

and it may easily be shewn that in the parabola, whose latus rectum is $4r$,

$$c = \sqrt{2.r\mu} = \sqrt{2r\mu}$$

$$\therefore v : v' :: \sqrt{(2r\mu)} : b\sqrt{\frac{\mu'}{a}}$$

$$:: \sqrt{2r} : \sqrt{(a\,.\,\overline{1 - e^2})}$$

when the absolute forces are the same.

But $r = a - ae = a(1 - e)$

$$\therefore v : v' :: \sqrt{2} : \sqrt{1 + e}.$$

480. Generally in a circle we have (440)

$$T = 2\pi . \sqrt{\frac{\rho}{F}} \propto \sqrt{\frac{r}{F}},$$

$$v = \frac{2\pi\rho}{T} \propto \frac{r}{T};$$

$\therefore$ if $v^3 \propto T^4$

$$\therefore T^{\frac{4}{3}} \propto \frac{r}{T},$$

$$\text{or } T \propto r^{\frac{3}{7}}$$

$$\text{and } F \propto \frac{r}{T^2} \propto \frac{r}{r^{\frac{6}{7}}} \propto r^{\frac{1}{7}}.$$

$$\text{Also } v \propto \frac{r}{T} \propto \frac{r}{r^{\frac{3}{7}}} \propto r^{\frac{4}{7}}.$$

481. Generally for conical revolutions, if the length of the string be l, and r the radius of the base, then the body being acted upon by three forces; (w) in the direction of gravity; another (S) in the direction of the string or its tension; and the third (F) towards the centre of the base of the cone, and therefore centripetal, we have,

$$\mathrm{W} : \mathrm{S} :: \sqrt{l^2 - r^2}) : l$$

$$\text{or } \mathrm{S} = \frac{l}{\sqrt{(l^2 - r^2)}} \mathrm{W} \ldots\ldots\ldots (a)$$

Again, by 440, the time of revolution in the circular base is

$$\mathrm{T} = 2\pi \sqrt{\frac{r}{\mathrm{F}}} \ldots\ldots\ldots\ldots (b)$$

$$\text{But F : gravity} :: r : \sqrt{(l^2 - r^2)} :: \frac{4\pi^2 r}{\mathrm{T}^2} : g$$

$$\therefore \mathrm{T} = \frac{2\pi\sqrt{(l^2 - r^2)}}{\sqrt{g}} \ldots\ldots\ldots (c)$$

From (c) it appears that the times of revolution, for different conical pendulums are always the same for the same altitude

$$\sqrt{(l^2 - r^2)}.$$

In the Problem

$$\mathrm{S} = 3\mathrm{W}$$

$$\therefore \sqrt{l^2 - r^2} = 3l$$

$$\text{and } \therefore \mathrm{T} = \frac{6\pi l}{\sqrt{g}}.$$

which expresses the required time in seconds, since $\frac{g}{2}$ is the space described in a vacuum by a falling body during that interval.

482. Generally let the perihelion distance of the comet be r, and the mean distance of the earth be a. Then, since the mean velocity of the earth is that by which a circle would be uniformly described in the time of the earth's revolution in its orbit, or in the time

$$\frac{2\pi a^{\frac{3}{2}}}{\sqrt{\mu}},$$

$\therefore$ this velocity is

$$v = \frac{2\pi}{T} = \frac{\sqrt{\mu}}{a^{\frac{1}{2}}}.$$

Also the velocity in a parabola is generally

$$v' = \frac{c}{p} = \frac{\sqrt{2r\mu}}{p}$$

and $p = \sqrt{r\varrho}$

$\therefore$ at the extremity of the latus-rectum, where $\varrho = 2r$

$$p = r\sqrt{2}$$

$$\therefore v' = \sqrt{\frac{\mu}{r}}$$

$$\therefore v : v' :: \frac{1}{a^{\frac{1}{2}}} : \frac{1}{\sqrt{r}} \ldots\ldots\ldots (a)$$

In the Problem

$a = 100, r = 64.$

$\therefore v : v' :: 8 : 1000$

$:: 1 : 125$

or $v' = 125. v.$

483. This very ingenious theorem is easily reduced to the following; *If* SAC *(Fig. 97.) be a parabolic area,* S *being the focus, and* A *the vertex of the parabola, and perpendiculars bisecting* AS and SC *meet in* H; *then* GH *is proportional to the area* ASC.

For the figure being completed, we have

$$HG+GR : RS+SM :: SC : CN$$

$$\therefore HG = \frac{SC}{CN} \times \left(RS + \frac{SC}{2}\right) - GR.$$

But $GR : GS :: CN : SN$

and $RS : GS :: CS : SN$

$$\therefore HG = \frac{SC}{CN} \times \left(\frac{GS.CS}{SN} + \frac{CS}{2}\right) - \frac{GS.CN}{SN}$$

$$= \frac{(SC^2 - CN^2)GS}{SN \times CN} + \frac{CS^2}{2CN}$$

$$= \frac{SN \times GS}{CN} + \frac{CS^2}{2CN}$$

$$= \frac{2SN.GS + CN^2 + SN^2}{2CN}$$

$$= \frac{SN \times AN}{2CN} + \frac{CN}{2}.$$

But by the nature of the parabola

$$CN^2 = 4AS \times AN;$$

$$\therefore GH = \frac{SN \times CN}{8AS} + \frac{CN}{2}$$

$$= \frac{(SN + 4AS) \times CN}{8AS}$$

$$= \frac{4AN - 3SN}{8AS} \times CN$$

$$= \frac{3}{4AS} \times \left(\frac{2}{3}AN \times CN - \frac{1}{2} SN.CN\right)$$

$$= \frac{3}{4\ AS} \times (\text{area ANC} - \triangle\ \text{SNC})$$

$$= \frac{3}{4AS} \times \text{area ASC}.$$

Consequently by Kepler's Law

GH $\propto$ time of comet's describing AC.

484. Since generally

$$dt = \frac{ds}{v}, \text{ and } v = \frac{c}{p}$$

$$\therefore dt = \frac{pds}{c}$$

But $ds : d\varrho :: \varrho : \sqrt{(\varrho^2 - p^2)}$,

$$\therefore dt = \frac{p\varrho d\varrho}{c \times \sqrt{(\varrho^2 - p^2)}} \text{. . . . .} (a)$$

Now in the parabola

$$p^2 = r\varrho$$

r being the shortest distance,

$$\therefore dt = \frac{\sqrt{r\varrho}.\varrho d\varrho}{c\sqrt{\varrho} \times \sqrt{(\varrho - r)}} = \frac{\sqrt{r}}{c} \times \frac{\rho d\rho}{\sqrt{(\varrho - r)}}$$

$$\therefore t = \frac{2\sqrt{r}}{c}\rho\sqrt{(\rho - r)} - \frac{2\sqrt{r}}{c}\int d\rho\,\sqrt{(\rho - r)}$$

$$= \frac{2\sqrt{r}}{c}\{\rho\sqrt{(\rho - r)} - \frac{2}{3}.(\rho - r)^{\frac{3}{2}}\}$$

there being no correction, if t is 0 when $\rho = r$.

But $c = \sqrt{(2r\mu)}$ by (453).

$\therefore$ the general expression for the describiug any arc of a parabola from the perihelion is

$$t = \sqrt{\frac{2}{\mu}} \times \sqrt{(\rho - r)} \times \left(\frac{\rho}{3} + \frac{2}{3}r\right) \ldots\ldots (b)$$

Let $\rho = 2r$. Then

$$t = \frac{4}{3}.\ \sqrt{\frac{2}{\mu}} \times r^{\frac{3}{2}} \propto r^{\frac{3}{2}} \ldots\ldots\ldots (c)$$

If we use the expression (a) for the ellipse, whose equation is

$$p^2 = \frac{b^2\rho}{2a - \rho}$$

we get

$$dt = \frac{b\rho d\rho}{c\sqrt{(2a\rho - b^2 - \rho^2)}}$$

and putting

$$\rho - a = u$$

it becomes

$$dt = \frac{b(a + u).du}{c\sqrt{(a^2e^2 - u)}}$$

ae being the eccentricity.

Hence

$$t = \frac{ba}{c}\int\frac{du}{\sqrt{(a^2e^2 - u^2)}} + \frac{b}{c}\int\frac{udu}{\sqrt{(a^2e^2 - u^2)}}$$

$$= \frac{ba}{c}.\sin.^{-1}\frac{u}{ae} - \frac{b}{c}\sqrt{(a^2e^2 - u^2)} + C.$$

Let $t = 0$, when $u = -ae$, or when $\rho = a - ae$

$$\text{then } C = -\frac{ba}{c}\sin.^{-1}(-1) = -\frac{ba}{c}.\left(-\frac{\pi}{2}\right) = \frac{ba}{c}.\frac{\pi}{2}$$

$$\therefore t = \frac{ba}{c} \times \left(\sin.^{-1}\frac{u}{ac} + \frac{\pi}{2}\right) - \frac{b}{c}\sqrt{(a^2e^2 - u^2)}$$

But $c = b\sqrt{\frac{\mu}{a}}$, and $u = \varrho - a$

$$\therefore t = \frac{a^{\frac{3}{2}}}{\sqrt{\mu}}\left(\sin.^{-1}\frac{\varrho - a}{ae} + \frac{\pi}{2}\right) - \sqrt{\frac{a}{\mu}} \times \sqrt{(a^2e^2 - \overline{\varrho - a}^2)}$$

the general expression for the time.

$$\text{Let } \varrho = \frac{b^2}{a} = a\,.\,(1 - e^2)$$

and we get *for the time between the vertex of the ellipse and the extremity of its latus rectum,*

$$T = \frac{a^{\frac{3}{2}}}{\sqrt{\mu}}(\sin.^{-1}) - e) + \frac{\pi}{2}\Big) - \sqrt{\frac{a}{\mu}}.\, ae\sqrt{(1 - e^2)}$$

$$= \frac{a^{\frac{3}{2}}}{\sqrt{\mu}}\left\{\frac{\pi}{2} - \sin^{-1} e - e\sqrt{(1 - e^2)}\right\} \ldots\ldots (d)$$

Again, let $\varrho = a$, then the time from the perihelion to the extremity of the minor-axis is

$$T = \frac{a^{\frac{3}{2}}}{\sqrt{\mu}}.\frac{\pi}{2} - \sqrt{\frac{a}{\mu}}.\, ae$$

$$= \frac{\frac{3}{2}}{\sqrt{\mu}} \times \left(\frac{\pi}{2} - e\right) \ldots\ldots (e)$$

Hence, and from expression (d) it follows that *if e or the ratio of the eccentricity to the major axis is given, then the Times from the vertex either to the extremity of the latus-rectum, or to that of the axis-minor, are in the sesquiplicate ratio of the major-axes, and the subduplicate ratio of the absolute forces.*

Again, let

$$\varrho = a + ae$$

then, $\frac{T}{2}$ being the semi-period

$$\frac{T}{2} = \frac{\pi a^{\frac{3}{2}}}{\sqrt{\mu}}$$

$$\text{or } T = \frac{2\pi a^{\frac{3}{2}}}{\sqrt{\mu}} \ldots\ldots\ldots\ldots\ldots\ldots (f)$$

the whole Periodic Time in an Ellipse.

Again, in expression (d)

let $e = 0$. Then

$$T = \frac{\pi}{2} \cdot \frac{a^{\frac{3}{2}}}{\sqrt{\mu}}$$

$$\text{and } 4T = \frac{2\pi a^{\frac{3}{2}}}{\sqrt{\mu}} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (g)$$

the Periodic Time in a circle when the force $\propto \frac{1}{\varrho^2}$ this also follows from (f).

Again in the expression (d), if $4r$ be the latus-rectum, we have

$$a = \frac{2r}{1-e^2}$$

$$\text{and } T = \frac{2\sqrt{2}\,.\,r^{\frac{3}{2}}}{\sqrt{\mu}} \times \frac{\frac{\pi}{2} - e\sqrt{(1-e^2)} - \sin.^{-1} e}{(1-e^2)^{\frac{3}{2}}}.$$

Now when the Ellipse becomes a Parabola, $e = 1$. Consequently the time from the vertex to the extremities of the latus rectum in the Parabola is

$$T = \frac{2\sqrt{2}\,.\,r^{\frac{3}{2}}}{\sqrt{\mu}} \times \frac{\frac{\pi}{2} - 0 - \frac{\pi}{2}}{(1-1)^{\frac{3}{2}}}$$

a Vanishing Fraction;

to find whose value, let

$$P = \frac{\pi}{2} - \sin.^{-1} e - e\sqrt{(1-e^2)}$$

$$Q = (1-e^2)^{\frac{3}{2}}$$

$$\text{Then } \frac{dP}{dQ} = \frac{-\frac{1}{\sqrt{1-e^2}} - \sqrt{1-e^2} + \frac{e^2}{\sqrt{1-e^2}}}{-3e\sqrt{1-e^2}}$$

$$= \frac{2(1-e^2)}{3e(1-e^2)} = \frac{2}{3e}$$

Hence the Fraction $= \frac{2}{3}$, when $e = 1$, and we have

$$T = \frac{4}{3}\sqrt{\frac{2}{\mu}} \cdot r^{\frac{3}{2}}$$

which is the same as (*c*).

This last case affords a striking example of the utility of the Theory of Vanishing Fractions.

485. The angular velocity $\propto \frac{1}{\varrho^2}$ (463)

$\therefore$ if v and v' denote the velocities of the earth at its mean distance a and perihelion distance $a - ae$, ae being the eccentricity, we have

$$v : v' :: \frac{1}{a^2} : \frac{1}{a^2(1-e)} :: 1 - e : 1. \qquad \text{Q. E. D.}$$

486. The centripetal force acting upon a point placed within a sphere $\propto \varrho$, see Newton's Prop. LXXIII. Consequently, if g be the force at the surface of the sphere, and R its radius, we have

$$F : g :: \varrho : R$$

$$\therefore F = \frac{g}{R} \cdot \varrho$$

$$\text{and } vdv = -Fd\varrho = -\frac{g}{R}\rho d\rho.$$

$$\therefore v^2 = \frac{g}{R} \cdot (C - \varrho^2)$$

Let $\varrho = R$, then $v = 0$, and

$$v^2 = \frac{g}{R}(R^2 - \varrho^2) \ldots\ldots\ldots\ldots (a)$$

Let $\rho = 0$. Then

$$V^2 = gR \ldots\ldots\ldots\ldots\ldots\ldots (b)$$

But the velocity (V') in a circle at the Earth's surface is that which would be acquired down $\frac{R}{2}$ with the force considered constant.

$$\therefore V'^2 = 2g \times \frac{R}{2} = gR$$

$$\therefore V' = V.$$

487. Generally, let $F' = \frac{n\mu}{\varrho^3}$. Then

$$\frac{n\mu}{\varrho^3} = \frac{c^2 dp}{p^3 dp}$$

$$\therefore \frac{n\mu}{\varrho^2} = \frac{c^2}{p^2} + C$$

But since the force in the circle is $\frac{\mu}{R^3}$ the initial velocity, *viz.*

$$\frac{c}{R} = \sqrt{\frac{\mu\theta}{R^3} \times R} = \frac{1}{R}\sqrt{\mu}$$

$$\therefore c = \sqrt{\mu}$$

and $\frac{n\mu}{\varrho^2} = \frac{\mu}{p^2} + C$

Let $\varrho = R$. Then $p = R$

and $C = \frac{\overline{n-1}\cdot\mu}{R^2}$

$$\therefore \frac{1}{p^2} = \frac{n}{\varrho^2} - \frac{n-1}{R^2}$$

$$= \frac{nR^2 - \overline{n-1}\,\varrho^2}{R^2\varrho^2}$$

and $p = \dfrac{R\varrho}{\sqrt{(nR^2 - (n-1)\varrho^2)}}$ . . . . (a)

the equation to the new Trajectory.

Also since

$$p : \rho :: \varrho d\theta : d\varrho$$

from similar triangles in the elemental figure, we have

$$d\theta = \frac{R d\varrho}{\varrho\sqrt{(nR^2 - \overline{n-1}\cdot\varrho^2)}}$$

$$\therefore \theta = \frac{1}{2\sqrt{n}} \cdot l \cdot \frac{\sqrt{(nR^2 - \overline{n-1} \cdot \varrho^2)} - R\sqrt{n}}{\sqrt{(nR^2 - \overline{n-1} \cdot \varrho^2)} + R\sqrt{n}} + C$$

see Hirsch's Integral Tables, *p.* 122.

Let $\theta = 0$, when $\varrho = R$, and we have

$$C = -l \cdot \frac{R - R\sqrt{n}}{R + R\sqrt{n}} = -l \cdot \frac{1 - \sqrt{n}}{1 + \sqrt{n}}$$

$$\therefore \theta = \frac{1}{2\sqrt{n}} \cdot l \cdot \left\{ \frac{1+\sqrt{n}}{1-\sqrt{n}} \cdot \frac{\sqrt{(nR^2 - \overline{n-1} \cdot \varrho^2)} - R\sqrt{n}}{\sqrt{(nR^2 - \overline{n-1} \cdot \varrho^2)} + R\sqrt{n}} \right\} \ldots\ldots (b)$$

the polar equation of the spiral.

Let $\varrho = 0$. Then

$$\theta = \frac{1}{2\sqrt{n}} \cdot \log. 0 = \infty.$$

or the number of revolutions will be infinite before the body falls into the centre, that is, it never reaches it, although it continually approaches the centre.

Again, let $\varrho = \frac{R}{2}$; then the body will have approached half way to the centre. In this case by (a)

$$p = \frac{R}{2\sqrt{\left(n - \frac{\overline{n-1}}{4}\right)}} = \frac{R}{\sqrt{(3n+1)}}.$$

Hence, the velocity at that point is

$$V = \frac{c}{p} = \sqrt{\mu} \times \frac{\sqrt{3n+1}}{R}$$

and the velocity in a circle at the distance $\frac{R}{2}$ is

$$V' = \sqrt{\frac{2\mu}{\left(\frac{R}{2}\right)^2} \times \frac{R}{2}} = \frac{1}{R}\sqrt{\mu}$$

$$\therefore V : V' :: \sqrt{(3n+1)} : 2 \ldots\ldots\ldots\ldots\ldots (c)$$

In the problem $n = 2$. Therefore

$$V : V' :: \sqrt{7} : 2.$$

488. If the force be the same in the circle and parabola, we have

$$v = \frac{c}{p} = \frac{\sqrt{2r\mu}}{\sqrt{r\rho}} = \sqrt{\frac{2\mu}{\rho}}$$

$$v' = \frac{c'}{R} = \frac{R}{R} \cdot \sqrt{\frac{\mu}{R}} = \sqrt{\frac{\mu}{R}} \quad (458)$$

$$\therefore v : v' :: \sqrt{\frac{2}{\rho}} : \sqrt{\frac{1}{R}}$$

for any distances with the same force.

In the problem

$$\rho = r, \text{ and } R = 2r$$

$$\therefore v : v' :: \sqrt{\frac{2}{r}} : \sqrt{\frac{1}{2r}}$$

$$:: 2 : 1.$$

489. Let t be the time required, and make $F = \frac{\mu}{\rho}$. Then

$$vdv = -\frac{\mu d\rho}{\rho}$$

$$\text{and } v^2 = 2\mu \times (la - l\rho) = 2\mu \cdot l \cdot \frac{a}{\rho}$$

a being the distance from the centre when the body begins its descent.

Let $\rho = a - \alpha$

α being the given space. Then

$$v^2 = 2\mu \times l \cdot \frac{a}{a - \alpha}$$

which gives the velocity acquired down the given space.

Again,

$$dt = \frac{d\rho}{v} = \frac{1}{2\mu} \cdot \frac{d\rho}{la - l\rho}$$

Let $l\,\frac{a}{\rho} = u.$

Then $\frac{a}{\rho} = e^u$

and $\varrho = ae^{-u}$

and $d\varrho = -a \,.\, e^{-u} du$

$$\therefore dt = -\frac{a}{2\mu} \times \frac{e^{-u}\,du}{u}$$

for the Integration of which see Lacroix's *Differential and Integral Calculus*, 4to edit., or Whewell's *Dynamics*, p. 15.

490. The tension is = the centripetal force necessary to retain the stone in the circle, or it is

$$F = \frac{4\pi^2 r}{T^2}$$

r being the length of the string.

$$\text{Also F : Weight} :: \frac{4\pi^2 r}{T^2} : g :: 4 : 1$$

by the question.

$$\text{Hence } T = \pi\sqrt{\frac{r}{g}}.$$

But $r = 2$ yards $= 6$ feet,

and $g = 32\frac{1}{6}$ feet,

and ∴ by substituting and performing the arithmetical operations we shall get the time required.

491. *Generally let it be required to find the velocity acquired through* $(n+1) \times R$ *to the earth's centre; R being its radius.*

For any distance ϱ we have

$$F : g :: \frac{1}{\varrho^2} : \frac{1}{R^2}$$

$$\therefore F = \frac{gR^2}{\varrho^2}$$

$$\therefore vdv = -Fd\varrho = -gR^2 \times \frac{d\varrho}{\varrho^2}$$

and $v^2 = 2gR^2 \times \left(\frac{1}{\varrho} - \frac{1}{(n+1)R}\right) \dots\dots\dots (a)$

Let $\varrho = nR$, and we have

$$v^2 = 2gR \times \left(\frac{1}{n} - \frac{1}{n+1}\right) = \frac{2gR}{n.(n+1)}$$

and $V = \sqrt{\frac{2gR}{n.(n+1)}} \dots\dots\dots\dots\dots\dots (b)$

the velocity acquiring in descending to the surface of the earth.

Again, since within the Earth's surface $F \propto \varrho$, we have

$$F : g :: \rho : R$$

$$\therefore F = \frac{g}{R} \times \varrho$$

and $v^2 = \frac{g}{R} \times (R^2 - \varrho^2) \dots\dots\dots\dots\dots\dots (a')$

Let $\varrho = 0$. Then

$$V'^2 = \frac{g}{R} . R^2$$

and $V' R = \sqrt{gR} \dots\dots\dots\dots (b')$

the velocity acquired from the surface to the centre.

Hence

$V + V' = \sqrt{gR} \times \left(1 + \frac{2}{\sqrt{n.n+1}}\right) \dots\dots\dots\dots (c)$

is the velocity required.

In the Problem

$$n = 1.$$

$$\therefore V + V' = \sqrt{gR} \times (1 + \sqrt{2})$$

492. Let a and b be the major and minor axes of the given ellipse. Then the greatest dist. $= a + \sqrt{a^2 - b^2} = a(1+e)$ is known.

Again, the Periodic Time in the Ellipse is (484)

$$T = \frac{2\pi a^{\frac{3}{2}}}{\sqrt{\mu}}$$

and in the circle for the same force it is (considering it the limit of an ellipse)

$$T' = \frac{2\pi a^{\frac{3}{2}}(1+e)^{\frac{3}{2}}}{\sqrt{\mu}}$$

$$\therefore T : T' :: 1 : (1+e)^{\frac{3}{2}}.$$

493. By (448) if v and v' denote the angular velocities of ϱ and p, we have

$$v : v' :: \frac{d\rho}{dp} : \frac{\varrho}{p}$$

$$\therefore v' = v \cdot \frac{\varrho}{p} \times \frac{dp}{d\rho}$$

But $v = \dfrac{c}{\varrho^2} = b\sqrt{\dfrac{\mu}{a}} \cdot \dfrac{1}{\varrho^2}$ (463 and 458)

$$\therefore v' = b\sqrt{\frac{\mu}{a}} \cdot \frac{dp}{p\varrho d\rho}.$$

But $p^2 = \dfrac{b^2\varrho}{2a-\varrho}$ in the ellipse.

$$\therefore \frac{dp}{d\varrho} = \frac{2ab^2}{(2a-\varrho)^2} = \frac{2ab^2p^4}{b^4\varrho^2}.$$

$$= \frac{2a}{b^2} \cdot \frac{p^4}{\rho^2}$$

$$\therefore v' = b\sqrt{\frac{\mu}{a}} \times \frac{2a}{b^2} \cdot \frac{p^3}{\varrho^3}$$

$$= \frac{2}{b} \cdot \sqrt{a\mu} \times \frac{p^3}{\rho^3}$$

$$= \frac{2}{b}\sqrt{a\mu} \times \frac{b^3\rho^{\frac{3}{2}}}{\rho^3 \times (2a-\rho)^{\frac{3}{2}}}$$

$$= 2b^2\sqrt{a\mu} \times \frac{1}{(2a\varrho - \varrho^2)^{\frac{3}{2}}}.$$

Hence v' is least when

$$2a\rho - \rho^2$$

is greatest, or when

$$\frac{d\,(2a\rho - \rho^2)}{d\rho} = 2a - 2\rho = 0$$

that is when

$$\rho = a;$$

or the angular velocity of p in the ellipse, the force being in the focus, is least at the extremity of the minor-axis.

494. By 487 when from $\frac{\mu}{\rho^3}$ the force suddenly becomes $\frac{n\mu}{\rho^3}$, we have

$$p = \frac{R\rho}{\sqrt{(aR^2 - \overline{n-1}\,.\,\rho^2)}}$$

$$\text{and } \theta = \frac{1}{2\sqrt{n}}\,.\,l\,.\left\{\frac{1+\sqrt{n}}{1-\sqrt{n}}\,.\,\frac{\sqrt{(nR^2 - \overline{n-1}\,\rho^2)} - R\sqrt{n}}{\sqrt{(nR^2 - \overline{n-1}\,.\,\rho^2)} + R\sqrt{n}}\right\}$$

Hence when $\rho = \infty$

$$\theta = \frac{1}{2\sqrt{n}} \times l\,.\,\frac{1+\sqrt{n}}{1-\sqrt{n}} = \frac{1}{\sqrt{n}}\,l\,\frac{1+\sqrt{n}}{1-n}$$

and the number of revolutions is

$$N = \frac{1}{2\pi\sqrt{n}}\,.\,l\,.\,\frac{1+\sqrt{n}}{1-n}.$$

In the problem, $n = \frac{1}{2}$.

$$\therefore N = \frac{1}{\pi\sqrt{2}}\,.\,l\,.\,(2+\sqrt{2})$$

$$= \frac{1}{\pi\sqrt{2}}\,.\,l\,.\,(\sqrt{2}) + \frac{1}{\pi\sqrt{2}}\,.\,l\,.\,(1+\sqrt{2})$$

But $l\,.\,(1+\sqrt{2}) = 2\left\{\frac{\sqrt{2}}{1+\sqrt{2}} + \frac{1}{3}\left(\frac{\sqrt{2}}{1+\sqrt{2}}\right)^3 + \frac{1}{5}\times\right.$ $\left.\left(\frac{\sqrt{2}}{1+\sqrt{2}}\right)^5 + \&c.\right\}$

Hence, if a few terms of this converging series be summed, and the sum be called S, we get

$$N = \frac{1}{2\pi\sqrt{2}} \times l.2 + \frac{\sqrt{2}}{\pi} \times S$$

which may be arithmetically computed to any degree of accuracy, by means of Logarithmic Tables. The whole number in the result will answer the problem.

495. Let a, b be the semiaxes of the Ellipse; then since (by 486)

$$F = \frac{\mu}{\varrho^2}$$

$$vdv = -Fd\rho = \frac{-\mu d\varrho}{\varrho^2}$$

$$\text{and } v^2 = 2\mu \times \left\{\frac{1}{\varrho} - \frac{1}{a + \sqrt{a^2 - b^2}}\right\} = \frac{2\mu}{r\varrho}(r - \rho)$$

$a + \sqrt{a^2 - b^2}$, which is the distance from the farther apside, being put $= r$.

Again,

$$dt = \frac{d\varrho}{v} = \sqrt{\frac{r}{2\mu}} \cdot \frac{\varrho d\varrho}{\sqrt{(r\varrho - \varrho^2)}}$$

$$\text{and } t = \sqrt{\frac{r}{2\mu}} \times \left\{\sqrt{(r\rho - \rho^2)} - \frac{r}{2} \times \int \frac{dr}{\sqrt{(r\rho - \rho^2)}}\right\}$$

$$= \sqrt{\frac{r}{2\mu}} \times \{\sqrt{r\varrho - \varrho^2}) - \frac{r}{2} \cdot \text{vers.}^{-1} \frac{2}{r}\rho + C\}$$

see Hirsch's *Integral Tables*, pp. 141 and 140.

Let $\varrho = r$; then

$$t = 0, \text{ and } c = \frac{r}{2} \cdot \text{vers.}^{-1} 2 = \frac{r\pi}{2}$$

$$\therefore t = \sqrt{\frac{r}{2\mu}} \times \left\{\sqrt{(r\rho - \rho^2)} + \frac{r}{2}\left(\pi - \text{vers.}^{-1} \frac{2\varrho}{r}\right)\right\}$$

Let $\varrho = 0$. Then the time to the focus from the farther apse is

$$T = \frac{\pi}{2\sqrt{2}} \cdot \frac{r^{\frac{3}{2}}}{\sqrt{\mu}}$$

Also by 484, the Periodic Time in the Ellipse is

$$T' = \frac{2\pi a^{\frac{3}{2}}}{\sqrt{\mu}}.$$

$$\therefore T : T' :: r^{\frac{3}{2}} : 4\sqrt{2}a^{\frac{3}{2}}.$$

496. The time from Perihelion to mean distance is (by 484 . e)

$$\frac{a^{\frac{3}{2}}}{\sqrt{\mu}} \times \left(\frac{\pi}{2} - e\right)$$

where

$$e = \frac{\sqrt{a^2 - b^2}}{a}$$

and the whole Periodic time is (484)

$$T = \frac{2\pi a^{\frac{3}{2}}}{\sqrt{\mu}}.$$

$\therefore$ the time from mean distance to mean distance through aphelion is

$$T' = T - 2 . \frac{a^{\frac{3}{2}}}{\sqrt{\mu}} \times \left(\frac{\pi}{2} - e\right)$$

$$= \frac{2a^{\frac{3}{2}}}{\sqrt{\mu}} \times \left(\pi - \frac{\pi}{2} + e\right)$$

$$= \frac{9a^{\frac{3}{2}}}{\sqrt{\mu}} \times \left(\frac{\pi}{2} + e\right).$$

Now if r be the radius required, since the Periodic Time in the circle is

$$\frac{2\pi r^{\frac{3}{2}}}{\sqrt{\mu}};$$

therefore by the question, we have

$$\frac{2\pi r^{\frac{3}{2}}}{\sqrt{\mu}} = \frac{2a^{\frac{3}{2}}}{\sqrt{\mu}} \times \left(\frac{\pi}{2} + e\right)$$

$$\therefore r^{\frac{3}{2}} = \frac{a^{\frac{3}{2}}}{\pi} \times \left(\frac{\pi}{2} + e\right)$$

$$= a^{\frac{3}{2}} \times \left(\frac{e}{\pi} \times \frac{1}{2}\right)$$

$$\text{and } r = a \times \left(\frac{e}{\pi} + \frac{1}{2}\right)^{\frac{2}{3}},$$

the radius required.

497. Let the given altitude be a. Then, since

$$F = \mu\varrho$$

we have

$$vdv = -\mu\varrho d\varrho$$

$$\text{and } v^2 = \mu(a^2 - \varrho^2).$$

let $\varrho = 0$; and the velocity acquired through a, or the velocity of the system is

$$V = a\sqrt{\mu}$$

Again, if $a - \varrho = x$ be the space described $\perp$ direction of the motion of the system, whilst the body acquires the velocity v, we have

$$dt = \frac{dx}{v} = \frac{dx}{\sqrt{\mu} \times \sqrt{(2ax - x^2)}}.$$

But since the motion of the system is uniform, if dy denote the distance moved through in the time dt, we have

$$dy = V \times dt\, a\sqrt{\mu} \times dt$$

$$\therefore \frac{dy}{a\sqrt{\mu}} = dt = \frac{dx}{\sqrt{\mu} . \sqrt{(2ax - x^2)}}$$

$$\text{and } \frac{y}{a} = \int \frac{dx}{\sqrt{(2ax - x^2)}} = \text{vers.}^{-1} \frac{x}{a} + C.$$

But when $x = 0$, $y = 0$, and $\therefore C = 0$.

Hence

$$y = a \,.\, \text{verse.}^{-1} \frac{x}{a}.$$

$$\text{or } x = a - a \cos. \frac{y}{a} \ldots\ldots \ldots\ldots \ldots\ldots \ldots (m)$$

the equation to the curve traced in the plane passing through the two directions of motion. Hence the curve is a *sinuoid*, which may easily be constructed by the formula

$$\tan.(\varphi) = \frac{dy}{dx} = \frac{a}{\sqrt{(2ax - x)}}$$

and the equation (m);

φ being the inclination of the tangent at any point to the line of abscissæ.

498. Let $a - \rho$ be the altitude fallen through in the time t, a being the finite distance from the centre at which the body begins its descent; then, by the question

$$t \propto (a - \rho)^n = \text{M} . (a - \rho)^n$$

$$\therefore dt = n\text{M} . (a - \rho)^{n-1} . d\rho = \frac{d\rho}{v}$$

$$\therefore v = \frac{1}{n\text{M} . (a - \rho)^{n-1}}.$$

Hence

$$dv = \frac{-d\rho}{n^2\text{M} . (a - \rho)^n}$$

$$\text{and } vdv = \frac{-d\rho}{n^3\text{M}^2 . (a - \rho)^{2n-1}} = -\text{F}d\rho$$

$$\therefore \text{F} = \frac{1}{n^3\text{M}^2 . (a - \rho)^{2n-1}} \propto \frac{1}{(a - \rho)^{2n-1}}$$

the law required.

499. If F and φ denote the centripetal and centrifugal forces respectively, we have, by 439,

$$\text{F} : \varphi :: \frac{\rho^3}{p^3} : \frac{d\rho}{dp}$$

$$\therefore \varphi = \frac{d\rho}{dp} \times \frac{p}{\rho^3} \times \text{F}.$$

$$\text{But F} = \frac{c^2 dp}{p^3 d\rho} \text{ (436)}$$

$$\therefore \varphi = \frac{c^2}{\rho^3} \dots\dots\dots\dots\dots\dots (a)$$

But if V and P be the initial velocity, and perpendicular upon the tangent, at any given point, we have

$$c = \frac{V}{P}$$

$$\text{and } \therefore \varphi = \frac{V^2}{P^2 \cdot \rho^3} \dots\dots\dots\dots\dots (a)$$

Hence for different parts of the same trajectory, we have

$$\varphi \propto \frac{1}{\rho^3} \dots\dots\dots\dots\dots\dots (b)$$

for different trajectories, having at the same one point of them the same velocity V

$$\varphi \propto \frac{1}{P^2 \rho^3} \dots\dots\dots\dots\dots\dots (c)$$

for different trajectories having for same one point of them the same $\perp$ upon the tangent P

$$\varphi \propto \frac{V^2}{\rho^3} \dots\dots\dots\dots\dots\dots (d)$$

and for different trajectories, having neither a common velocity nor a common $\perp$ upon the tangent

$$\varphi \propto \frac{V^2}{P^2 \cdot \rho^3} \dots\dots\dots\dots\dots\dots (e)$$

500. The velocity in general is

$$v = \frac{c}{p}.$$

Hence since at the extremity of the latus rectum $\left(\frac{2b^2}{a}\right)$ of the ellipse

$$p^2 = \frac{b^2 \rho}{2a - \rho} = \frac{b^2 \times \frac{b^2}{a}}{2a - \frac{b^2}{a}} = \frac{a^2 \cdot (1 - e^2)^2}{1 + e^2}$$

where $e^2 = \frac{a^2 - b^2}{a^2}$, we have

$$V = \frac{c}{a} \cdot \frac{\sqrt{(1 + e^2)}}{1 - e^2} \dots\dots\dots (a)$$

Again, in the circle, $p = \rho$, and if $\rho = a - ae =$ the perihelion distance, the velocity in the circle is

$$V' = \frac{c'}{\rho} = \frac{c'}{a \cdot (1 - e)}$$

$$\therefore \frac{V}{V'} = \frac{c}{c'} \times \frac{\sqrt{(1 + e^2)}}{1 + e}.$$

But if μ be the absolute force we have generally in the ellipse whose axes are 2A, and 2B (see 453,)

$$c = B \sqrt{\frac{\mu}{A}}$$

and $\therefore$ for the above ellipse

$$c = b \sqrt{\frac{\mu}{a}} = \sqrt{a \cdot (1 - e^2)\mu}.$$

and for the circle (considering it the limit of an ellipse)

$$c' = \sqrt{a \cdot (1 - e)\mu}.$$

Hence

$$\frac{V}{V'} = \sqrt{(1 + e)} \times \frac{\sqrt{(1 + e^2)}}{1 + e}$$

$$= \frac{\sqrt{(1 + e^2)}}{\sqrt{(1 + e)}} \dots\dots\dots\dots\dots (b)$$

Now let e become Indefinite, and let L be the Limit; then putting

$$\frac{V}{V'} = L + l$$

l being the Indefinite part of the ratio, we have

$$(L + l)^2 = \frac{1 + e^2}{1 + e}$$

$$\text{or } L^2 + 2Ll + l^2 + e(L + l)^2 = 1 + e^2$$

and equating Definite Quantities, (see *Wright's Commentary on the Principia of Newton*, sect. I.)

$$L^2 = 1, \text{ or } L = 1$$

for the limit of $\frac{V}{V'}$ whether the eccentricity become Indefinitely great or Indefinitely small.

501. By (495,) the time down any space $r - \varrho$ is

$$t = \sqrt{\frac{r}{2\mu}}\left\{\sqrt{(r\varrho - \varrho^2)} + \frac{r}{2}\left(\pi - \text{vers.}^{-1}\frac{2\varrho}{r}\right)\right.$$

Hence if $\varrho = 0$, then the time through r is

$$T = \sqrt{\frac{r}{2\mu}} \times \frac{r\pi}{2} = \frac{r^{\frac{3}{2}}\pi}{2\sqrt{2\mu}}.$$

The periodic time in a circle whose radius is r and $F \propto \frac{1}{\varrho^2}$, is also

$$T' = \frac{2\pi r^{\frac{3}{2}}}{\sqrt{\mu}}$$

$$\therefore T : T' :: \frac{1}{2\sqrt{2}} : 2$$

$$:: 1 : 4\sqrt{2}.$$

OTHERWISE

By prop. XXXVI. and IV. of the *Principia*,

$$T : t\ (\text{time in} \odot \text{ rad.} = \frac{SA}{2}) :: 1 : 2$$

$$\text{and } t : T' :: \left(\frac{r}{2}\right)^{\frac{3}{2}} : r^{\frac{3}{2}} :: 1 : 2\sqrt{2}$$

$$\therefore T : T' :: 1 : 4\sqrt{2} \text{ as before.}$$

502. If the force be $\frac{\mu}{\varrho^2}$

we have (484)

$$T = \frac{2\pi a^{\frac{3}{2}}}{\sqrt{\mu}}.$$

But by the question

$$\frac{\mu}{d^2} : g :: f : 1$$

$$\therefore \mu = (d)^2 gf$$

$$\text{and } T = \frac{2\pi}{d\sqrt{fg}} \cdot a^{\frac{3}{2}}$$

the time required.

503. **The equation to the opposite Hyperbola is**

$$p^2 = \frac{b^2\rho}{\rho - 2a}$$

a and b being the semiaxes, p and ρ the $\perp$ upon the tangent and radius vector from the focus.

$$\therefore \frac{2pdp}{d\rho} = \frac{b^2}{\rho - 2a} - \frac{b^2\rho}{(\rho - 2a)^2}$$

$$= -\frac{2ab^2p^4}{b^4\rho^2}$$

$$= -\frac{2a}{b^2} \cdot \frac{p^4}{\rho^2}$$

$$\therefore F = \frac{c^2dp}{p^3d\rho} = -\frac{2ac^2}{b^2} \cdot \frac{1}{\rho^2}$$

$$\propto \frac{1}{\rho^2} \text{ and is } \textit{repulsive.}$$

504. If the distance of the centre of the ⊙ from the centre of co-ordinates be a, and the radius of the ⊙ be r, then we easily know that

$$\pm p = \frac{r^2 - a^2 + \rho^2}{2r}.$$

Hence

$$\frac{dp}{d\rho} = \frac{\rho}{r}$$

$$\text{and } F = \frac{c^2dp}{p^3d\rho} = \pm \frac{8c^2r^2 \times \rho}{(r^2 - a^2 + \rho^2)^3}$$

according as the centre of force falls within or without the tangent.

505. Since $vdv = -\mathrm{F}d\varrho$, and $\mathrm{F} = \frac{\mu}{\rho^2}$ we have

$$v^2 = 2\mu \times \left(\frac{1}{\varrho} - \frac{1}{a}\right)$$

a being the distance at which the body begins to fall. If $a = \infty$, then $\frac{1}{a} = 0$, and we have

$$v = \sqrt{\frac{2\mu}{\varrho}}$$

Now if the force be constant for the space ϱ, we have

$$v' = \sqrt{2\mathrm{F}\varrho} = \sqrt{\frac{2\mu}{\rho}}$$

$$\therefore\ v = v'.$$

506. Since the force is in the focus we have

$$\mathrm{F} = \frac{\mu}{\rho^2}$$

and the velocity down any portion x of the radius vector ρ, beginning from the curve is

$$v = \sqrt{2\mu} \times \sqrt{\frac{1}{\rho - x} - \frac{1}{\rho}} \text{ see 505.}$$

But the velocity in the curve is

$$v = \frac{c}{p}$$

$$\therefore\ \frac{c^2}{p^2} = 2\mu \cdot \left(\frac{1}{\varrho - x} - \frac{1}{\rho}\right) = 2\mu \times \frac{x}{\rho(\varrho - x)}.$$

Hence

$$x = \frac{c^2}{2\mu} \times \frac{\varrho^2}{p^2\left(1 + \frac{c^2}{2\mu} \times \frac{\varrho}{p^2}\right)}.$$

But $p^2 = \frac{b^2\varrho}{2a - \rho}$, and $c^2 = b^2 \cdot \frac{\mu}{a}$, (453,)

$\therefore$ after substitution and the requisite reductions we get

$$x = \frac{\rho(2a - \varrho)}{4a - \rho} \dots\dots\dots\dots\dots\dots (a)$$

Hence when $x =$ *maximum* or *minimum*, we have

$$\frac{dx}{d\varrho} = \frac{2a - \rho}{4a - \varrho} - \frac{\varrho}{4a - \varrho} + \frac{\varrho \,.\, (2a - \varrho)}{(4a - \varrho)^2} = 0$$

$$\text{or } 2(a - \varrho) \,.\, (4a - \varrho) + \rho \,.\, (2a - \varrho) = 0$$

whence we get

$$\varrho^2 - 8a\varrho = - 8a^2$$

and solving the equation

$$\varrho = 4a \pm a2\surd 2 = 2a\,(2 \pm \surd 2)$$

which being substituted in (a) will give either *maximum* or *minimum* values of x. To ascertain whether they are maximum or minimum, we have

$$\frac{dx}{d\varrho} = \frac{\varrho^2 - 8a\varrho + 8a^2}{(4a - \varrho)^2}$$

$$\therefore \frac{d^2x}{d\varrho^2} = \frac{2\varrho - 8a}{(4a - \varrho)^2} + \frac{2(\varrho^2 - 8a\varrho + 8a^2)}{(4a - \rho)^3}$$

$$= - \frac{16a^2}{(4a - \varrho)^3}$$

Hence at the points indicated by

$$\rho = 2a \,.\, (2 + \surd 2)$$

we have

$$\frac{d\,x}{d\rho^2} = - \frac{16a^2}{(\mp a2\surd 2)^3}$$

$$= \pm \frac{1}{a\surd 2}$$

and $\therefore$

$$\rho = 2a \cdot (2 + \surd 2)$$

gives the maximum value of x

$$\text{and } \varrho = 2a \,.\, (2 - \surd 2)$$

gives the *minimum*, (see *Simpson's Fluxions*, Appendix to the New Edition, or *Lacroix*.)

These values of x are

$$x = \frac{\rho \,.\, (2a - \varrho)}{4a - \varrho} = 2a \,.\, (2 \pm \surd 2) \times \frac{2a - 2a \,.\, (2 \pm \surd 2)}{4a - 2a \,.\, (2 \pm \surd 2)}$$

$$= 2(1 \pm \surd 2) \times (1 \pm \surd 2)\, a = 2(1 \pm \surd 2)^2 a.$$

507. Let $F = \mu\varrho^m$, m and μ being at present unknown but determinate. Then

$$vdv = - Fd\varrho = - \mu$$

$$\text{and } v^2 = \frac{2\mu}{m+1} \times (a^{m+1} - \varrho^{m+1})$$

$$\therefore v = \sqrt{\frac{2\mu}{m+1}} \times \sqrt{(a^{m+1} - \varrho^{m+1})}$$

$$\therefore dt = \frac{-d\varrho}{v} = - \sqrt{\frac{m+1}{2\mu}} \times \frac{d\varrho}{\sqrt{(a^{m+1} - \varrho^{m+1})}}$$

$$\text{and } t = - \sqrt{\frac{m+1}{2\mu}} \times \int \frac{d\varrho}{\sqrt{(a^{m+1} - \varrho^{m+1})}}$$

But

$$(a^{m+1} - \varrho^{m+1})^{-\frac{1}{2}} = a^{-\frac{m+1}{2}}\, 1 - \beta)^{-\frac{1}{2}}, \frac{\varrho^{m+1}}{a^{m+1}} \text{ being put } = \beta,$$

$$= a^{-\frac{m+1}{2}} \times \{1 - \frac{1}{2}\beta + \frac{1}{2}\,\frac{3}{4}\beta^2 + \frac{1.3.5}{2.4.6}\beta^3 + \&c.\}$$

$$\therefore - a^{\frac{m+1}{2}} \sqrt{\frac{2\mu}{m+1}} \times t = \int \{ d\varrho + \frac{1}{2} . \frac{\varrho^{m+1} d\varrho}{a^{m+1}} + \frac{3}{2.4} . \frac{\varrho^{2m+2} d\varrho}{a^{2m+2}} + \&c \}$$

$$= \varrho + \frac{\varrho^{m+2}}{2.(m+2)a^{m+1}} + \frac{3}{2\,.\,4} . \frac{\varrho}{(2m+3)a^{2m+2}} + \&c. + C.$$

Let $t = 0$, when $\varrho = a$ and we have C

$$= - a\,(1 + \frac{1}{2\,.\,(m+2)} + \&c.)$$

$$\therefore - a^{\frac{m+1}{2}} \sqrt{\frac{2\mu}{m+1}} \times t = \varrho + \frac{\rho^{m+2}}{2\,.\,(m+2)a^{m+1}} + \&c.$$

$$- a(1 + \frac{1}{2\,.\,(m+2)} + \&c.) \ldots\ldots (a)$$

Let now $\rho = 0$. Then the time to the centre is

$$T = \{1 + \frac{1}{2(m+2)} + \&c.\} . \sqrt{\frac{m+1}{2\mu}} + a^{\frac{1-m}{2}}$$

and consequently for different altitudes (a),

$$T \propto a^{\frac{1-m}{2}}.$$

But by the question

$$T \propto a^n$$

$$\therefore \frac{1-m}{2} = n$$

and $m = 1 - 2n$.

Hence if $T \propto a^n$, we have

$$F \propto \varrho^{1-2n}. \qquad \text{Q. E. D.}$$

Ex. If $n = \frac{3}{2}$. Then

$$F \propto \frac{1}{\varrho^2}$$

which we know to be the case from other and more direct principles.

508. Let a and b be the semi-axes of the ellipse; then the velocity at the extremity of the minor axis is

$$v = \frac{c}{p} = \frac{b\sqrt{\frac{\mu}{a}}}{b} = \sqrt{\frac{\mu}{a}}$$

μ being the absolute force.

Again, since

$$vdv = -Fd\varrho = -\frac{\mu d\rho}{\varrho^2}$$

$$\therefore v^2 = 2\mu\left(\frac{1}{\varrho} - C\right)$$

But since when $\rho = a$, $v = \sqrt{\frac{\mu}{a}}$

$$\therefore \frac{\mu}{a} = 2\mu.\left(\frac{1}{a} - C\right)$$

$$\text{and } C = \frac{\mu}{a}$$

$$\therefore v^2 = 2\mu.\left(\frac{1}{\rho} - \frac{1}{a}\right).$$

$$\text{Hence } dt = \frac{d\varrho}{v} = \sqrt{\frac{a}{2\mu}}.\frac{\varrho d\varrho}{\sqrt{(a\varrho - \varrho^2)}}$$

$$\therefore t = \sqrt{\frac{a}{2\mu}} \times \{\sqrt{(a\rho - \varrho^2)} - \frac{a}{2}.\text{vers.}^{-1}.\frac{2}{a}\rho + C\}$$

See *Hirsch's Tables*, p. 141.

Let $\rho = a$, then $t = 0$, and

$$C = \frac{a}{2}. \text{vers.}^{-1} 2 = \frac{a\pi}{2}$$

$$\therefore t = \sqrt{\frac{a}{2\mu}} \times \left\{\sqrt{(a\varrho - \varrho^2)} + \frac{a}{2} \times \left(\pi - \text{vers.}^{-1}\frac{2\varrho}{a}\right)\right\}$$

Let $\varrho = 0$; then

$$F = \frac{a^{\frac{3}{2}}\pi}{2\sqrt{2} \times \sqrt{\mu}}$$

the whole time of descent to the focus.

Again, by 484, the time of the body's moving from the nearer apside to the extremity of the axis minor is

$$T' = \frac{a^{\frac{3}{2}}}{\sqrt{\mu}} \times \left(\frac{\pi}{2} - e\right).$$

$$\therefore T : T' :: \frac{\pi}{2.\sqrt{2}} : \frac{\pi}{2} - e$$

$$:: 1 : \sqrt{2} - 2\sqrt{2} \times \frac{e}{\pi}.$$

509. Let R be the distance from the centre of force at which the body is projected; then since

$$vdv = - Fd\varrho = - \frac{\mu}{\varrho^n} d\rho$$

$$\therefore v^2 = \frac{\mu}{n-1}. \left(\frac{1}{\varrho^{n-1}} - \frac{1}{a^{n-1}}\right).$$

Let $a = \infty$, and $\varrho = R$, then

$$V^2 = \frac{\mu}{n-1}. \frac{1}{R^{n-1}}.$$

But since the velocity in any curve is that which would be acquired by the bodys's descent along $\frac{1}{4}$ chord of curvature (PV) with the force continued constant, we have

$$V^2 = 2 F. \frac{PV}{4} = \frac{\mu}{2R^n}. PV = \frac{\mu}{n-1}. \frac{1}{R^{n-1}}$$

by the question. Hence

$$PV = \frac{2}{n-1} R$$

or PV : R :: 2 : $n - 1$.

510. If the centre of force be any where about a circle, we learn from (504) that

$$F = \pm \frac{8c^2r^2\varrho}{(r^2 - a^2 + \varrho^2)^3}$$

r being the radius of the circle, and a the distance of the centre of force from that of the circle.

Now when the force is in the circumference $a = r$, and p is positive

$$\therefore F = \frac{8c^2r^2}{\varrho^5} = \frac{\mu}{\varrho^5}$$

$$\therefore c = \frac{\sqrt{\mu}}{2\sqrt{2r}}.$$

Again the equation to the circle is

$$p = \frac{r^2 - a^2 + \rho^2}{2r} = \frac{\varrho^2}{2r}$$

in this case. Hence

$$v = \frac{c}{p} = \frac{2rc}{\varrho^2}$$

$$\therefore dt = \frac{-d\varrho}{v} = \frac{\varrho^2 d\varrho}{2rc}$$

Hence

$$t = \frac{1}{2.3rc} \times (C + \varrho^3).$$

Let $t = 0$, when $\varrho = 0$;

$$\text{then } t = \frac{\varrho^3}{2.3rc} = \frac{\sqrt{2}\rho^3}{3\sqrt{\mu}}.$$

Let $\varrho = 2r$; then

$$2t = \frac{16\sqrt{2}}{3\sqrt{\mu}} . r^3 \propto r^3. \qquad \text{Q. E. D.}$$

511. Let the earth's radius be R, and $\frac{g}{2}$ the space fallen through in a second at its surface; also let p be the periodic time of the Moon. Then if ρ be the distance of the moon from the earth, and F the force of the earth's attraction upon the moon, we have

$$F : g :: \frac{1}{\rho^2} : \frac{1}{R^2}$$

$$\text{or } F = \frac{R^2 g}{\rho^2}.$$

Now by (440), we get

$$p = 2\pi \sqrt{\frac{\rho}{F}} = \frac{2\pi}{R\sqrt{g}} \times \rho^{\frac{3}{2}}$$

$$\therefore \rho^3 = \frac{R^2 g}{4\pi^2} p^{\frac{2}{3}}$$

$$\text{and } \rho = \frac{R^{\frac{2}{3}} g^{\frac{1}{3}}}{4^{\frac{1}{3}} \pi^{\frac{2}{3}}} p^{\frac{2}{3}}$$

the distance required in feet, g being equal to $32\frac{1}{6}$ feet, and p expressed in seconds.

512. Generally the time in the parabola is (see 484).

$$t = \sqrt{\frac{2}{\mu}} \times (\rho - r)^{\frac{1}{2}} \left(\frac{\rho}{3} + \frac{2}{3} r\right)$$

r being $=$ SA and $\rho =$ SP;

and the time of falling from rest through any space $\rho - x$ when $F = \frac{\mu}{\rho^2}$ is (495).

$$t' = \sqrt{\frac{\rho}{2\mu}} \times \left\{(\rho x - x^2)^{\frac{1}{2}} + \frac{\rho}{2}\left(\pi - \text{vers.}^{-1} \frac{2x}{\rho}\right)\right\}$$

and when $x = 0$

$$t' = \frac{\pi}{2\sqrt{2\mu}} \cdot \rho^{\frac{3}{2}}$$

$$t' : t :: \frac{\pi}{2\sqrt{2}} \rho^{\frac{3}{2}} : \frac{\sqrt{2}}{3} \sqrt{(\rho - r)} \times (\rho + 2r)$$

$$:: \frac{3}{4} \cdot \varrho^{\frac{3}{2}} : \sqrt{(\varrho - r) \cdot (\varrho + 2r)}$$

the analogy required.

513. Let x be the angle between the tangent or direction of the body's motion and ϱ; then

$$p = \varrho . \sin. \psi.$$

$$\text{and } \frac{dp}{d\varrho} = \sin. \psi + \varrho \cos. \psi. \frac{d\psi}{d\varrho}$$

$$\text{But } v^2 : v'^2 :: \frac{d\varrho}{dp} : \frac{\varrho}{p} \text{ (438).}$$

v and v' being the velocities in a curve, and a circle at the same distance. Hence

$$v^2 : v'^2 :: \sin. \psi : \sin. \psi : \sin. \psi + \varrho \cos. \psi. \frac{d\psi}{d\varrho}$$

$$\text{or } v^2 : v'^2 - v^2 :: \sin. \psi : \rho \cos. \psi \frac{d\psi}{d\varrho}$$

$$:: \tan. \psi ; \varrho . \frac{d\psi}{d\varrho}$$

$$\therefore d\psi = \frac{v'^2 - v^2}{v^2} . \frac{\tan. \psi}{\varrho} . d\varrho \ldots\ldots\ldots (a)$$

Hence, since ψ cannot exceed 90°, if $d\varrho$ be considered positive, $d\psi$ is positive, or negative, or ψ is increasing or decreasing, according as v' is > or < than v. If $d\varrho$ be negative or ϱ be decreasing, then $d\psi$ is positive or negative, that is, ψ is increasing, or decreasing, according as v is > or < v'.

These results indicate a defect in the enunciation of the problem.

514. Let R be the earth's radius.

Then since the velocity of a body revolving at the surface is

$$v = \sqrt{\frac{\mu R}{R^2}} = \sqrt{\frac{\mu}{R}}$$

we have for the new trajectory

$$c = P \times V = b \sqrt{\frac{\mu}{a}}.$$

But the body being projected at an $\angle$ of 45° degrees with the horizon or $90^\circ - 45^\circ = 45^\circ$ with R, we have

$$P = R. \sin. 45^\circ = \frac{R}{\sqrt{2}}$$

$$\therefore \frac{R}{\sqrt{2}} \times \sqrt{\frac{\mu}{R}} = b\sqrt{\frac{\mu}{a}}$$

$$\text{or } \frac{R}{2} = \frac{b^2}{a} \dots\dots\dots (1)$$

or R = *the latus rectum of the Ellipse.*

Again, since the equation to an ellipse is generally

$$p^2 = \frac{b^2\rho}{2a-\rho}$$

we have

$$p^2 = \frac{b^2 R}{2a - R}$$

$$\text{or } \frac{R^2}{2} = \frac{\frac{aR^2}{2}}{2a-R}$$

$$\therefore 2a - R = a$$

$$\therefore a = R \dots\dots\dots (2)$$

$$\text{Hence } b^2 = \frac{R^2}{2}$$

$$\text{and } b = \frac{R}{\sqrt{2}} \dots\dots\dots (3)$$

Hence then the equation to the trajectory is

$$p^2 = \frac{\frac{R^2\rho}{2}}{2R-\rho} \dots\dots\dots (4)$$

and it may therefore be described.

Again, the equation to an ellipse between ρ and θ is

$$\rho = \frac{b^2}{a + ae\cos.\theta}$$

ae being $= \sqrt{a^2 - b^2}$, and θ measured from the shortest distance.

$\therefore$ in this case we have

$$\rho = \frac{R^2}{2(R + \frac{R}{\sqrt{2}}\cos.\theta)}$$

$$= \frac{R}{2 + \sqrt{2}.\cos.\theta} \dots\dots\dots\dots (5)$$

Hence, to find the angular distance between the body's departure from and to the earth's surface, we have

$$\cos.\theta = \frac{R - 2\rho}{\rho\sqrt{2}}$$

$$\text{and } \therefore \theta = \cos.^{-1}\left(\pm\frac{R - 2\rho}{\rho\sqrt{2}}\right) \dots\dots\dots\dots 6.$$

Let $\rho = R$; then

$$\theta = \cos.^{-1}\left(\mp\frac{1}{\sqrt{2}}\right) = 135^\circ \text{ or } 45^\circ$$

Hence the distance required is

$$135^\circ - 45^\circ = 90^\circ.$$

The same result may more easily be obtained geometrically from considering Fig. 95, and Problem (459).

515. The least tension will evidently be at the highest, and the greatest tension at the lowest point of the circle. It is also plain that the centrifugal force in the circle must be just equal at the highest point to counteract the gravitation, and it will therefore be equal to g. Hence, at the highest point if n be the number of particles in the revolving body, the tension will be

$$t = ng = w, \text{ the weight} \dots\dots (1)$$

Again, since in a circle the centrifugal force = centripetal, and velocity generally in any curve = that acquired down $\frac{1}{4}$ chord of curvature, therefore the velocity in the circle when the force is g is

$$v = \sqrt{2.g.\frac{R}{2}} = \sqrt{gR}$$

R being the length of the string.

In the descent, moreover, from the highest to the lowest point, the velocity acquired will be $\sqrt{2g.(2R)} = 2\sqrt{gR}$. Hence the velocity of the body at the lowest point is

$$v' = 3\sqrt{gR}$$

Now, generally the centrifugal force arising from the angular

velocity $\propto$ (angular velocity) $2 \times \frac{1}{\text{dist.}} \propto$ (linear velocity) ^{2} in the same circle. Hence if φ be the centrifugal force at the lowest point, we have

$$g : \varphi :: v^2 : v'^2 :: g : 9g :: 1 : 9$$

$$\therefore \varphi = 9g$$

and the Tension $= n\varphi = 9. ng = 9$W, and not 6 W as the enunciation has it.

516. In the reciprocal spiral we have

$$\rho = \frac{a}{\theta}$$

$$\therefore \frac{d\rho}{d\theta} = -\frac{a}{\theta^2} = -\frac{\rho^2}{a}$$

and $\frac{d\rho^2}{\rho^4 d\theta^2} = -\frac{1}{a}$

Hence $d. \frac{d\rho^2}{\rho^4 d\theta^2} = 0$

and by 436 we have

$$F = \frac{c^2}{\rho^3} - 0 = \frac{c^2}{\rho^3} = \frac{\mu}{\rho^3}$$

$$\therefore c = \sqrt{\mu}$$

Hence

$$dt = \frac{\rho^2 d\theta}{c} = \frac{\rho^2 d\theta}{\sqrt{\mu}}.$$

But if u be the area of any spiral, we have

$$du = \frac{\rho^2 d\theta}{2} = \frac{cdt}{2}$$

$\therefore$ in this case

$$du = \frac{\sqrt{\mu}}{2}. dt.$$

and $u = \frac{\mu}{2} t.$

since u is the area described in the time t.

Hence when t is given $u \propto \mu$ *the absolute force;* and when also the absolute force is given

u is constant for all reciprocal spirals. Q. E. D.

517. The angular velocity about the first focus is (468)

$$\frac{c}{\rho^2}$$

and the angular vel. about the other focus : $\frac{c}{\rho^2} :: \rho : 2a - \rho$.

Hence the angular velocity about the second focus is

$$V = \frac{c.}{\rho(2a-\rho)} = \frac{b\sqrt{\frac{\mu}{a}}}{\rho(2a-\rho)}$$

Now, the mean angular velocity V′ is that by which 360° would be uniformly described in the periodic time in the ellipse, or

$$V' = \frac{2\pi}{T} = \frac{2\pi}{\frac{2\pi a^{\frac{3}{2}}.}{\sqrt{\mu}}} = \frac{\sqrt{\mu}}{a^{\frac{3}{2}}}$$

Hence when $V = V'$, we have

$$\frac{b}{\sqrt{a} \times \rho(2a-\rho)} = \frac{1}{a^{\frac{3}{2}}}$$

and $\rho.(2a-\rho) = ab$

$$\therefore \rho^2 - 2a\rho = -ab$$

$$\therefore \rho = a \pm \sqrt{a^2 - ab}$$

which, by means of the equation of the ellipse determines two of the required points.

The other two are immediately opposite to these.

If β denote the given angular velocity, we have

$$\theta = t \times \beta$$

But $vdv = -\frac{\mu}{\rho^2} d\rho$

$$\therefore v = \sqrt{2\mu}.\sqrt{\left(\frac{1}{\rho} - \frac{1}{a}\right)}$$

and $dt = \frac{-d\rho}{v} = \sqrt{\frac{a}{2\mu}} \times \frac{-\rho d\rho}{\sqrt{.(a\rho - \rho^2)}}$

and $t = \sqrt{\frac{a}{2\mu}} \times \left\{ \sqrt{(a\rho - \rho^2)} + \frac{a}{2}\left(\pi - \text{vers.}^{-1}\frac{2\rho}{a}\right)\right\}$

see *Hirsch's Tables*, p. 141.

Hence

$$\theta = \beta\sqrt{\frac{a}{2\mu}} \times \left\{ \sqrt{(a\rho - \rho^2)} + \frac{a}{2}\left(\pi - \text{vers.}^{-1}\frac{2\rho}{a}\right)\right\}$$

the equation to the curve described.

Now the area being u we have

$$du = \frac{\rho^2 d\theta}{2} = \frac{\beta\rho^2 . dt}{2}$$

$$= \frac{\beta}{2}\sqrt{\frac{a}{2\mu}} \times - \frac{\rho^3 d\rho}{\sqrt{(a\rho - \rho^2)}}.$$

$$\frac{2\sqrt{2\mu}\, u}{\beta\sqrt{a}} = \left(\frac{\rho^2}{3} - \frac{5a\rho}{12} + \frac{5a^2}{8}\right)\sqrt{(a\rho - \rho^2)}$$

$$- \frac{5a^3}{16}\text{ vers.}^{-1}\frac{2\rho}{a} + \text{C}.$$ See *Hirsch*, p. 141.

Let $\rho = a$. Then $u = 0$ and $\text{C} = \frac{5a^3}{16}\pi$.

$$\therefore u = \frac{\beta}{2}\sqrt{\frac{a}{2\mu}}\left\{ \cdot \left(\frac{\rho^2}{3} - \frac{5a\rho}{12} + \frac{5a^2}{8}\right)\sqrt{(a\rho - \rho^2)}\right.$$

$$\left. + \frac{5a^3}{16} \cdot \left(\pi - \text{vers.}^{-1}\frac{2\rho}{a}\right)\right\}.$$

Let $\rho = 0$. Then the whole area is

$$u = \frac{5a^3\pi}{16} \times \frac{\beta}{2}\sqrt{\frac{a}{2\mu}}.$$

But since the whole time of descent is

$$\text{T} = \frac{a^{\frac{3}{2}}\pi}{2\sqrt{2\mu}}$$

$$\text{and } 2\pi = \text{T} \times \beta = \frac{a^{\frac{3}{2}}\pi}{2\sqrt{2\mu}} \times \beta$$

$$\therefore \beta = \frac{4\sqrt{2\mu}}{a^{\frac{3}{2}}}$$

and we finally get for the whole area described

$$u = \frac{5a^2\pi}{8}.$$

519. Since $v = \frac{c}{p}$, the greatest and least velocities are at the nearer and farther vertices, or when $p = a - ae$, or $= a + ae$. Hence, if x be the $\perp$ upon the tangent when the velocity is an harmonic mean between the greatest and least velocities, we have

$$\frac{1}{a(1-e)},\ \frac{1}{x} \text{ and } \frac{1}{a(1+e)}$$

in harmonic progression. Hence

$$a(1-e),\ x, \text{ and } a(1+e)$$

are in arithmetic progression, and we have

$$x - a(1-e) = a(1+e) - x$$

$$\therefore\ x = a$$

Hence $a^2 = \frac{b^2\rho}{2a - \rho}$ gives

$$\rho = \frac{2a^3}{a^2 + b^2}$$

which gives the point required.

520. The velocity down $(\rho - a)$, $F = \frac{\mu}{\rho^2}$, is,

$$v = \sqrt{\frac{2\mu}{a}}\cdot\sqrt{\frac{a-\rho}{\rho}}\ (517)$$

Also, if the force be considered constant the velocity acquired down $\rho - a$ is

$$v' = \sqrt{\frac{2\mu}{\rho^2}\cdot(a-\rho)}$$

$$\therefore\ v : v' :: \sqrt{\frac{1}{a}} : \sqrt{\frac{1}{\rho}} :: \sqrt{\rho} : \sqrt{a}$$

But $\text{chord.}^2\theta : \sin.^2\theta :: 2R - \text{versine } \theta : 2R$

$$\therefore\ v : v' :: \text{chord. } \theta : \sin.\ \theta$$

θ being the arc whose versine is $a - \rho$ and diameter a.

521. Let the area of the paper to be doubled down be constantly equal to a^2; then since the corner of the Δ is a right

$\angle$, if the locus required be referred to the corner by the radius vector ϱ and angle θ originating with either edge of the paper, it may easily be proved that

$$a^2 = \frac{\varrho}{2} \cdot \frac{\text{hypothenuse}}{2}$$

$$= \frac{\varrho}{4} \times \frac{\varrho}{2} \cdot (\tan. \theta + \cos. \theta)$$

$$= \frac{\varrho^2}{8} \cdot \left(\frac{\sin. \theta}{\cos. \theta} + \frac{\cos. \theta}{\sin. \theta}\right)$$

$$= \frac{\varrho^2}{8} \frac{\sin.^2\theta + \cos.^2\theta}{\sin. \theta. \cos. \theta} = \frac{\varrho^2}{4} \cdot \frac{1}{\sin. 2\theta}$$

$$\therefore \varrho^4 = 4a^2. \sin. 2\theta$$

the equation to the *Lemniscata*.

Again (436)

$$\mathrm{F} = \frac{c^2}{\varrho^3} - \frac{c^2}{2} \cdot \frac{d. \frac{d\varrho^2}{\varrho^4 d\theta^2}}{d\varrho}$$

But $\frac{d\varrho}{d\theta} = \frac{4a^2 \cos. 2\theta}{\rho}$

$$\therefore \frac{d\rho^2}{\rho^4 d\theta^2} = \frac{16a^4 \cos.^2 2\theta}{\varrho^6}$$

$$= \frac{16a^4}{\rho^6} - \frac{\varrho^4}{16a^4} \cdot \frac{16a^4}{\varrho^6}$$

$$= \frac{16a^4}{\varrho^6} - \frac{1}{\varrho^2}$$

$$\therefore \mathrm{F} = \frac{c^2}{2} \cdot \frac{6.16a^4}{\varrho^7} = \frac{48c^2a^4}{\varrho^7} \propto \frac{1}{\varrho^7}.$$

522. If the force $= \mu\rho$, then $vdv = -\mu\varrho d\varrho$ and $v^2 = \mu. (a^2 - \varrho^2)$

$a - \varrho$ being the space fallen through.

But if the force be considered constant, we have

$$v'^2 = 2\mu\varrho. (a - \varrho)$$

$$\therefore v : v' :: \sqrt{(a+\varrho)} : \sqrt{2\varrho}$$

$$:: \text{sine } \theta ; \text{chord. } \theta$$

where θ is the arc whose *radius* is a, and versine $a - \rho$, as is easily shown on drawing the figure.

523. Let the force be

$$F = \frac{\mu}{\rho^4}$$

$$\text{Then } vdv = -Fd\rho = -\frac{\mu d\rho}{\rho^4}$$

$$\text{and } v^2 = \frac{2}{3}\mu\left(\frac{1}{\rho^3} - \frac{1}{a^3}\right)$$

and if a be infinite

$$v^2 = \frac{2}{3}\frac{\mu}{\rho^3}$$

Hence if at the apse $\rho = R$, the velocity of projection is

$$\frac{c}{R} = \sqrt{\frac{2}{3}\frac{\mu}{R^3}}$$

$$\therefore c = \sqrt{\frac{2}{3}}\sqrt{\frac{\mu}{R}}.$$

Now by 436

$$d\theta = \frac{cd\rho}{\rho\sqrt{(-c^2 - 2\rho^2\int Fd\rho)}}$$

$$= \frac{cd\rho}{\rho\sqrt{\left(\frac{2\mu}{-3R} + \frac{2\mu}{3\rho}\right)}}$$

$$= \frac{d\rho}{\rho\sqrt{\left(\frac{1}{\rho} - \frac{1}{R}\right)}} = \frac{d\rho}{\sqrt{\left(\rho - \frac{\rho^2}{R}\right)}}$$

$$= \sqrt{R} \times \frac{d\rho}{\sqrt{(R\rho - \rho^2)}}.$$

Hence $\theta = \sqrt{R}\,.\,\text{vers.}^{-1}\frac{\rho}{R} + C.$

Let $\theta = 0$, when $\rho = R$; then

$$C = -\sqrt{R}\,.\,\text{vers.}^{-1}\,1 = -\sqrt{R}\,.\,\frac{\pi}{2}.$$

$$\therefore \frac{\theta}{\sqrt{R}} + \frac{\pi}{2} = \text{vers.}^{-1}\frac{\rho}{R}$$

$$\text{or } \frac{\rho}{R} = \text{vers.}\left(\frac{\theta}{\sqrt{R}} + \frac{\pi}{2}\right)$$

$$= 1 + \cos.\frac{\theta}{\sqrt{R}}$$

the equation to the orbit.

524. Since the body is projected from an aspe, the velocity parallel to the plane will always be that of projection, which we call β; then if y represent the $\perp$ to the plane, and x the corresponding abscissa along it, we have

$$dx = V \times dt$$

$$\therefore dt = \frac{dx}{\beta}$$

$$\text{also } vdv = -\frac{\mu}{y^n}dy$$

$$\therefore v^2 = \frac{2\mu}{n-1} \cdot \frac{a^{n-1} - y^{n-1}}{a^{n-1}\,y^{n-1}}$$

$$\text{and } dt = \frac{dy}{v} = \sqrt{\frac{n-1}{2\mu}} \times a^{\frac{n-1}{2}} \cdot \frac{y^{\frac{n-1}{2}}dy}{\sqrt{(a^{n-1}-y^{n-1})}} = \frac{dx}{\beta}.$$

$$\therefore x = \beta\sqrt{\frac{n-1}{2\mu}} \cdot a^{\frac{n-1}{2}} \times \int \frac{y^{\frac{n-1}{2}}dy}{\sqrt{(a^{n-1}-y^{n-1})}} \ldots\ldots (a)$$

which can be integrated in all cases wherein

$$\frac{\frac{n-1}{2}+1}{n} \text{ or } \frac{\frac{n-1}{2}+1}{n} - \frac{1}{2}$$

$$\text{that when } \frac{n+1}{2n} \text{ or } \frac{1}{2n}$$

is a whole number, by assuming in the first case

$${}^{n-1} - y^{n-1} = u^2$$

and in the other case

$$a^{n-1} - y^{n-1} = u^2 y^{n-1}.$$

To apply this equation to the case specified in the problem, we have $n = 0$, and $\therefore$

$$x = \beta \sqrt{\left(-\frac{1}{2\mu}\right)} \cdot a^{-\frac{1}{2}} \int \frac{y^{-\frac{1}{2}} dy}{\sqrt{(a^{-1} - y^{-1})}}$$

$$= \beta \cdot \sqrt{\left(-\frac{1}{2\mu}\right)} \int \frac{dy}{\sqrt{(y-a)}}$$

$$= \frac{\beta}{2} \sqrt{\left(-\frac{1}{2\mu}\right)} \times \sqrt{(y-a)} + \mathrm{C}.$$

But $x = 0$, when $y = a$; $\therefore$ C $= 0$; and

$$x = \frac{\beta}{2} \cdot \frac{1}{\sqrt{(2\mu)}} \times \sqrt{(a - y)}$$

$$\text{and } x^2 = \frac{\beta^2}{8\mu} \cdot (a - y) \ldots\ldots\ldots\ldots\ldots (b)$$

the equation to the common parabola, whose latus rectum is $\frac{\beta^2}{8\mu}$, *ordinate* x *and abscissa measured along the axis from the vertex* $a - y$.

525. The force being constant, we have

$$vdv = - \mathrm{F} d\varrho$$

$$\text{and } v^2 = 2\mathrm{F}(a - \varrho)$$

$$\therefore dt = \frac{d\varrho}{v} = \frac{1}{\sqrt{2\mathrm{F}}} \times \frac{-d\varrho}{\sqrt{(a-\varrho)}}$$

$$\therefore t = \sqrt{\frac{2}{\mathrm{F}}} \cdot \sqrt{(a-\varrho)}$$

and when $a - \varrho = \frac{r}{2}$

r being the radius of the circle, we have

$$\mathrm{T} = \sqrt{\frac{2}{\mathrm{F}}} \cdot \sqrt{\frac{r}{2}} = \sqrt{\frac{r}{\mathrm{F}}}.$$

Again, in the circle, the periodic time is (440)

$$\mathrm{T}' = 2\pi \sqrt{\frac{r}{\mathrm{F}}}$$

$$\therefore T : T' :: 1 : 2\pi$$

$$:: 1 : 2. \quad \frac{\text{circumference}}{\text{diameter}}$$

$$:: \text{rad.} : \text{circumference}.$$

526. If QR be the deflection from the tangent at P parallel to the ordinate PM, and QT be $\perp$ to PM; then by *Newton's* Princip. prop. VI. cor. I. we have

$$F \propto \frac{QR}{SP^2 \times QT^2} \propto \frac{QR}{QT^2}$$

since S is infinitely distant, and therefore SP is constant.

Now if PC be the normal meeting the axis in C; then from similar $\triangle$s PMC, PZT, ZRQ, we have

$$PR^2 : QT^2 :: PC^2 : PM^2.$$

But by the nature of the parabola

$$QR \times (QR + 2QN) = \frac{L^2 \times RP^2}{L^2 + 4PM^2}$$

$$\text{and } MC = \frac{L}{2}$$

L being the latus rectum; also

$$PC^2 = PM^2 + \frac{L^2}{4} = \frac{4PM^2 + L^2}{4}.$$

Hence

$$\frac{QR}{QT^2} = \frac{1}{4PM^2} \times \frac{L^2 \times (L^2 + 4PM^2)}{(L^2 + 4PM^2)(QR + 2QN)}$$

$$= \frac{L^2}{4(QR + 2QN)} \times \frac{1}{PM^2}$$

and the limit of QR + 2QN is 2PM.

Therefore

$$\frac{QR}{QT^2} = \frac{L^2}{8} \cdot \frac{1}{PM^3} \text{ in the limit};$$

$$\text{and } F \propto \frac{QR}{QT^2} \propto \frac{1}{PM^3}.$$

527. If the ordinate DF (see Fig. to Prop. XXXIX. of Princip.) be always taken proportional to the force, or propor-

tional to $\frac{1}{CD^3}$, and AB be that value of it at the distance CA, at which the body commences its descent; then, by *Newton*, the square of the velocity acquired through AD is proportional to the area ABFD, or in symbols.

$$v^2 = A \times ABFD$$

But by the question, if CE be put $= x$, and CA $= a$; AB $= b$, and DF $= y$; then

$$b : y :: \frac{1}{a^3} : \frac{1}{x^3}$$

$$\therefore y = \frac{b}{a^3 \,.\, x^3} \ldots\ldots\ldots\ldots\ldots\ldots \ldots\ldots (a)$$

the equation of the curve BF.

And if we recognise a curve from this equation, whose area is known, the velocity is known. But if not, we must find the area by some method, either exactly or approximately. The general method is that of first finding the fluxion or increment of the area, and then inversely seeking the integral or sum of the increments. The fluxion of an area is ydx, y being the ordinate of the curve, and dx the fluxion of the abscissa; hence

$$v^2 = A \times \int ydx$$
$$= A \times \frac{b}{a^3} \int \frac{-dx}{x^3}$$

taken between $x = a$, when $v = 0$, and $x = x$.

$$\text{But} \int \frac{-dx}{x^3} = \frac{1}{2} \left(\frac{1}{x^2} - \frac{1}{a^2}\right)$$

$$\therefore v^2 = \frac{A.b}{2.a^5} \times \frac{a^2 - x^2}{x^2}$$

which gives the velocity acquired down $a - x$, when a is determined.

Now, if at any given distance as the radius R, for instance, of the attracting body, it be observed that in one second the descent from rest towards its centre is $\frac{r}{2}$ feet, then supposing the force constant during that time, and $= 1$, we have ($v^2 = 2Fs$)

$$v^2 = 2 \times \frac{r}{2} = r.$$

Hence when $x = \text{R}$, we have

$$\text{A}.\ \frac{b}{2a^5}.\ \frac{a^2-\text{R}^2}{\text{R}^2} = r$$

which gives A, and therefore *the velocity is fully determined in feet.*

Again, if the ordinate DL (Fig. *Newton*) be always taken reciprocally proportional to $\sqrt{(\text{ABFD})}$ or to the velocity, then the velocity down AD will be proportional to the area of ADLV, the value of DL at A when ABFD is 0 being infinite, and AT, therefore an asymptote to the curve VL.

Now at the given distance or value of x, viz. α, we have

$$\text{V}^2 = \frac{\text{A}b}{a^5}.\ \frac{a^2-\alpha^2}{\alpha^2}$$

Hence taking the ordinate β of the time curve at the distance α from the centre $= \frac{1}{\text{V}}$ we have

$$\beta = \frac{a^{\frac{5}{2}}\alpha}{\sqrt{\text{A}b}.\ \sqrt{(a^2-\alpha^2)}}$$

And if $\text{DL} = y'$, we get

$$\therefore\ y' = \frac{a^{\frac{5}{2}}x}{\sqrt{(\text{A}b)}.\ \sqrt{(a^2-x^2)}} \quad \ldots\ldots\ldots\ldots\ (b)$$

the equation to the time curve.

The area of this curve or the time is

$$\text{T} = -\int y'dx = \frac{-a^{\frac{5}{2}}}{\sqrt{\text{A}b}} \int \frac{xdx}{\sqrt{(a^2-x^2)}}$$

between $x = a$ and $x = x$. But

$$\int \frac{-xdx}{\sqrt{a^2-x^2}} = \sqrt{(a^2-x^2)} + \text{C}.$$

$$\therefore\ \text{T} = \frac{a^{\frac{5}{2}}}{\sqrt{\text{A}b}}.\ \sqrt{(a^2-x^2)}$$

gives the time in seconds.

528. When the centripetal force $=$ the centrifugal force, we have (439)

$$\frac{\varrho^3}{p^3} = \frac{d\varrho}{dp}.$$

And the paracentric velocity is a maximum or minimum when (see 452)

$$\frac{1}{p^2} - \frac{1}{\varrho^2} \text{ max. or min.}$$

$$\text{or when } \frac{dp}{p^3} = \frac{d\varrho}{\varrho^3}$$

$\therefore$ when the centripetal force $=$ the centrifugal, the paracentric velocity is a *maximum or minimum*.

529. The subtense $\propto$ F. Hence if F denote the force at the earth's surface, and then since $F \propto \frac{1}{\varrho^2}$, at the distance of nR where R is the radius of the earth, the force is

$$\frac{1}{n^2R^2} \times R^2F = \frac{F}{n^2}.$$

Hence, since at the surface the subtense is found by experiment to be $16\frac{1}{12}$ feet $= \frac{g}{2}$, we have

$$\sigma : \frac{g}{2} :: \frac{F}{n^2} : F :: 1 : n^2$$

$$\therefore \sigma = \frac{g}{2n^2}$$

the subtense at the distance of n radii from the centre.

In the problem $n = 21$.

$$\therefore \sigma = \frac{16\frac{1}{12}}{441} \text{ feet}$$

$$= .0364 \text{ feet.}$$

The moon is distant from the earth about 60 R. Hence

$$n = 60;$$

And it hence appears that the subtense at the moon in a minute is about as great as that at the earth in a second.

528. When the force is in the centre, the periodic time is

$$T = 2\pi\sqrt{\frac{r}{F}} \ldots\ldots\ldots\ldots\ldots\ldots (a)$$

when in the circumference the force is (504)

$$F = \frac{8c^2\pi^2}{\varrho^5} = \frac{\mu'}{\varrho^5}$$

$$\therefore v^2 = 2 \times \frac{F\rho}{4} = \frac{F\varrho}{2} = \frac{\mu'}{2} \cdot \frac{1}{\varrho^4}$$

since the chord of curvature $= \rho$.

Hence

$$dt = \frac{ds}{v} = \sqrt{\frac{2}{\mu'}} \times \varrho^2 ds$$

But since $\varrho = 2 \sin. \frac{s}{2}$, to rad. $= r$.

$$\therefore \frac{s}{2} = \sin.^{-1} \frac{\varrho}{2}$$

$$\text{and } ds = \frac{d\varrho}{\sqrt{\left(1 - \frac{\varrho^2}{4r^2}\right)}}$$

$$\therefore dt = \sqrt{\frac{2}{\mu'}} \times \frac{\varrho^2 d\varrho}{\sqrt{\left(1 - \frac{\rho^2}{4r^2}\right)}}$$

But

$$\int \frac{\varrho^2 d\varrho}{\sqrt{\left(1 - \frac{\varrho^2}{4r^2}\right)}} = -2r^2\varrho\sqrt{\left(1 - \frac{\varrho^2}{4r^2}\right)} + 2r^2 \int \frac{d\varrho}{\sqrt{\left(1 - \frac{\varrho^2}{4r^2}\right)}}$$

$$= -2r^2\varrho\sqrt{\left(1 - \frac{\varrho^2}{4r^2}\right)} + 4r^3 \sin.^{-1}\frac{\varrho}{2r} + C$$

see *Hirsch's* Tables, pages 119 and 118.

Let $t = 0$, when $\rho = 0$, and $C = 0$. Hence

$$t = \sqrt{\frac{2}{\mu}} \times 2r^2 \times \left\{ -\varrho\sqrt{\left(1 - \frac{\varrho^2}{4r^2}\right)} + 2r \sin.^{-1}\frac{\varrho}{2r} \right\}$$

Let $\varrho = 2r$. Then the whole time (T) of revolution is $2t$, and we have

$$T = 8r^3\sqrt{\frac{2}{\mu'}} \times \frac{\pi}{2} = 4\pi r^3\sqrt{\frac{2}{\mu'}} \ldots\ldots\ldots\ldots (b)$$

Hence, since by the question T = T', we have

$$2\pi\sqrt{\frac{r^6}{\mu}} = 4\pi r^3\sqrt{\frac{2}{\mu'}}$$

$$\text{or } \sqrt{\frac{\mu'}{\mu}} = 2\sqrt{2}$$

$$\therefore \mu : \mu' :: 0 : 1 : 8$$

the ratio required.

Otherwise,

By Prop. VII. cor. 2, of *Newton's* Prin. we have (see *Newton's* Fig.)

$$\text{F at R} : \text{F at S} :: \text{RP}^2 \times \text{SP} : \text{SG}^2.$$

Now, let S be in the centre of the circle, and R be in the circumference; then we have

$$\text{SG} : 2\text{SP} :: \text{SP} : \text{PR}$$

$$\text{or SG} = \frac{2\,\text{SP}^2}{\text{PR}}$$

Hence

$$\text{F at S}' : \text{F at R} :: \text{RP}^2 \times \text{SP} : \frac{8\text{SP}^6}{\text{RP}^3}$$

$$:: \frac{1}{\text{SP}^5} : \frac{8}{\text{RP}^5}.$$

But if μ and μ' be the absolute forces, we have

$$\text{F at S} : \text{F at R} :: \frac{\mu}{\text{SP}^5} : \frac{\mu'}{\text{RP}^5}$$

$$\therefore \mu : \mu' :: 1 : 8$$

as before.

529. The velocity in an ellipse at any point of it where the $\perp$ on the tangent is p, is

$$v = \frac{c}{p}$$

c being $= b\sqrt{\frac{\mu}{a}}$ (453). Hence at the mean distance, where $p = b$, we have

$$v = \sqrt{\frac{\mu}{a}} \ldots\ldots\ldots\ldots\ldots\ldots (a)$$

Again, in a circle the velocity is

$$v' = \frac{c'}{r}$$

where c' (considering the ⊙ a limit of an ellipse) is $\sqrt{r\mu'}$. Hence

$$v' = \frac{\sqrt{r\mu'}}{r} = \sqrt{\frac{\mu'}{r}} \ldots\ldots\ldots\ldots (b)$$

Hence if the absolute forces, μ, μ', be the same in the circle and ellipse, and the radius (r) of the circle be equal to the semi-major-axis (a) of the ellipse, we have

$$v' = v$$

or the velocity in the ellipse at its mean distance is the same as that in a circle whose radius is the mean distance, and centre of force the same as that in the ellipse. Q. E. D.

Again, in the ellipse the periodic time is (484)

$$T = \frac{2\pi a^{\frac{3}{2}}}{\sqrt{\mu}}$$

and in the circle it is (440)

$$T' = 2\pi\sqrt{\frac{a}{F}} = \frac{2\pi a^{\frac{3}{2}}}{\sqrt{\mu}}$$

$$\therefore T' = T \qquad \text{Q. E. D.}$$

530. If the variable force be

$$F = \mu\varrho$$

we have

$$v^2 = 2\mu\int -\varrho d\varrho = \mu . (a^2 - \rho^2)$$

a being the whole distance to the centre.

Hence the velocity acquired in falling to the centre is

$$V = a\sqrt{\mu} \ldots\ldots\ldots\ldots (a)$$

Again, if the force at the beginning of the descent, viz. μa, be considered constant, the velocity acquired down the whole distance with half this force is

$$V' = \sqrt{2 \times F \times \text{dist.}} = \sqrt{2 . \frac{\mu a}{2} \times a} = a\sqrt{\mu}$$

$$\therefore V' = V. \qquad \text{Q. E. D.}$$

531. If R and r be the radii of the globe and wheel of the epicycloid, the equation referred to the centre of the globe is easily proved to be

$$p^2 = \frac{(R+2r)^2}{(R+2r)^2 - R^2} \times (\rho^2 - R^2) \ldots\ldots\ldots (1)$$

$$= A(\varrho^2 - R^2)$$

$$\therefore pdp = A\varrho d\varrho$$

$$\text{and } F = \frac{c^2 dp}{p^3 d\varrho} = \frac{c^2 A\varrho}{p^4} = \frac{c^2}{A} \cdot \frac{\varrho}{(\varrho^2 - R^2)^2}$$

$$\therefore F \propto \frac{\varrho}{(\rho^2 - R^2)^2}$$

the law of force required.

532. Let CA or CB $= a$, and make CD $= b$. Then since DE $= a$, by the hypothesis we have

$$CE = \sqrt{a^2 - b^2}.$$

Now the force at C varying as the distance, let it be

$$F = \mu\varrho$$

Hence the velocity down any space $a - \varrho$ is

$$v = \sqrt{2\int - \mu\varrho d\varrho} = \sqrt{\mu}\sqrt{(a^2 - \rho^2)}$$

Hence the velocities of the balls when they arrive at D and E are respectively

$$v = \sqrt{\mu} . \sqrt{(a^2 - b^2)}, \text{ and } v' = \sqrt{\mu}\sqrt{(a^2 - \overline{a^2 - b^2})} = \sqrt{\mu} \times b,$$

Now the bodies being perfectly elastic, move after impact with the planes with the same velocities as they impinge.

Therefore the initial velocities at the distances b and $\sqrt{a^2 - b^2}$ are

$$\sqrt{\mu} . \sqrt{(a^2 - b^2)}, \sqrt{\mu} . b$$

and since the directions of the initial movements, or tangents of the orbits, are inclined to the distances at angles of $180^\circ - 45^\circ$; $\therefore$ the initial perpendiculars on the tangents are respectively

$$P = R. \sin. 45^\circ \text{ and } P' = R' \sin. 45^\circ$$

$$\text{or } P = \frac{b}{\sqrt{2}}, \text{ and } P' = \frac{\sqrt{(a^2 - b^2)}}{\sqrt{2}}.$$

Again, since the force varies as the distance the orbits are ellipses with the force in the centre, and the equation to the ellipse referred to its centre is

$$p^2 = \frac{\alpha^2\beta^2}{\rho^2 + \alpha^2 - \beta^2}$$

α and β being the semi-axes. Hence

$$\left.\begin{aligned} P^2 &= \frac{b^2}{2} = \frac{\alpha^2\beta^2}{b^2 + \alpha^2 - \beta^2} \\ \text{and } P'^2 &= \frac{a^2 - b^2}{2} = \frac{\alpha'^2\beta'^2}{a^2 - b^2 + \alpha'^2 - \beta'^2} \end{aligned}\right\} \ldots\ldots (1)$$

Again, since

$$F = \mu\rho = \frac{c^2dp}{p^3d\rho} \quad (436)$$

we have for the two orbits,

$$\mu\rho = \frac{c^2\rho}{\alpha^2\beta^2}, \text{ and } \mu\rho = \frac{c'^2\rho}{\alpha'^2\beta'^2}$$

$$\therefore c = \alpha\beta\sqrt{\mu} \text{ and } c' = \alpha'\beta'\sqrt{\mu}.$$

$$\text{But } v = \frac{c}{P}, \text{ and } v' = \frac{c'}{P'}$$

$$\therefore \sqrt{\mu}\sqrt{(a^2 - b^2)} = \frac{\alpha\beta\sqrt{\mu}}{\frac{b}{\sqrt{2}}}, \text{ and } \sqrt{\mu}.\, b = \frac{\alpha'\beta'\sqrt{\mu}}{\frac{\sqrt{(a^2-b^2)}}{\sqrt{2}}}$$

$$\therefore \alpha\beta = b\sqrt{\frac{a^2 - b^2}{2}}, \text{ and } \alpha'\beta' = b\sqrt{\frac{a^2 - b^2}{2}} \ldots (2)$$

Hence, and from Equat. (1) we have

$$\left.\begin{aligned} \alpha^2 - \beta^2 &= (a^2 - b^2) - b^2 = a^2 \\ \text{and } \alpha'^2 - \beta'^2 &= b^2 - (a^2 - b^2) = 2b^2 - a^2 \end{aligned}\right\} \ldots\ldots (3)$$

and from these four equations it is easy to find by the solution of quadratic equations the values of α, β; α', β'; and therefore to construct the orbits.

533. Let a be the distance of the earth from the Moon, μ, μ' their quantities of matter; then if ρ be the distance from the earth at which their attractions are equal, we have

$$\frac{\mu}{\rho^2} = \frac{\mu'}{(a-\rho)^2}$$

whence

$$\varrho = \frac{a\sqrt{\mu}}{\sqrt{\mu} + \sqrt{\mu'}} \ldots\ldots\ldots (a)$$

Again, if r, r' be any distances from the earth and moon of the body in the curve of equal attraction, we have

$$\frac{\mu}{r^2} = \frac{\mu'}{r'^2}$$

But if x and y denote the rectangular co-ordinates of the required curve originating in the earth's centre, and measured along the line joining the earth and moon, we have

$$r^2 = x^2 + y^2$$

$$\text{and } r'^2 = (a - x)^2 + y^2$$

$$\therefore \mu \,.\, (a - x)^2 + \mu y^2 = \mu' x^2 + \mu' y^2$$

$$\therefore y^2 = \frac{\mu' x^2 - \mu.(a-x)^2}{\mu - \mu'}$$

$$= -x^2 + \frac{2a\mu x}{\mu - \mu'} - \frac{\mu a^2}{\mu - \mu'} \ldots\ldots\ldots (b)$$

Let $u = x - \frac{a\sqrt{\mu}}{\sqrt{\mu} + \sqrt{\mu'}}$, and the equation is transformable to

$$y^2 = \frac{2a\sqrt{\mu\mu'}}{\mu - \mu'}x - x^2 \ldots\ldots (c)$$

consequently the curve is a circle whose radius is

$$\frac{a\sqrt{(\mu\mu')}}{\mu - \mu'}.$$

534. If F, F' denote the forces, v and v' the velocities, and R, R' the distances; then since

$$v^2 : v'^2 :: \mathrm{F} \times \mathrm{R} : \mathrm{F'} \times \mathrm{R'}$$

But the velocity in any curve at any point of it is equal to that which would be acquired down $\frac{1}{4}$ chord of curvature at that point.

Hence if c, c' denote these chords at the points of projection, we have

$$v^2 = 2\mathrm{F} \times \frac{c}{4}, \text{ and } v'^2 = 2\mathrm{F'}.\frac{c'}{4}$$

$$\therefore c \times F : c' \times F' :: FR : F'R'$$

$$\text{or } c : c' :: R : R'.$$

Hence if P, P′ (Fig. 98) denote the points of projection, S, S′ the centres of force similarly situated, that is making equal $\angle$s, with the directions of projection or tangents PR, R′R′ of the orbits PQW, P′Q′W′, and PV, P′V′ the chords of curvature, we have

$$PV : P'V' :: PS : P'S'$$

From S, S′ draw SQ, S′Q′ making equal angles with SP, S′P′, and meeting the circles of curvature in Q, Q′ and join PQ, P′Q′ and produce SQ, S′Q′ to R, R′. Then

$$\angle PQV \; (= \angle YPS = \angle Y'P'S') = \angle P'Q'V'.$$

Hence putting $\angle PSQ \; (= \angle P'S'Q') = \alpha$

and $\angle PQV \; (= \angle P'Q'V') = \beta$

and the $\angle VQS = \theta$, and $\angle V'Q'S' = \theta'$

we get

$$QV : SV :: \sin. \alpha : \sin. \theta$$

$$\text{and } PV : QV :: \sin. \beta : \sin. (\alpha + \beta + \theta)$$

$$\therefore PV : SV :: \sin. \alpha . \sin. \beta : \sin. \theta . \sin. (\alpha + \beta + \theta).$$

similarly,

$$P'V' : S'V' :: \sin. \alpha . \sin. \beta : \sin. \theta' . \sin. (\alpha + \beta + \theta')$$

$$\text{and } PV : SV :: P'V' : S'V'$$

$$\therefore \sin \theta \times \sin. (\alpha + \beta + \theta) = \sin. \theta' . \sin. (\alpha + \beta + \theta')$$

But generally $\cos. (P - Q) - \cos. (P + Q) = 2 \sin. P \sin. Q.$

$\therefore \cos. (\alpha + \beta) - \cos. (\alpha + \beta + 2\theta) = \cos. (\alpha + \beta) - \cos. (\alpha + \beta + 2\theta').$

$$\therefore \cos. (\alpha + \beta + 2\theta) = \cos. (\alpha + \beta + 2\theta')$$

$$\text{and } \alpha + \beta + 2\theta = \alpha + \beta + 2\theta'$$

$$\text{or } \theta = \theta'$$

that is

$$\angle VQS = \angle V'Q'S'$$

Hence $\angle PQS = \angle P'Q'S'$

and $\angle S = \angle S'$

$\therefore \triangle SPQ, S'P'Q'$ are similar.

Or $PQ : P'Q' :: PS : P'S'$

And by Lemma 7th of Principia, the arcs and chords are ultimately in a ratio of equality.

$$\left.\begin{array}{r}\therefore \text{ arc PQ : arc P}'\text{Q}' :: \text{PS : P}'\text{S}' \\ \text{also QS : Q}'\text{S}' :: \text{PS : P}'\text{S}'\end{array}\right\} \ldots\ldots (1)$$

Again if Qq, $Q'q'$ be the next elemental arcs subtending equal $\angle QSq, Q'S'q'$; and $yQr, y'Q'r'$ the tangents or directions of motion at Q, Q'; then since the velocities at Q, Q' are as $\frac{1}{Sy}$, $\frac{1}{S'y'}$, or as $\frac{1}{R}$, $\frac{1}{R'}$, or as $\frac{1}{SY}$, $\frac{1}{SY'}$, or as the velocities at P, P' and $\angle yQS$ ($= \angle yQP + \angle PQS = \angle PVQ + \angle PQS = \angle P'V'Q' + \angle P'Q'S' = \angle y'Q'P' + \angle P'Q'S'$) $= \angle y'Q'S'$, or the directions of motions are similarly situated with respect to the distances SQ, S'Q, we have, by what has already been demonstrated,

$$\left.\begin{array}{r}\text{arc } Qq : \text{arc } Q'q' :: QS : Q'S' :: PS : P'S' \\ \text{and } qS : q'S :: QS : Q'S' :: PS : P'S'\end{array}\right\}$$

and so on through the whole extent of the orbits. Hence, by the composition of ratios we have, supposing $\angle PSW = \angle P'S'W'$.

$$\left.\begin{array}{r}\text{arc PW : arc P}'\text{W}' :: \text{PS : P}'\text{S}' \\ \text{and SW : S}'\text{W}' :: \text{PS : P}'\text{S}'\end{array}\right\} \ldots\ldots (2)$$

But curves referred to a centre are similar, when the radii-vectores containing equal angles are always proportional.

Therefore the orbit PW is similar to the orbit P'W'.

Q. E. D.

535. Since the velocity is finite, the law required must be inverse. Hence assuming

$$F = \frac{\mu}{\varrho^m}$$

we have

$$v^2 = 2\int - F d\varrho = \frac{2\mu}{m-1} \times \frac{1}{\varrho^{m-1}},$$

the velocity being 0 when $\varrho = \infty$.

Now the velocity in a circle at the distance ϱ is

$$v' = \sqrt{F\rho} = \sqrt{\mu} \cdot \frac{1}{\rho^{\frac{m-1}{2}}}$$

Hence, by the question,

$$\frac{2\mu}{m-1} \times \frac{1}{\varrho^{m-1}} = \frac{\mu}{n} \times \frac{1}{\varrho^{m-1}}$$

$$\therefore \frac{2n}{m-1} = 1$$

$$\text{or } m = 2n + 1,$$

$$\therefore F \propto \frac{1}{\varrho^{2n+1}}.$$

536. In the logarithmic spiral

$$p = \frac{b}{a} \cdot \varrho, \text{ and } \theta = \frac{b}{\sqrt{(a^2-b^2)}} \cdot l \cdot \frac{\rho}{a}$$

$$\therefore F = \frac{c^2 dp}{p^3 d\rho} = c^2 \times \frac{a^2}{b^2} \frac{1}{\rho^3} = \frac{\mu}{\rho^3},$$

$$\therefore c = \frac{b}{a} \sqrt{\mu}.$$

$$\text{Hence } dt = \frac{\varrho^2 d\theta}{c} = \frac{\rho^2}{c} \times \frac{b}{\sqrt{(a^2-b^2)}} \cdot \frac{-d\rho}{\varrho}$$

$$= \frac{a}{\sqrt{\mu} \cdot \sqrt{(a^2-b^2)}} \cdot -\varrho d\varrho.$$

$$\therefore t = \frac{a}{2\sqrt{\mu} \cdot \sqrt{(a^2-b^2)}} \cdot (a^2 - \varrho^2)$$

t being supposed $= 0$, when $\varrho = a$.

Hence the whole time to the centre is

$$T = \frac{a^3}{2\sqrt{\mu}\sqrt{(a^2-b^2)}}.$$

Now in the circle whose radius is a, the Periodic Time is (440)

$$T' = 2\pi \sqrt{\frac{a}{F}} = 2\pi \cdot \sqrt{\frac{a^4}{\mu}} = 2a^2 \pi \sqrt{\frac{1}{\mu}}$$

$$\therefore T : T' :: a : 4\pi\sqrt{(a^2 - b^2}.$$

537. The force being $\frac{\mu}{\varrho^3}$ the velocity acquired down r,

the radius of the circle is found from

$$v^2 = 2\mu \, . \int -\rho^3 d\rho = \frac{\mu}{2}(r^4 - \rho^4), \text{ which gives}$$

$$v = \sqrt{\frac{\mu}{2}} \times r^2 \text{ .... } (a)$$

when $\rho = 0$. Now at the centre let the force become repulsive, and we have

$$v'^2 = 2\mu \, . \int \rho^3 \, d\rho$$
$$= \frac{\mu}{2} \rho^4$$

the correction being zero. Hence when $\rho = r$,

$$v' = \sqrt{\frac{\mu}{2}} \times r^2.$$

Hence the whole velocity acquired by falling and rising is

$$V = v + v' = \sqrt{2\mu} \times r^2.$$

But the velocity in the circle, the force being the same is

$$V' = \sqrt{2F \times \frac{r}{2}} = \sqrt{Fr} = \sqrt{\mu \times r^4} = r^2\sqrt{\mu}$$

$$\therefore V : V' :: \sqrt{2} : 1 \text{ and not } :: 1 : 1.$$

The mistake in the enunciation probably arose from adding v^2 and v'^2, instead of v and v'.

538. If μ and μ' be the absolute forces, and t, t' the Periodic Times, according as the centre of force is in the centre or circumference of the equal circles, then by 428, we have

$$t = \frac{2\pi r^3}{\sqrt{\mu}} \text{ and } t' = 4\pi r^3 \sqrt{\frac{2}{\mu'}}$$

$$\therefore t : t' :: \sqrt{\mu'} : 2\sqrt{2} \times \sqrt{\mu}$$

But by the question $\mu' = 4\mu$,

$$\therefore t : t' :: 1 : \sqrt{2}.$$

539. Let the force be

$$F = \frac{\mu}{\rho^5}$$

and R the apsidal distance. Then since

$$v^2 = 2\mu \int \frac{-d\rho}{\rho^5} = \frac{\mu}{2} \frac{1}{\rho^4} \ldots\ldots \text{from } \infty \text{ to } \rho.$$

$$\text{and } V^2 = \frac{\mu}{2} \cdot \frac{1}{R^4}$$

between the limits of $\rho = \infty$ and $\rho = R$.

Hence the velocity of projection is

$$V = \frac{1}{R^2} \sqrt{\frac{\mu}{2}} \ldots\ldots\ldots\ldots\ldots\ldots (a)$$

Now (436)

$$d\theta = \frac{cd\rho}{\rho\sqrt{(-c^2 - 2\rho^2 \int Fd\rho)}}$$

$$= \frac{cd\rho}{\rho\sqrt{\left(-c^2 + 2\rho^2 \times \frac{\mu}{4} \times \frac{1}{\rho^4}\right)}}$$

$$= \frac{cd\rho}{\rho\sqrt{\left(-c^2 + \frac{\mu}{2} \cdot \frac{1}{\rho^2}\right)}}$$

$$= \frac{cd\rho}{\sqrt{\left(-c^2\rho^2 + \frac{\mu}{2}\right)}}$$

$$\text{But } c = R \times V = R \times \sqrt{\frac{\mu}{2}} \times \frac{1}{R^2} = \sqrt{\frac{\mu}{2}} \times \frac{1}{R}$$

$$\therefore d\theta = \frac{1}{R} \times \frac{d\rho}{\sqrt{-\frac{\rho^2}{R^2} + 1}} = \frac{d\rho}{\sqrt{(R^2 - \rho^2)}}$$

$$\therefore \theta = \sin.^{-1} \frac{\rho}{R} + C.$$

Let $\theta = 0$, when $\rho = R$, and we have

$$\theta = \sin.^{-1} \frac{\rho}{R} - \frac{\pi}{2} \ldots\ldots\ldots\ldots (a)$$

$$\therefore \rho = R \times \sin.\left(\frac{\pi}{2} + \theta\right) = R \cos.\theta \ldots\ldots\ldots (b)$$

the equation to the spiral.

Now

$$dt = \frac{\varrho^2 d\theta}{c} = -\frac{\varrho^2 d\varrho}{c\sqrt{(R^2-\varrho^2)}}$$

$$\text{and } t = \frac{1}{2c} \times \left\{ \rho\sqrt{R^2-\varrho^2} - R^2 \sin.^{-1}\frac{\varrho}{R} \right\} + C$$

Let $t = 0$, when $\rho = R$ and we have

$$c = \frac{1}{2c} \times R^2 . \frac{\pi}{2}$$

$$\therefore t = \frac{1}{2c} \left\{ \rho\sqrt{(R^2-\rho^2)} + R^2 . \overline{\frac{\pi}{2} - \sin.^{-1}\frac{\rho}{R}} \right\}$$

Let $\varrho = 0$. Then the whole time to the centre is

$$T = \frac{R^2\pi}{2c} = \frac{R^2\pi}{2.\sqrt{\frac{\mu}{2}} \times \frac{1}{R}} = \frac{R^3\pi}{\sqrt{2\mu}}.$$

Again, the Periodic Time in a circle whose radius is $\frac{R}{2}$ is (440)

$$T' = 2\pi \sqrt{\frac{\frac{R}{2}}{F}} = \pi\sqrt{\frac{2}{\mu}} \times R^3$$

$$\therefore T : T' :: \frac{1}{\sqrt{2}} : \sqrt{2} :: 1 : 2.$$

540. Let the distance of the vertex of the parabola from its focus be a, then the velocity in the curve is

$$v = \frac{c}{p} = \frac{\sqrt{2a\mu}}{\sqrt{2a\varrho}} = \sqrt{\frac{\mu}{\varrho}} \text{ (453)}$$

and the velocity parallel to the axis

$$: \text{vel. in curve} :: dx : ds$$
$$:: dx : \sqrt{(dx^2 + dy^2)}$$
$$:: 1 : \sqrt{\left(1 + \frac{dy^2}{dx^2}\right)}$$

But $y^2 = 4ax$. Therefore

$$\frac{dy^2}{dx^2} = \left(\frac{2ax}{y}\right)^2 = ax$$

$$\therefore v' : v = \sqrt{\frac{\mu}{\varrho}} :: 1 : \sqrt{(1+ax)}$$

$$\therefore v' = \frac{dx}{dt} = \sqrt{\frac{\mu}{\rho . (1+ax)}}$$

$$\text{But } \rho = \sqrt{(y^2 + \overline{x-a}|^2)}$$
$$= \sqrt{(4ax + x^2 + a^2 - 2ax)}$$
$$= a + x$$

$$\therefore \frac{dx}{dt} = \sqrt{\frac{\mu}{(a+x) . (1+ax)}}$$

$$\text{and } dt = \frac{dx}{\sqrt{\mu}} . \sqrt{(a+x) . (1+ax)} \ldots\ldots\ldots (1)$$

Again, if β be the velocity with which the system moves, and y' the ordinate of the *curve-in-space* at the end of the time t, the whole increment of y' during dt is

$$dy' = \beta dt \pm dy$$

according as the velocity $\perp$ to the axis arising from the motion in the curve, is in the same or opposite direction to that of the system.

$$\text{But } dy = \sqrt{\frac{a}{x}} . dx;$$

$$\therefore dy' = \frac{\beta}{\sqrt{\mu}} dx \sqrt{(a+x)(1+ax)} \pm \sqrt{\frac{a}{x}} dx.$$

Now by Hirsch's Tables, p. 160,

$$\int \frac{dx}{\sqrt{1+\left(a+\frac{1}{a}\right)x + x^2}} = l. \left\{2x + a + \frac{1}{a} + 2\sqrt{\left(1+\overline{a+\frac{1}{a}}\, x\right.}\right.$$
$$\left.\left. + x^2\right)\right\}$$

and by p. 172, of the same book

$$\int dx \left(1 + \overline{a + \frac{1}{a}} . x + x^2\right) = \frac{2x + a + \frac{1}{a}}{4} \times \sqrt{\left(1 + a + \frac{1}{a} x + x^2\right)}$$

$$+ \frac{4 - \left(a + \frac{1}{a}\right)^2}{8} \times \frac{dx}{\int \sqrt{\left(1 + a + \frac{1}{a} x + x^2\right)}}$$

Hence, putting $a + \frac{1}{a} = A$, and $\sqrt{(1 + a + \frac{1}{a} x + x^2)} = X$

$$y' = \frac{\beta}{\sqrt{\mu}} \times \left\{ \frac{2x + A}{4} X - \frac{\left(a - \frac{1}{a}\right)^2}{8} l. (2x + A + 2X) \right\} + C$$

And making $y' = 0$ when $x = 0$, we get

$$C = \frac{-\beta}{\sqrt{\mu}} \left\{ \frac{A}{4} - \frac{\left(a - \frac{1}{a}\right)^2}{8} . lA \right\}$$

$$\therefore y' = \frac{\beta}{4\sqrt{\mu}} \times \left\{ \left(2x + a + \frac{1}{a}\right) \sqrt{\left(1 + a + \frac{1}{a} x + x^2\right)} \right.$$

$$\left. - \sqrt{a + \frac{1}{a}} - \frac{\left(a - \frac{1}{a}\right)^2}{2} l. \frac{2x + a + \frac{1}{a} + 2\sqrt{(1 + a + \frac{1}{a} x + x^2)}}{a + \frac{1}{a}} \right\}$$

the equation to the required curve in space.

541. The velocity in a circle, when the force is F, and radius $= r$, is

$$v = \sqrt{2F \times \frac{r}{2}}$$

$$= \sqrt{\mu \times \varphi r}$$

where μ is the absolute force, and φr that same function of the radius (r) according to which the central force attracts at different distances.

Hence, at the same distance, and for the same law of force denoted by φ, we have

$$v \propto \sqrt{\mu} \propto (\text{absol. force}^{\frac{1}{2}}).$$

542. Since $v\,dv = -F dx$.

$$\therefore F = \frac{-v\,dv}{dx}$$

But if $v^2 = A \times \frac{a - x}{x}$; then

$$\frac{-2vdv}{dx} = \mathrm{A} \times \left(\frac{1}{x} + \frac{a-x}{x^2}\right)$$

$$= \frac{\mathrm{A}a}{x^2}$$

$$\therefore \mathrm{F} = \frac{2\mathrm{A}a}{x^2} \propto \frac{1}{x^2}$$

the law required.

543. Let a, b be the given semi-axes, and T the given periodic time. Then since the force tends to the focus, if ρ and θ be the polar co-ordinates, it is easily shewn (see *Newton*, Prop. 1.) that the areas described by ρ are proportional to the times of description; and because the area of the ellipse is πab, we have

$$\mathrm{T} : \pi ab :: dt : \frac{\rho^2 d\theta}{2}$$

$$:: \frac{ds}{v} : \frac{\rho^2 d\theta}{2}$$

$$\therefore v = \frac{2\pi ab\, ds}{\mathrm{T}\rho^2 d\theta} = \frac{2\pi ab}{\mathrm{T}} . \frac{\sqrt{\left(\rho^2 + \frac{d\rho^2}{d\theta^2}\right)}}{\rho^2}$$

But the equation to the ellipse is

$$\rho = \frac{a.(1-e^2)}{1 + e\cos.\theta}$$

$$\therefore \rho^2 + \frac{d\rho^2}{d\theta^2} = \frac{2a\rho^3 - \rho^4}{a^2(1-e^2)}$$

$$\text{And } v = \frac{2\pi ab}{\mathrm{T}} \times \frac{\sqrt{\left(\frac{2a}{\rho} - 1\right)}}{a\sqrt{(1-e^2)}}$$

$$= \frac{2\pi b}{\mathrm{T}\sqrt{(1-e^2)}} \times \sqrt{\frac{2a-\rho}{\rho}} = \frac{2\pi a}{\mathrm{T}}\sqrt{\frac{2a-\rho}{\rho}}$$

which gives the velocity at any point of the ellipse.

544. By 486 we have

$$\mathrm{F} = \frac{c^2\, dp}{p^3 dx}$$

But by the question

$$p^2 = \frac{b^2x^4}{a^4+x^4}$$

$$\therefore \frac{pdp}{dx} = \frac{2b^2x^3}{a^4+x^4} - \frac{2b^2x^7}{(a^4+x^4)^2}.$$

$$= \frac{2b^2x^3}{(a^4+x^4)^2} \times a^4$$

$$= \frac{2b^2a^4x^3}{b^4x^8} \times p^4$$

$$\therefore F = \frac{2c^2a^4}{b^2x^5} \propto \frac{1}{x^5}.$$

545. If v and v' denote the velocity in the curve, and that in a circle at the same distance, we have generally (438)

$$v^2 : v'^2 :: \frac{d\varrho}{dp} : \frac{\varrho}{p}$$

Hence when $v = v$

$$\frac{d\rho}{\rho} = \frac{dp}{p}$$

And $l\varrho = lp + C$

But when $\varrho = R$, let $p = P$; then

$$C = lR - lP = l\frac{R}{P}$$

And we have

$$l\varrho = l \times p \times \frac{R}{P}$$

$$\therefore p = \frac{P}{R} \times \varrho$$

the equation to a *Logarithmic Spiral.*

546. If $(a - \varrho)$ be the space described in a time *towards* a centre where the force is F, we have

$$vdv = -Fd\rho$$

$$\therefore v^2 = -2\int Fd\varrho$$

And if v be supposed to vary as the n^{th} power of the space described, n being an integer, we have

$$-2\int F d\varrho = A^2 \times (a - \rho)^{2n}$$

$$\therefore\ -2F d\rho = 2n A^2 (a - \rho)^{2n-1} d\varrho$$

$$\therefore\ F = -nA^2 . (a - \rho)^{2n-1}$$

or the force is repulsive, and varies as $(a - \varrho)^{2n-1}$

But since the motion is *towards* the centre, the force must be attractive. Hence the force is both attractive and repulsive at the same time, which is absurd. Therefore the velocity cannot vary as $(a - \varrho)^n$.

Again, let n be a proper fraction of the form $\frac{1}{m}$. Then

$$v^2 = A^2 . (a - \varrho)^{\frac{2}{m}}$$

$$\text{And } F = -\frac{1}{m} A^2 (a - \rho)^{\frac{2}{m} - 1}$$

which may be negative, and therefore F positive or attractive. In this general case then the velocity may $\propto (a - \varrho)^{\frac{1}{m}}$

$$\text{When } v = A . (a - \varrho)^{\frac{1}{3}}$$

$$F = \frac{1}{m} A^2 . (a - \rho)^{\frac{2}{3} - 1}$$

$$= \frac{1}{m} \frac{A^2}{(a - \varrho)^{\frac{1}{3}}} \propto \frac{1}{(a - \varrho)^{\frac{1}{3}}}$$

the negative root being always taken.

$$\text{Also} \quad dt = \frac{-d\varrho}{v} = \frac{-d\varrho}{A (a - \rho)^{\frac{1}{3}}}$$

$$\therefore\ t = \frac{3}{2A} \times (a - \varrho)^{\frac{2}{3}} + C.$$

But $t = 0$, when $\varrho = a$. Therefore

$$t = \frac{3}{2A} \times (a - \varrho)^{\frac{2}{3}} \propto (a - \varrho)^{\frac{2}{3}}$$

547. Let R and P be the radius vector, and $\perp$ on tangent at point of projection. Then

$$P = R \times \sin. 60^\circ = R \frac{\sqrt{3}}{2}.$$

Again, since the force is

$$F = \frac{\mu}{\rho^2}$$

the velocity down ρ from infinity to dist. $= R$, is

$$v = \sqrt{\frac{2\mu}{R}}$$

And the velocity of projection by the question is

$$V = \frac{v}{\sqrt{3}} = \sqrt{\frac{2\mu}{3R}} \dots\dots\dots (1)$$

Again,

$$V = \frac{c}{P} = \frac{b\sqrt{\frac{\mu}{a}}}{r\sqrt{\frac{3}{2}}} = \sqrt{\frac{2\mu}{3R}} \quad \text{(see 453)}$$

$$\therefore \frac{b}{\sqrt{a}} = \sqrt{R}$$

$$\text{and } \frac{b^2}{a} = R = \frac{L}{2} \dots\dots\dots\dots\dots\dots (2)$$

L being the *latus rectum.*

Hence the body is projected from the extremity of the latus rectum.

Again, since the general equation to an ellipse is

$$p^2 = \frac{b^2\rho}{2a - \rho}$$

$$\therefore \frac{3}{4}R^2 = \frac{aR^2}{2a - R}$$

$$\therefore a = \frac{3}{2}R \dots\dots\dots\dots\dots \dots (3)$$

the *semiaxis major.*

Hence

$$b = \sqrt{aR} = R\sqrt{\frac{3}{2}} \dots\dots\dots\dots (4$$

the *semiaxis minor,*

and $\sqrt{a^2-b^2} = \sqrt{\frac{9}{4}R^2-\frac{3}{2}R^2}$

$$= \frac{R}{2}\sqrt{3} \ldots\ldots\ldots\ldots\ldots (5)$$

the *eccentricity.*

Now the Periodic Time is (484)

$$T = \frac{2\pi a^{\frac{3}{2}}}{\sqrt{\mu}} = \frac{3\pi R^{\frac{3}{2}}}{\sqrt{2\mu}} \ldots\ldots\ldots\ldots\ldots (6).$$

Moreover the apsides are found by drawing through the centre of force a $\perp$ to R, and equal to $\frac{R}{2}$ and to $2a - \frac{R}{2} = \frac{5}{2}R$.

548. (1) Let $F = \frac{\mu}{\sqrt{\rho}}$. Then

$$vdv = -Fd\rho = -\frac{\mu d\rho}{\sqrt{\rho}}$$

$$\therefore v^2 = 4\mu(a^{\frac{1}{2}} - \rho^{\frac{1}{2}})$$

$$\therefore dt = -\frac{d\rho}{v} = -\frac{-d\rho}{2\sqrt{\mu}\sqrt{(a^{\frac{1}{2}} - \rho^{\frac{1}{2}})}}$$

Let $\rho^{\frac{1}{2}} = a^{\frac{1}{2}}u$. Then

$$d\rho = 2audu$$

and $dt = \dfrac{-a^{\frac{3}{4}}udu}{\sqrt{\mu}\sqrt{(1-u)}}$

$$\therefore t = -\frac{a^{\frac{3}{4}}}{\sqrt{\mu}} \times \left(\frac{1}{3}\sqrt{1-u}-1\right)(2\sqrt{(1-u)}+C,$$

see Hirsch's Tables, p. 94.

Let $t = 0$, when $\rho = a$ or $u = 1$ and $C = 0$.

$$\therefore t = -\frac{2a^{\frac{3}{4}}}{\sqrt{\mu}} \times \left(\frac{1}{3}\sqrt{\left(1-\frac{\rho^{\frac{1}{2}}}{a^{\frac{1}{2}}}\right)}-1\right)\sqrt{\left(1-\frac{\rho^{\frac{1}{2}}}{a^{\frac{1}{2}}}\right)}$$

the time through $a - \rho$.

(2) Let $F = \frac{\mu}{\rho^{\frac{1}{3}}}$. Then

$$v^2 = 2\int - \frac{\mu d\rho}{\rho^{\frac{1}{3}}} = 2\mu . (a^{\frac{2}{3}} - \rho^{\frac{2}{3}})$$

and $dt = \frac{-d\rho}{v} = \frac{-d\rho}{\sqrt{2\mu} . (a^{\frac{2}{3}} - \rho^{\frac{2}{3}})^{\frac{1}{2}}}$

Let $\rho^{\frac{2}{3}} = u^2 a^{\frac{2}{3}}$. Then

$\rho = au^3$

and $d\rho = 3\,au^2 du$.

Hence

$$dt = \frac{-3au^2 du}{\sqrt{2\mu}\, a^{\frac{1}{3}}\, (1 - u^2)^{\frac{1}{2}}}$$

$$\therefore t = - \frac{3a^{\frac{2}{3}}}{\sqrt{2\mu}} \int \frac{u^2 du}{(1-u^2)^{\frac{1}{2}}}$$

$$= - \frac{3a^{\frac{2}{3}}}{\sqrt{2\mu}} \times \left\{ \frac{-u\sqrt{(1-u^2)}}{2} + \frac{1}{2} \sin.^{-1} u + C \right\}.$$

Let $t = 0$, when $u = 1$, and we have $C = - \frac{1}{2} . \frac{\pi}{2}$

$$\therefore t = - \frac{3a^{\frac{2}{3}}}{2\sqrt{2\mu}} \times \left\{ \left(\sin.^{-1} . \left(\frac{\rho}{a}\right)^{\frac{1}{3}} - \frac{\pi}{2}\right) - \left(\frac{\rho}{a}\right)^{\frac{1}{3}} . \times \sqrt{\left(1 - \left(\frac{\rho}{a}\right)^{\frac{2}{3}}\right)} \right\}.$$

To know in what cases dt is integrable see Prob. (522).

549. Let $F = \frac{\mu}{\rho^5}$. Then

$$v^2 = 2\mu \int \frac{-d\rho}{\rho^5},$$

$$= \frac{\mu}{2} . \left(\frac{1}{\rho^4} - \frac{1}{a^4}\right)$$

and when $a = \infty$

$$v = \sqrt{\frac{\mu}{2}} \times \frac{1}{\varrho^2}.$$

Hence if R be the value of ϱ at the point of projection, and P the $\perp$ on the tangent, we have

$$\frac{c}{P} = \frac{5}{3} V = \frac{5}{3} \sqrt{\frac{\mu}{2}} \times \frac{1}{R^2}.$$

$$\text{But } P = R \times \frac{2\sqrt{6}}{5}$$

$$\therefore c = 2\sqrt{\frac{\mu}{3}} \times \frac{1}{R}.$$

Hence, and by 436, we have

$$d\theta = \frac{2}{R}\sqrt{\frac{\mu}{3}} \frac{d\rho}{\rho\sqrt{\left(-\frac{4\mu}{3R^2} + \frac{2\mu}{4}\varrho^2 \times \frac{1}{\varrho^4}\right)}}$$

$$= \frac{2\sqrt{2}}{R\sqrt{3}} \times \frac{d\varrho}{\sqrt{(1-\frac{8}{5R^2}\varrho^2)}} \ldots\ldots\ldots\ldots (a)$$

$$\therefore \theta = \frac{2\sqrt{2}}{R\sqrt{3}} \times \left\{C + \sqrt{\frac{3R^2}{8}} \times \sin.^{-1}\varrho\sqrt{\frac{8}{3R^2}}\right\}.$$

Let $\theta = 0$, when $\varrho = 0$, then $C = 0$, and we get

$$\theta = \sin.^{-1}\varrho\sqrt{\frac{8}{3R^2}}$$

$$\therefore \varrho = \frac{R}{2}\sqrt{\frac{3}{2}} \sin.\theta \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

the equation to the circle whose radius is

$$\frac{R}{4}\sqrt{\frac{3}{2}},$$

and origin of ϱ in the circumference.

Hence there are two apses, one at the centre of force, and another at the extremity of the diameter passing through the centre of force.

Hence

$$dt = \frac{\varrho^2 d\theta}{c} = \frac{R\varrho^2}{2\sqrt{\frac{\mu}{3}}} \times \frac{2\sqrt{2}}{R\sqrt{3}} \times \frac{\varrho^2 d\rho}{\sqrt{(1-\frac{8}{3R^2}\varrho^2)}}$$

and the integral (see *Hirsch's Tables*, p. 119,) taken between r, r' the values of ϱ which comprise the given angle, will give the time of describing that angle. This presenting no difficulty, we leave to the Student.

550. Since the body is projected from an aspe, the velocity of projection is

$$V = \frac{c}{R}$$

and since

$$F = \frac{\mu}{\varrho^n}$$

$$\therefore V^2 = 2\mu \int \frac{-d\varrho}{\varrho^n}$$

taken between $\varrho = \infty$ and $\varrho = R$.

$$\therefore V^2 = \frac{2\mu}{n-1} \times \frac{1}{R^{n-1}}$$

$$\therefore c^2 = \frac{2\mu}{n-1} \times \frac{1}{R^{n-3}}$$

$$\text{and } c = \sqrt{\frac{2\mu}{n-1}} \times \frac{1}{R^{\frac{n-3}{2}}}$$

Hence, and by (436,)

$$d\theta = \frac{c\,d\varrho}{\varrho\sqrt{\left(-c^2 + 2\varrho^2 \cdot \frac{\mu}{n-1} \cdot \frac{1}{\varrho^{n-1}}\right)}}$$

$$= \frac{c\varrho d\varrho^{\frac{n-5}{2}} d\varrho}{\sqrt{\left(-c^2\varrho^{n-3} + \frac{2\mu}{n-1}\right)}}$$

$$= \frac{c\varrho^{\frac{n-5}{2}} d\varrho}{\sqrt{(c^2 . R^{n-3} - c^2 . \varrho^{n-3})}}$$

$$= \frac{\varrho^{\frac{n-5}{2}} d\varrho}{\sqrt{(R^{n-3} - \varrho^{n-3})}}$$

Let $\rho^{\frac{n-3}{2}} = u R^{\frac{n-3}{2}}$. Then

$$du = \frac{\frac{n-3}{2}}{R^{\frac{n-3}{2}}} \cdot \rho^{\frac{n-5}{2}} d\rho$$

$$\text{and } d\theta = \frac{2}{n-3} \times \frac{du}{\sqrt{(1-u^2)}}$$

$$\therefore \theta = \frac{2}{n-3} \sin.^{-1} u + C$$

Let $\theta = 0$, when $u = \left(\frac{\rho}{R}\right)^{\frac{n-3}{2}} = 1$,

and we have $C = -\frac{2}{n-3} \cdot \frac{\pi}{2}$

Hence

$$\sin^{-1} u = \frac{\pi}{2} + \frac{n-3}{2}\theta$$

$$\therefore \left(\frac{\rho}{R}\right)^{\frac{n-3}{2}} = \sin.\left(\frac{\pi}{2} + \frac{n-3}{2}\theta\right)$$

$$= \cos. \frac{n-3}{2}\theta$$

the equation to the Trajectory.

Let the body fall into the centre or $\rho = 0$. Then n must be > 3, and we have

$$\text{Cos.} \frac{n-3}{2}\theta = 0$$

$$\text{or } \frac{n-3}{2}\theta = \frac{\pi}{2}$$

$$\therefore \theta = \frac{2\pi}{2n-6}$$

or the number of revolutions will be $\frac{1}{2n-6}$.

Again, let the body fly off to infinity, or $\rho = \infty$. Then n must be < 3, and the number of revolutions will be $\frac{1}{6-2n}$.

551. Let the velocity $= \frac{1}{m}$th of that acquired from ∞.

Then if R and P be the radius and $\perp$ on tangent at the point of projection; by the preceding problem we easily get

$$\frac{c}{\mathrm{P}} = \frac{1}{m} \times \sqrt{\mu} \times \frac{1}{\mathrm{R}}$$

$$\text{and } c = \frac{\mathrm{P}}{\mathrm{R}} \sqrt{\frac{\mu}{m^2}}$$

and proceeding as in 550 there will be no difficulty in obtaining the orbit and time required.

CONSTRAINED MOTION.

552. Let r be the radius of the sphere; then $r =$ radius of the base, and $2r$ the length of its circumscribing cylinder. Again, if S, S′ denote the required momentums of Inertia of the sphere and cylinder, referred to their common axis, we have (by the form $\frac{\pi}{2}\int y^4 dx$, (see *Bridge*, vol. II., p. 53, or *Creswell's Translation of Venturoli*, p. 123,)

$$S = \frac{8\pi r^5}{15},\ S' = \pi r^5$$

$$\therefore\ S : S' :: 8 : 15.$$

553. Let AD $= a$, BD $= b$; then the distance of the centre of gravity from A is (since AG : GB :: B : A, and ∴ AG : $a + b$ :: B : A + B)

$$AG = \frac{B}{A + B} \times (a + b)$$

And its distance from D is

$$k = \frac{B}{A + B} \cdot (a + b) \backsim a = \frac{Bb \backsim Aa}{A + B}.$$

Again, the *Momentum of Inertia* of A, B, with respect to the point D, is

$$S = Aa^2 \backsim Bb^2$$

∴ the distance of the centre of Spontaneous Rotation from D is (see *Venturoli*, p. 145)

$$\frac{S}{Mk} = \frac{Aa^2 \backsim Bb^2}{Aa \backsim Bb},$$

which determines the position of the centre of Spontaneous Rotation.

To find the locus of this centre described during an entire revolution.

Since by the definition its linear motion round the center of gravity = that of the centre of gravity along a $\perp$ to AB at D in every position, (this latter motion being uniform) therefore the locus is a cycloid, the radius of whose generating circle is

$$\frac{S}{Mk} - k = \frac{Aa^2 - Bb^2}{Aa - Bb} - \frac{Aa - Bb}{A + B}$$

$$= \frac{ABb^2 - ABa^2 + 2B^2b^2}{(A + B)(Bb - Aa)}$$

$$= \frac{B}{A + B} \; \frac{b^2.A + 2B - Aa^2}{Bb - Aa}$$

and its base passes through D $\perp$ to AB.

554. If S denote the moment of Inertia of the rod, M its mass, and k the distance of its centre of gravity from the axis of suspension; then the distance of the centre of oscillation from that axis is (See *Venturoli's* or *Whewell's Dynamics.*)

$$a = \frac{S}{Mk} = \frac{\frac{l^3}{3}}{M \times \frac{l}{2}} = \frac{2l}{3}$$

the length of the simple pendulum that would oscillate seconds.

Again, let the distance from the axis of the point in which w is placed be x. Then the distance of the center of oscillation from the axis of suspension of this compound pendulum is

$$a' = \frac{S'}{M'k'} = \frac{W \times \frac{4l^2}{9} + w\,x^2}{(W+w) \times \frac{2lW + 3wx}{3(W+w)}}$$

$$= \frac{4l^2W + 9wx^2}{3(2lW + 3wx)}$$

But since the time of an oscillation when the length of the pendulum is L is

$$T = \pi \sqrt{\frac{L}{g}}$$

$$\therefore a' = \frac{T^2 g}{\pi^2} = \frac{g}{\pi^2} \times \tfrac{1}{4} \text{ by the question.}$$

Hence

$$\frac{3g}{4\pi^2} \times (2lW + 3wx) = 4l^2 W + 9wx^2$$

$$\therefore x^2 - \frac{g}{4\pi^2} x = \frac{glW}{6\pi^2 w} - \frac{4l^2 W}{9w}$$

$$= \frac{lW}{18\pi^2 w} \times (3g - 8\pi^2 l)$$

which gives the distance required.

555. Let R, r be the radii of the wheel and axle; then since ry, Rp denote the efforts of y and p to descend, the moving force is

$$M = Rp - ry.$$

Moreover the Inertiæ of p and y are pR^2, yr^2. Hence the accelerating force is

$$F = \frac{pR - yr}{pR^2 + yr^2} \times r \propto \text{velocity}$$

when the time is given.

Hence the momentum generated is

$$\frac{pR - yr}{pR^2 + yr^2} \times ry = \text{maximum}$$

$$\therefore u = \frac{pRy - ry^2}{pR^2 + r^2 y} = \text{max.}$$

and putting $du = 0$, we get

$$\frac{pR - 2ry}{pR^2 + r^2 y} - \frac{r^2(pRy - ry^2)}{(pR^2 + r^2 y)^2} = 0$$

$$\therefore (pR - 2ry)(pR^2 + r^2 y) = r^2 y (pR - ry)$$

$$\therefore (pR - ry)\, pR^2 = ry\,(pR^2 + r^2 y)$$

$$\therefore p^2 R^3 = 2prR^2 y + r^3 y^2$$

$$\therefore y^2 + 2p.\frac{R^2}{r^2} y = p^2.\frac{{}^2R^3}{r^3}$$

$$\therefore y = -\frac{pR^2}{r^2} \pm \sqrt{\frac{p^2 R^4}{r^4} + \frac{p^2 R^3}{r^3}}$$

$$= -\frac{pR^2 + pR^{\frac{3}{2}}.\sqrt{(R + r)}}{r^2}$$

the value required.

556. Let h be the vertical abscissa measured from the lowest point of the height from which the body begins its descent, and x any other when the body shall have acquired the velocity v. Then since the body has descended vertically through $h - x$, its velocity the cycloid is

$$v = \sqrt{2g \cdot (h - x)}.$$

Hence this being resolved into horizontal and vertical velocities we get for the former

$$v' = \frac{v \cdot x}{\sqrt{2rx}} = \sqrt{\frac{g}{r} \cdot x \cdot (h - x)}$$

r being the radius of the generating circle.

$$\therefore x \cdot (h - x) = \text{maximum} = m$$

$$\therefore dm = 0 \text{ gives}$$

$$x = \frac{h}{2}$$

and $v' = \frac{h}{2} \sqrt{\frac{g}{r}}$ for the maximum.

557. Since for the centre of gravity of a solid of revolution the distance of that centre from the origin of x is

$$\frac{\int xy^2 dx}{\int y^2 dx}$$

$\therefore$ by the question, we have

$$\frac{\int xy^2 dx}{\int y^2 dx} = \frac{2}{3} x$$

$$\therefore xy^2 dx = \frac{2}{3} dx \int y^2 dx + \frac{2}{3} xy^2 dx$$

$$\therefore xy^2 = 2 \int y^2 dx$$

and $y^2 dx + 2xydy = 2y^2 dx$

$$\therefore ydx = 2xdy$$

and $\frac{2dy}{y} = \frac{dx}{x}$

$$\therefore l. y^2 = l. x + lc = lcx$$

and $y^2 = cx$

and the curve required is the common parabola.

558. Denoting by l, S, and k, the distance of the centre of oscillation from the point of suspension, the momentum of Inertia, and distance of the centre of gravity from the point of suspension we have (See *Venturoli.*)

$$l = \frac{S}{Mk}$$

Now, supposing generally the density to $\propto x^n$, we have

$$k = \frac{\int x dM}{M}$$

But $M = A \times \frac{x^{n+1}}{a^{n+1}}$

A being the mass at the distance a.

$$\therefore k = \frac{n+1}{n+2} x$$

Also $S = \int x^2 dM = \frac{\overline{n+1} \cdot A}{a^{n+1}} \int x^{n+2} dx$

$$= \frac{n+1}{n+3} \frac{A}{a^{n+1}} x^{n+3} = \frac{n+1}{n+3} Mx^2$$

$$\therefore l = \frac{n+2}{n+3} x = \frac{n+2}{n+3} a$$

for the given length a.

Hence

$$t = \pi \sqrt{\frac{l}{g}} = \pi \sqrt{\frac{n+2}{n+3} \cdot \frac{a}{g}}.$$

In the question $n = 1$. In this case

$$t = \pi \sqrt{\frac{3a}{4g}}.$$

559. Let a be the given base, x the length of the inclined plane; the height $= \sqrt{(x^2 - a^2)}$, and since

$$x = \frac{g}{2} Ft^2 = \frac{g}{2} \frac{\sqrt{a^2 - x^2}}{x} \cdot t^2$$

$$\therefore t^2 = \frac{2x^2}{g\sqrt{(a^2 - x^2)}} = \text{maximum}$$

and $\therefore \frac{a^2 - x^2}{x^4} = \text{minimum}$

$$\therefore -\frac{2xdx}{x^4} - \frac{4dx \times (a^2 - x^2)}{x^3} = 0$$

whence

$$x^2 + 2a^2 - 2x^2 = 0$$

$$\text{or } x = a\sqrt{2}$$

and therefore the required Δ is isosceles.

560. Let r be the radius of the globe; then since the mass is given the density $\propto$ inversely as the magnitude. Hence calling x, x' the densities, and A, A′ the magnitudes of the globe and cylinder, we have

$$x : x' :: \frac{1}{A} : \frac{1}{A'}$$

$$\text{But } A = \frac{2}{3} A'.$$

$$\therefore x : x' :: 3 : 2.$$

561. Let the radius of the inner section of the shell be x. Then its mass will be

$$\pi hr^2 - \pi hx^2 = \pi h.(r^2 - x^2)$$

$$= \text{(by the question) } \frac{4}{3}\pi r^3.$$

$$\therefore x^2 = \frac{r^2}{3h}(3h - 4r)$$

$$\text{and } x = r \times \sqrt{\frac{3h - 4r}{3h}}$$

Hence the thickness of the shell is

$$r - x = \frac{r}{\sqrt{3h}} \times \{\sqrt{3h} - \sqrt{(3h - 4r)}\}$$

It is also plain that h cannot be $< \frac{4r}{3}$.

562. Since, V, V′, V″ are the absolute velocities of the bodies A, A′ A″, the relative velocities of A, A′; A, A″ are

$$V \backsim V' \quad V \backsim V''$$

and the times of *first rencontre* between A, A′; A, A″ are respectively

$$t = \frac{p}{V \backsim V'},\ t' = \frac{p}{V \backsim V''} \ \ldots\ldots\ (a)$$

p being the periphery of the *curve* described.

Again, let T, T′, T″, be the times in which A, A′, A″, each complete a revolution, then we have

$$p = VT = V'T' = V''T''$$

$$\text{and } t = \frac{p}{\frac{p}{T} \backsim \frac{p}{T'}} = \frac{TT'}{T \backsim T'} \ \ldots\ldots\ (b)$$

$$t' = \frac{p}{\frac{p}{T} \backsim \frac{p}{T''}} = \frac{TT''}{T \backsim T''} \ \ldots\ldots\ (c)$$

and (t) (t') *being integers the time of the rencontre of all three bodies is the least common multiple of* t and t', or of

$$\frac{TT'}{T \backsim T'} \text{ and } \frac{TT''}{T \backsim T''}$$

In the same manner if $T_1, T_2, T_3 \ldots\ldots T_n$ *be the times in which n bodies perform each an entire revolution with uniform velocities* $V_1, V_2, \ldots\ldots V_n$, *then the interval between any two successive simultaneous rencontres of them is the least common multiple* of

$$\frac{T_1\,T_2}{T_1 \backsim T_2},\quad \frac{T_1\,T_3}{T_1 \backsim T_3} \quad \&c. \quad \frac{T_1\,T_{n-1}}{T_1 \backsim T_{n-1}}$$

when these are integers.

By way of example, *let it be required to find the interval between two simultaneous rencontres of the bodies* A, A′, A″, *whose velocities are*

$$V,\ V + \frac{p}{m},\ \ V + \frac{p}{n}$$

p being the periphery of the curve round which they run, and m, n, any integers whatever.

In this case

$$T = \frac{p}{V},\ T' = \frac{p}{V + \frac{p}{m}},\ T'' = \frac{p}{V + \frac{p}{n}}$$

$$\frac{TT'}{T - T'} = m, \text{ and } \frac{TT''}{T - T''} = n.$$

and the time of rencontre is the least common multiple of m and n.

Again, if t and t' be not integers, then since during the interval between two successive rencontres of A, A′, A″, there must have occurred a certain number w of rencontres between A, A′ and a certain number w', between A and A″ *exactly*; therefore we have the time required

$$x = w \times \frac{TT'}{T - T'} = w' \times \frac{TT''}{T - T''}$$

w and w' being positive whole numbers.

Hence

$$w = w' \times \frac{T''}{T'} \times \frac{T - T'}{T - T''} \dots\dots\dots (d)$$

which will give the values of w and w', from whence we can determine x.

Ex. *Required the interval between any two successive simultaneous rencontres of the hour, minute, and second hand, of a clock.*

Here

$$T = 12 \text{ hours} = 12 \times 60 \text{ minutes},$$
$$T' = 1 \text{ hour} = 60 \text{ minutes},$$
$$\text{and } T'' = 1 \text{ minute}.$$

$$\therefore w = w' \times \frac{1}{60} \times \frac{11 \times 60}{729} = w' \times \frac{11}{729}$$

$$\therefore w = 11, \text{ and } w' = 729.$$

Hence

$$x = 11 \times \frac{12 \times 60 \times 60}{11 \times 60} = 12 \text{ hours}.$$

Consequently the hands are together every twelve hours, or at 12 o'clock only.

As another example we may find the times of conjunction or opposition of one, two, three, or more of the planets, supposing them to move in circles (which they do nearly), and their sidereal periods to be known.

We have insisted somewhat at length upon this question, in order to expose an error which has crept into several of the MSS. in circulation at Cambridge, wherein it is asserted that *the interval*

between two successive conjunctions of three planets is the least common multiple of the synodical periods of any of them. This cannot be the case, as we have seen, unless those periods, viz., the expression (a), (b), are integers.

563. Let r be the radius of the circle, α the given angular velocity; then if y denote an ordinate $\perp$ to the axis of revolution, the linear velocity of the particle ds at the extremity of y is

$$v = \alpha y$$

and the momentum of this particle ds is

$$dM = \alpha y \times ds$$

$$= \alpha y \times \frac{rdy}{\sqrt{(r^2 - y^2)}} = \frac{\alpha rydy}{\sqrt{(r^2 - y^2)}}$$

$$\therefore M = -\alpha r.\sqrt{(r^2 - y^2)} + C$$

Let $y = 0$, then $C = \alpha r^2$

and $M = \alpha r^2 - \alpha r\sqrt{(r^2 - y^2)}$

Let $r = y$. Then

$$4M = 4\alpha r^2 \quad \ldots\ldots\ldots\ldots \quad (a)$$

the momentum of the whole circumference.

Now the momentum of the sides of the square is

$$2\int \alpha r ds = 2\alpha \times 2r \times r = 4\alpha r^2$$

the integral being taken between $s = o$ and $s = 2r$.[1]

Hence then the momentum s of the circumference and sides of the square are equal.

564. Let a, b, be the respective lengths of the arms to which are appended the weights p, q. Then the moving forces are

$$a, \frac{b}{a} q.$$

Hence the accelerating force at the extremity of the arm a in the direction of gravity is

$$F = \frac{p - \frac{b}{a} q}{p + \frac{b^2}{a^2} q}$$

the weight q when placed at the extremity b, being $\frac{b}{a}q$ and the inertia of q being $\frac{b^2}{a^2}q$.

Hence the linear velocity in the arc described is

$$v = \sqrt{2gFy}$$

y being the sine of the arc described.

But the angular velocity $\frac{1}{a}\times$ the linear velocity, and is therefore

$$v' = \frac{1}{a}\sqrt{2gFy}.$$

Let $y = a$; then

$$v' = \sqrt{\frac{2gF}{a}} = \sqrt{2g} \times \frac{ap-bg}{a^2p+b^2g}$$

is the velocity required.

565. Since the reflecting plane is inclined to the horizon at half a right angle, the bodies after impact, will move uniformly along the horizon (friction not being considered) with the velocity acquired. Hence putting a, x, = the altitudes fallen through by the bodies A and B, and b = the distance moved along the horizon concourse, we get

$$v = \sqrt{2ga},\ v' = \sqrt{2gx}$$

for the velocities acquired;

$$\text{and } t = \sqrt{\frac{2a}{g}} + \frac{b}{\sqrt{2ga}} = \sqrt{\frac{2x}{g}} + \frac{b}{\sqrt{2gx}}$$

gives

$$x - \frac{2a+b}{2\sqrt{a}}\sqrt{x} = -\frac{b}{2}$$

which being resolved, we have

$$\sqrt{x} = \frac{2a+b\pm\sqrt{(4a^2+4ab+b^2-8ab)}}{4\sqrt{a}}$$

$$= \sqrt{a} \text{ or } \frac{b}{2\sqrt{a}}$$

and therefore

$$x = a \text{ or } \frac{b^2}{4a}.$$

566. Let P be the weight at the circumference of the wheel, W that at the circumference of the axle, and w that of the wheel and axle; also let $w \times R^2$ be the moment of inertia of the system referred to the axis of rotation, R being determined by experiment (see *Venturoli*, p. 140), r and x the radii of the axle and wheel; then if v denote the velocity of P at the end of the time t, the velocities of

P, W, w are respectively

$$v, \frac{r}{x} v, \frac{R}{x} v$$

and at the end of $t + dt$, these actual velocities are

$$v + dv, \frac{r}{x}.(v + dv), \frac{R}{x}(v + dv).$$

Again, the impressed velocities of the bodies are

$$v + gdt, \frac{r}{x} v - gdt, \frac{R}{x} v$$

gravity having no effect upon the system; and the distances from the axis are

$$x, r, R.$$

Hence this Table

Mass.	Imp. vel.	Actual V.	Dist. from ax.
P	$v + gdt$	$v + dv$	x
W	$\frac{rv}{x} - gdt$	$\frac{r}{x}.(v+dv)$	r
w	$\frac{R}{x} v$	$\frac{R}{x}.(v+dv)$	R

But by *D'Alembert's* Principle

$$xP.(v + gdt) + rW.\left(\frac{rv}{x} - gdt\right) + \frac{R^2}{x} wv =$$

$$\left(xP + rW.\frac{r}{x} + RW.\frac{R}{x}\right)w + dv)$$

$$\therefore \frac{dv}{dt} = \frac{g(xP - rW)}{xP + \frac{r^2W}{x} + \frac{R^2w}{x}}$$

$$\text{or } F = \frac{gx.(xP - rW)}{x^2P + r^2W + R^2w} \; \ldots\ldots\ldots (1)$$

the accelerating force of P.

Hence the accelerating force of W is

$$F = \frac{gr.(xP - rW)}{x^2P + r^2W + R^2w} \; \ldots\ldots\ldots (2)$$

and if a be the given space through which W is to be raised we have

$$T^2 = \frac{a}{F} = \frac{a}{gr}.\frac{x^2P + r^2W + R^2w}{xP - rW} = \text{maximum}$$

by the question.

$$\therefore \frac{x^2P + r^2W + R^2w}{xP - rW} = \text{max.} = u.$$

$$\therefore \frac{du}{dx} = \frac{2xP}{xP - rW} - \frac{P.(x^2P + r^2W + R^2w)}{(xP - rW)^2} = 0$$

$$\text{which gives } x = \frac{rW \pm \sqrt{r^2W^2 + (r^2W + R^2w)P}}{P}$$

567. The distance of the centre of gyration from the axis of rotation is generally (see *Vince, Simpson.*)

$$R = \sqrt{\frac{\int z^2 ds}{s}}$$

where ds is the element of the body at the distance z from the axis.

Now x being the radius of the given circle, and dx the breadth of an annulus at the distance x from the axis, we have

$$ds = 2\pi x dx$$

$$\text{and } \therefore R = \sqrt{\frac{2\pi\int x^3 dx}{s}}$$

$$= \sqrt{\frac{2\pi x^4}{4s}}$$

Let $x = r$, then $s = \pi r^2$
and we have

$$R = \sqrt{\frac{r^2}{2}} = \frac{r}{\sqrt{2}},$$

which gives the centre of gyration required.

568. Let a, b, be the lengths of the arms of the lever; A, B their weights, and $\alpha =$ the $\angle$ between them. Then since when in equilibrio, the vertical line passing through the point of suspension also passes through their centre of gravity, we easily get the distance (k) of this centre of gravity from the axis of suspension, *viz.*

$$k = \frac{a \sin. \beta}{2b\text{B} \sin. \alpha} \times \sqrt{\{1 + (a\text{A} + b\text{B} \cos. \alpha)^2\}}$$

where β denotes the angle contained by a, and the line joining the extremities of the lever.

Again, the mass is

$$\text{M} = \text{A} + \text{B}.$$

Again, if x denote the distance of any point in the arm a from the point of suspension, then the momentum of inertia of that point is

$$dx \times x^2 \times \sin.^2 \delta$$

δ being the angle whose tangent is

$$\frac{b\text{B} \cos. \alpha + a\text{A}}{b\text{B} \sin. \alpha}.$$

Hence the whole momentum of inertia of x is

$$\frac{x^3}{3} \sin.^2 \delta$$

and that of the arm a is

$$\frac{\text{A}^3}{3} \sin.^2 \delta.$$

Similarly it may be shewn that

$$\frac{\text{B}^3}{3} \sin.^2 (\alpha + \delta).$$

is the momentum of inertia of the other arm.

Hence the distance of the centre of oscillation from the point of suspension is (*Venturoli*).

$$\frac{S}{Mk} = \frac{\frac{A^3}{3}\sin.^2\delta + \frac{B^3}{3}\sin.^2(\alpha+\delta)}{(A+B)\frac{a\sin.\beta}{2bB\sin.\alpha} \times \sqrt{\{1+(aA+bB\cos.\alpha)^2\}}}$$

the length of the simple pendulum. But the time of an oscillation in the simple pendulum whose length is l, (see *Venturoli*) is

$$t = \frac{\pi\sqrt{l}}{\sqrt{g}}$$

the time required.

569. Let w be the given weight of the rod, a its length, and T the given time of oscillation; then the distance of the centre of oscillation from that of suspension is

$$l = \frac{S}{Mk} = \frac{\left(\frac{a}{2}-x\right)^3 \times \frac{1}{3} + \left(\frac{a}{2}+x\right)^3 \times \frac{1}{3}}{ax}$$

x being the distance of the centre of gravity of the rod from the point of suspension. Hence

$$x^2 - lx = -\frac{a^2}{12}$$

$$\text{and } x = \frac{l}{2} \pm \sqrt{\frac{3l^2-a^2}{12}}$$

$$\text{But } T = \pi\sqrt{\frac{l}{g}}, \therefore l = \frac{T^2 g}{\pi^2}$$

Hence

$$x = \frac{T^2 g}{2\pi^2} \pm \sqrt{\frac{\frac{3T^4g^2}{\pi^4} - a^2}{12}}.$$

570. Let w be the given weight of the cylinder, r the radius of its base, and b its length; also let w' be the weight of the chain, l its whole length, and a that of the part thereof un-

wound at the commencement of the motion; then since the inertia of the cylinder is

$$\frac{1}{2}\pi r^4 a = \frac{1}{2} wr^2,$$

that of the chain

$$w' r^2$$

and the moving force after the space x has been described

$$\frac{w'a}{l} + \frac{w'x}{l}$$

therefore the accelerating force is

$$F = \frac{gw'(a+x)r^2}{l(w'r^2 + \frac{1}{2} wr^2} = \frac{g2w'(a+x)}{l(2w'+w)}$$

But if v be the velocity acquired

$$vdv = -Fdx$$

$$\therefore v^2 = \frac{2w'g}{l(2w'+w)} \times \{(a+x)^2 - a^2\}$$

which gives the velocity for any descent x.

Hence

$$t = \int \frac{dx}{v} = \sqrt{\frac{l \times (2w'+w)}{2w'g}} \times \int \frac{dx}{\sqrt{(2ax - x^2)}}$$

$$= \sqrt{\frac{l.2w'+w}{2gw'}}. \text{ hyp. log. } \frac{a+x+\sqrt{(2ax+x^2)}}{a}$$

Let $x = l - a$; then the time of unwinding the length $l - a$ is

$$t = \sqrt{\frac{l.2w'+w}{2gw'}} \text{ hyp. log. } \frac{l + \sqrt{(l^2 - a^2)}}{a}$$

the time required.

571. If v, v' denote the velocities of the oscillating and revolving bodies at the distance x from the lowest point of the circle, then we have

$$v = \sqrt{2g.(h-x)},\ v' = \sqrt{2g.(h'-x)}.$$

Again, let t, t' denote the times in which the arc $2\pi l - s$ measured from the highest point, s being that part of the semicircle which corresponds to the abscissa x; then

$$dt = \frac{-ds}{v} = \frac{-ldx}{\sqrt{2g(h-x)}.\sqrt{(2lx - x^2)}}$$

$$\text{and } dt' = \frac{-ds}{v'} = \frac{-ldx}{\sqrt{2g.(h'-x)}.\sqrt{(2lx-x^2)}}$$

which being integrated between

$x = h$ and 0

$x = 2l$ and 0

will give t and t'.

$$\text{But } dt = \frac{l}{\sqrt{2gl}} \times \frac{-dx}{\sqrt{(lx-x^2)}} \times \left(1 - \frac{x}{2l}\right)^{-\frac{1}{2}}$$

$$= \frac{l}{\sqrt{4gl}} \times \frac{-dx}{\sqrt{(hx-x^2)}} \times \left\{1 + \frac{1}{2} \cdot \frac{x}{2l} + \frac{1.3}{2.4} \cdot \frac{x^2}{(2l)^2} + \&c.\right\}$$

$$\text{and } dt' = \frac{l}{\sqrt{2gh'}} \cdot \frac{-dx}{\sqrt{(2lx-x^2)}} \times \left\{1 + \frac{1}{2} \cdot \frac{x}{h'} + \frac{1.3}{2.4} \cdot \frac{x^2}{h'^2} + \&c.\right\}$$

of which the general terms are

$$\sqrt{\frac{l}{4g}} \times \frac{1.3 \times \ldots\ldots 2n. + 1}{2.4\ldots\ldots 2m + 2} \times \frac{-x^m dx}{\sqrt{(hx-x^2)}}$$

$$\text{and } \sqrt{\frac{l^2}{2gh'}} \times \frac{1.3 \times \ldots\ldots 2m+1}{2.4\ldots 2m+2} \times \frac{-x^m dx}{\sqrt{(2lx-x^2)}}.$$

Now

$$\int \frac{-x^m dx}{\sqrt{(hx-x^2)}} = \frac{x^{m-1}}{m}\sqrt{(hx-x^2)} + \frac{h}{2}. \frac{2m-1}{m} \int \frac{-x^{m-1}dx}{\sqrt{(hx-x^2)}}$$

and if we write

H_m, H_{m-1}, &c. for the integrals of

$$\frac{-x^m dx}{\sqrt{(hx-x^2)}}, \frac{-^{m-1}dx}{\sqrt{(hx-x^2)}}, \&c.$$

when $x = h$, we get

$$H_m = \frac{1}{2}h.\frac{2m-1}{m} H$$

$$H_{m-1} = \frac{1}{2}h.\frac{2m-3}{m-1}. H_{m-2}$$

&c. = &c.

$$H_0 = \int \frac{-dx}{\sqrt{(hx-x^2)}} = \cos.^{-1} \frac{2x-h}{h} = \pi$$

$$\therefore H_m = \left(\frac{1}{2}h\right) \pi \times \frac{1.3\ldots(2m-3)\,(2m-1)}{1.2.3\ldots\ldots m}$$

Hence the time of the semi-oscillation is

$$t = \sqrt{\frac{l}{4g}} \times \pi \times \{1 + \left(\frac{1}{2}\right)^2 \frac{h}{2l} + \left(\frac{1.3}{2.4}\right)^2 \frac{h^2}{4l^2} + \&c.\}$$

Similarly it may be shewn that

$$t' = \sqrt{\frac{l^2}{2gh'}}, \times \pi \times \{1 + \left(\frac{1}{2}\right)^2 \frac{2l}{h'} + \left(\frac{1.3}{2.4}\right)^2 \left(\frac{2l}{h'}\right)^2 + \&c.\}$$

$$\text{But } \frac{h}{2l} = \frac{2l}{h'} = m^2$$

$$\therefore t : t' :: \sqrt{\frac{l}{2}} : \sqrt{\frac{l^2 m^2}{2l}} :: 1 : m. \qquad \text{Q. E. D.}$$

To find the velocity V *and periodic time* T *in the circle, not considering the accelerations due to gravity, we have*

$$\text{V} = \sqrt{2g.(h' - 2l)}$$

$$= \sqrt{2g\left(\frac{4l^2}{h} - 2l\right)}$$

$$= 2\sqrt{\frac{gl}{h} \times (2l - h)}$$

$$\text{Hence T} = \frac{2\pi l}{\text{V}} = \frac{\pi\sqrt{hl}}{\sqrt{g.(2l-h)}}$$

Hence moreover since the periodic time in a circle is generally (440.)

$$2\pi\sqrt{\frac{\text{R}}{\text{F}}}$$

R being the radius and F the centripetal force, the centripetal force in this case, or the tension of the string is,

$$\tau = \frac{4g\,(2l - h)}{h}.]$$

572. Let A, B, be the weights of bodies, B being that of the one projected; also let a be the distance of B from the ring before projection, and ρ its distance after it has been in motion during the time t, and make θ = the angle between ρ and a, and let β be the velocity of projection. Then, if u denote the velocity

along ρ and S be the retarding force upon B arising from the inertia, we have

$$S \times B = A \times \frac{du}{dt}$$

But $dt = \frac{d\varrho}{u}$.

$$\therefore S = \frac{A}{B} \cdot \frac{udu}{d\varrho}.$$

Again, the centrifugal force is

$$\varphi = \frac{c^2}{\varrho^3} (499) = \frac{a^2\beta^2}{\rho^3}$$

$\therefore$ the centripetal force is

$$F = \varphi - S = \frac{a^2\beta^2}{\varrho^3} - \frac{A}{B} \cdot \frac{udu}{d\varrho}$$

$$\therefore udu = Fd\varrho = \frac{a^2\beta^2 d\varrho}{\rho^3} - \frac{A}{B} . udu$$

$$\text{and } udu = \frac{Ba^2\beta^2}{A + B} \cdot \frac{d\varrho}{\varrho^3}$$

Hence

$$F = \frac{a^2\beta^2 B}{A+B} \times \frac{1}{\rho^3}$$

$$= \frac{c^2 dp}{p^3 d\varrho} \text{ (see 436)}$$

$$\therefore \frac{B}{A + B} \cdot \frac{d\varrho}{\varrho^3} = \frac{dp}{p^3}$$

$$\text{and } C + \frac{B}{A + B} \cdot \frac{1}{\varrho^2} = \frac{1}{p^2}.$$

But when $\varrho = a$, $p = a$.

$$\therefore C = \frac{1}{a^2} \times \left(1 - \frac{B}{A + B}\right) = \frac{A}{(A+B)a^2}$$

Hence

$$p^2 = \frac{a^2.(A+B).\varrho^2}{Aa^2 + A\varrho^2} \ldots\ldots.(1)$$

the equation to the curve, which is therefore one of *Cotes' Spirals*.

Again, in all curves

$$p = \frac{\varrho^2 d\theta}{\sqrt{(\varrho^2 d\theta^2 + d\varrho^2)}}$$

$$\therefore \frac{\varrho^2 d\theta^2}{\varrho^2 d\theta^2 + d\varrho^2} = \frac{a^2.(A+B)}{Ba^2 + A\varrho^2}$$

Hence

$$d\theta = a\sqrt{\frac{A+B}{A}} \cdot \frac{d\varrho}{\varrho\sqrt{(\varrho^2 - a^2)}}$$

$$\therefore \theta = \sqrt{\frac{A+B}{A}} \times \sec.^{-1} \frac{\varrho}{a} \ldots\ldots (2)$$

which gives the angle θ.

Hence

$$\varrho = a.\sec.\left(\theta\sqrt{\frac{A}{A+B}}\right) \ldots\ldots\ldots\ldots (3)$$

the polar equation to the spiral.

573. Let w be the given weight of the cylinder; then the accelerating force down the plane is

$$F = \frac{w}{w + w.\frac{R^2}{r^2}} \times \sin.\theta.$$

R being the distance of the centre of gyration from the axis of rotation, and θ the inclination of the plane.

$$\therefore F = \frac{r^2}{r^2 + \frac{1}{2}r^2} \sin.\theta = \frac{2}{3}\sin.\theta.$$

Hence, $F = \frac{2}{3}$ force of gravity.

574. Let W be the weight of the cylinder, r the radius of its base, then its inertia is $\frac{1}{2}W \times r^2$, and the accelerating force is

$$F = \frac{Pr^2}{Pr^2 + \frac{1}{2}Wr^2} = \frac{2P}{2P+W}$$

But $t = \sqrt{\frac{2}{g} \cdot \frac{s}{F}} = \sqrt{\frac{2}{g} \cdot \frac{48 \times (2P+W)}{2P}} = 2$ by the question.

$$\therefore\ 12.(2P + W) = Pg$$

$$\therefore\ W = \frac{gP}{12} - 2P = \frac{P}{12} \times (g - 24).$$

575. Supposing m' not equal to m, and putting $cm' = x$, $ac = cb = a$, then since the efficacy of m in opposing the motion of m', is measured by

$$\frac{mx}{\sqrt{(a^2 + x^2)}}$$

the whole moving force is

$$m' - \frac{2mx}{\sqrt{(a^2 + x^2)}}$$

and the mass moved is

$$2m + m'$$

$$\therefore\ F = \left\{m' - \frac{2mx}{\sqrt{(a^2 + x^2)}}\right\} \times \frac{1}{2m + m'}$$

But at the time the velocity is a maximum, the accelerating force $= 0$.

$$\therefore\ \frac{2mx}{\sqrt{a^2 + x^2}} = m'$$

$$\text{And } \therefore\ x = \frac{am'}{\sqrt{(4m^2 - m'^2)}},$$

the distance required.

576. The moving force is

$$P - Q.\frac{r'}{r} - Q'.\frac{r''}{r}$$

and the mass is

$$P + Q\frac{r'^2}{r^2} + Q'.\frac{r''^2}{r^2}$$

$$\therefore\ F = \frac{r^2P - Qrr' - Q'rr''}{r^2P + Qr'^2 + Q'r''^2}$$

Hence

$$t = \sqrt{\frac{2s}{gF}} \text{ is known.}$$

577. Let r be the radius of either globe, M its mass, k, k' the distances of their centres of gravity from the axis of rotation l, l' the distances of their centres of oscillation from the same, when the bodies are unconnected; then since (*Creswell's Translat. of Venturoli*, p. 141), the length of the compound pendulum is

$$L = \frac{Mkl + Mk'l'}{Mk + Mk'} = \frac{kl + k'l'}{k + k'}$$

and by the question

$$k = 3r,\ k' = 5r.$$

Therefore

$$L = \frac{3l + 5l'}{8}.$$

But see *Venturoli*.

$$l = k + \frac{2}{5}\frac{r^2}{k} = 3r + \frac{2r}{15} = \frac{47}{15}r$$

$$l' = k' + \frac{2}{5}.\frac{r^2}{k} = 5r + \frac{2}{25}r = \frac{127}{25}r$$

$$\therefore L = \frac{174}{40}r = \frac{87}{20}r.$$

578. Let a be the length of the rod; then supposing generally $\frac{a}{n}$ the distance of the point of suspension from its extremity, the length of the pendulum is (see *Venturoli*)

$$l = \frac{S}{Mk} = \frac{\dfrac{a^3}{3n^3} + \dfrac{\left(a - \dfrac{a}{n}\right)^3}{3}}{a.\left(\dfrac{a}{2} - \dfrac{a}{n}\right)}$$

$$= \frac{2.a(n^2 - 3n + 3)}{3n(n-2)}.$$

Let $n = \infty$. Then

$$l' = \frac{2}{3}a$$

$$\therefore l' : l :: n.(n-2) : n^2 - 3n + 3.$$

Let $n = 4$ as in the question; then

$$l' : l :: 8 : 7.$$

579. Since the centre of Initial Rotation is distant from that point of impact by the same interval as the centre of oscillation, considering that point as the point of suspension, the distance required is (*Venturoli*)

$$l = \frac{S}{Mk} = \frac{Aa^2 + Bb^2}{(A+B) \times \dfrac{Aa - Bb}{A + B}} = \frac{Aa^2 + Bb^2}{Aa - Bb}$$

A, B being the masses, and a, b the lengths of the arms of the lever.

580. Generally, if P denote the power moving the system, whose weight is w, acting at the distance r from the axis of rotation, then the force which accelerates P is

$$F = \frac{Pr^2}{Pr^2 + W.R^2}$$

R being the distance of the centre of gynation from the axis.

But s denoting the space described in the time t, we have

$$s = gFt$$

$$\text{and } gFt : \theta :: 2\pi r : 360^\circ$$

$$\therefore \theta = 360^\circ \times \frac{gFt}{2\pi r}$$

or the body moves at the rate of $\frac{gFt}{2\pi r}$ revolutions in a second.

Hence, by the question,

$$\frac{gFt}{2\pi r} = 10$$

$$\therefore t = \frac{20\pi r}{gF} = \frac{20\pi r}{g} \cdot \frac{Pr^2 + WR^2}{Pr^2}$$

But $P = 1$, $r = 1$, $W = 100$, $R = 2\sqrt{\frac{1}{2}} = \sqrt{2}$

$$\therefore t = \frac{20\pi}{g} \cdot \frac{1 + 200}{1 \times 1} = \frac{201 \times 20 \times \pi}{g}$$

$$= \frac{4020 \times 3.14159}{32\frac{1}{6}}$$

$$= 6'.32''.6173 \text{ nearly.}$$

581. The moving force is

$$P - W.$$

Hence the accelerating force is

$$F = \frac{P - W}{P + W}$$

That part of P's weight which is sustained is

$$P - \frac{P - W}{P + W} \times P = \frac{2PW}{P + W}$$

And it is evident that the same part of W is sustained. Hence the whole pressure on the axis is

$$\frac{4PW}{P + W}$$

582. Generally the time of an oscillation through an arc, the versed sine of half of which is (h), is (see 571.)

$$t = \pi \sqrt{\frac{l}{g}} \times \{1 + \left(\frac{1}{2}\right)^2 \cdot \frac{h}{2l} + \left(\frac{1.3}{2.4}\right)^2 \cdot \left(\frac{h}{2l}\right)^2 + \&c.\}$$

l being the radius or length of the pendulum.

Now A the arc being small compared with the whole circumference, we have

$$h = \frac{A^2}{2l},$$

and if n be the number of degrees in this arc, we get

$$= \frac{2\pi l n}{360^\circ}$$

$$\therefore h = \frac{4\pi^2 l^2 n^2}{2l.\ 360^2} = \frac{2\pi^2 . l n^2}{360^2}$$

$$= \frac{l n^2}{6565} \text{ nearly.}$$

But for small arcs

$$t = \pi \sqrt{\frac{l}{g}} \times \left(1 + \frac{h}{8l}\right) \text{ nearly,}$$

And the true time is

$$t' = \pi \sqrt{\frac{l}{g}}$$

$\therefore$ the error in time is

$$t - t' = \pi . \sqrt{\frac{l}{g}} \times \frac{h}{8l}$$

$$= t' \times \frac{n^2}{52520} \text{ nearly.}$$

Hence, if the pendulum oscillating through n degrees keeps true time, the error arising from its vibrating through N degrees is

$$t' \times \frac{N^2 - n^2}{52520} \text{ nearly.}$$

In the question, we have

$$n = 2^\circ, \ N = 2^\circ \ 10' = 2\frac{1^\circ}{6} = \frac{13^\circ}{6}.$$

$\therefore$ the error is

$$t' \times \frac{\frac{169}{36} - 4}{52520} = \frac{25t'}{36 \times 52520} = \frac{5t'}{36 \times 10504}.$$

583. The accelerating force down the plane is sin. θ, and the mass + inertia $= w + \frac{w}{2} = \frac{3}{2} w$, θ being the inclination of the plane. Hence the moving force is

$$\frac{3}{2} \sin \theta \times w$$

to counteract which by means P, we must have

$$P = \frac{3}{2}. \sin. \theta \times w$$

$$\therefore \sin \theta = \frac{2P}{3w}$$

which gives the ratio of the height of the plane to its length.

If $P = \frac{w}{10}$ as in the question, then

$$\sin. \theta = \frac{2}{30} = \frac{1}{15}.$$

584. Let a be the length of the rod, then S being the moment of inertia, M the mass, and k the distance of the centre

of gravity from the axis of suspension, the length of the first pendulum (see *Venturoli*) is

$$L = \frac{S}{Mk} = \frac{\frac{a^2}{3} + \frac{1}{2}\frac{a^2}{9}}{\left(1+\frac{1}{2}\right)\frac{4}{9}a} = \frac{7}{12}a$$

and that of the second is

$$l = \frac{\frac{a^2}{3}}{1 \times \frac{a}{2}} = \frac{2}{3}a.$$

But the time of oscillation $\propto$ (length)$^{\frac{1}{2}}$.

$$\therefore T : t :: \sqrt{\frac{7}{12}} : \sqrt{\frac{2}{3}}$$

$$:: \sqrt{7} : 2\sqrt{2}.$$

585. Take any zone contained between two sections indefinitely near to each other, made by planes $\perp$ axis of rotation, and distant from the centre by

$$x \text{ and } x + dx;$$

then the momentum of inertia of this zone is

$$dS = 2\pi y^3 dz = 2\pi y^3 . \frac{rdx}{y}$$

$$= 2\pi r\,(r^2 - x^2)dx$$

$$\therefore S = 2\pi r\left(r^2 x - \frac{x^3}{3}\right)$$

Let $x = r$. Then

$$S = 2\pi r\left(r^3 - \frac{r^3}{3}\right) = \frac{4\pi r^4}{3}$$

$\therefore$ for the whole sphere, we have

$$2S = \frac{8\pi r^4}{3} = 4\pi r^2 \times \frac{2r^2}{3}$$

$$= \text{surface of sphere} \times \frac{2r^2}{3}$$

$\therefore$ the distance of the centre of gyration from the axis of rotation, is $r\sqrt{\frac{2}{3}}$ (*Venturoli.*)

586. Let x be the weight required, λ, k, M; λ', k', M$'$ the distances of the centres of oscillation from the axis of rotation, those of the centres of gravity and the masses of the respective parts d, $l-d$.

Also, let λ'', k'', M$''$, λ''', k''', x be those of the weights added to the lower and upper extremities of the rod; then the length of the pendulum is

$$\mathrm{L} = \frac{\mathrm{M}k\lambda + \mathrm{M}'k'\lambda' + \mathrm{M}''k''\lambda'' + xk'''\lambda'''}{\mathrm{M}k + \mathrm{M}'k' + \mathrm{M}''k'' + xk'''}$$

$$= \frac{\mathrm{M}k\lambda + \mathrm{M}'k'\lambda'}{\mathrm{M}k + \mathrm{M}'k'}$$

by the question. Now

$$\mathrm{M} = \frac{w}{l}d, \ \mathrm{M}' = \frac{w}{l}(l-d), \ \mathrm{M}'' = a$$

$$k = \frac{d}{2}, k' = \frac{l-d}{2}, \ k'' = l-d, k''' = d,$$

$$\lambda = \frac{2}{3}d, \lambda' = \frac{2}{3}(l-d), \lambda'' = l-d, \lambda''' = d.$$

Hence

$$\mathrm{L} = \frac{\frac{wd^3}{3l} + \frac{w}{3l}(l-d)^3 + a(l-d)^2 + xd^2}{\frac{w}{2l}d^2 + \frac{w}{2l}(l-d)^2 + a(l-d) + xd}$$

$$= \frac{2}{3} \times \frac{l^2 - 3ld + 3d^2}{l - 2d}$$

and

$$x = \frac{al(l-d)}{d} \times \frac{l - 3d}{2l^2 - 9ld + 12d^2}.$$

OTHERWISE.

Let L, L$'$ be the lengths of the two pendulums. Then

$$L = \frac{S}{Mk} = \frac{\frac{d^3}{3} + \frac{(l-d)^3}{3}}{l \times \left(\frac{l}{2} - d\right)} = \frac{2}{3} \cdot \frac{l^2 - 3ld + 3d^2}{l - 2d}.$$

Again,

$$L' = \frac{\frac{d^3}{3} + \frac{(l-d)^3}{3} + a(l-d)^2 + xd^2}{(l+a+x)k'}$$

But $k' \times (l + a + x) = l.\left(\frac{l}{2} - d\right) + a\,(l - d) + xd$ by property of the centre of gravity.

$\therefore$ since by the question $L = L'$, we have

$$\frac{2}{3} \cdot \frac{l^2 - 3ld + 3d^2}{l - 2d} = \frac{2}{3} \cdot \frac{l.(l^2 - 3ld + 3d^2) + 3a(l-d)^2 + 3d^2x}{l.(l-2d) + 2a(l-d) + 2dx}$$

and putting $l^2 - 3ld + 3d^2 = P$, and $l - 2d = Q$

$$\frac{P}{Q} = \frac{lP + 3a\,(l-d)^2 + 3xd^2}{lQ + 2a\,(l-d) + 2xd}$$

Hence

$$x = a.\frac{l-d}{d} \cdot \frac{3(l-d)Q - 2P}{2P - 3Qd}$$

$$= \frac{al.(l-d)}{d} \times \frac{l - 3d}{2l^2 - 9ld + 12d^2}$$

as before.

587. Generally

$$L = \frac{S'}{Mk} = \frac{S + Mk^2}{Mk} = \frac{S}{Mk} + k$$

see *Venturoli*,' p. 120, where S denotes the momentum of inertia referred to the axis passing through the centre of gravity, M the mass, and k the distance of the centre of gravity from the axis of rotation.

But $S = Mr^2$

$$\therefore\ L = \frac{Mr^2}{Mk} + k = \frac{r^2 + k^2}{k}$$

$$\therefore\ k = \frac{L \pm \sqrt{(L^2 - 4r^2)}}{2}$$

which indicates two axes of rotation.

Now by the question

$$k = \frac{r}{2}.$$

$\therefore$ L $= \frac{5r}{2}$ and the two points of suspension are given by

$$k = \frac{\frac{5r}{2} \pm \sqrt{\left(\frac{25r^2}{4} - 4r^2\right)}}{2} = \frac{5r \pm 3r}{4}$$

$$= 2r \text{ or } \frac{r}{2}.$$

Hence it also appears that L cannot be less than $2r$.

When, moreover, L $= 2r$,

$k = r$, or the point of suspension is in the circumference.

588. Let x, y, be the co-ordinates of the point required, referred to the tangent at the vertex of the parabola, whose latus-rectum is a; then

$$x^2 = ay.$$

Again ρ being the radius-vector corresponding to x, y, it is evident that the direction of reflection will be along ρ; and since by the question the body must strike the vertex of the parabola, therefore the time through ρ with the velocity acquired before impact continued uniform, must equal the time through y by the force of gravity. Hence

$$\frac{\rho}{\sqrt{2g(b-y)}} = \sqrt{\frac{2y}{g}}$$

b being the length of the axis of the parabola.

$$\therefore \rho = 2\sqrt{y.(b-y)}$$

But $\rho = y + \frac{a}{4}$ by the property of the parabola,

$$\therefore y^2 + \frac{a}{2}y + \frac{a^2}{16} = 4by - 4y^2$$

$$\therefore y^2 + \frac{(a-8b)y}{10} = -\frac{a^2}{80}$$

$$\therefore y = -\frac{a-8b}{20} \pm \sqrt{\left\{\frac{(a-8b)^2}{400} - \frac{a^2}{80}\right\}}$$
$$= \frac{8b-a}{20} \pm \frac{\sqrt{(16b^2 - 4ab - a^2)}}{10}.$$

Hence it is evident that
$$16b^2 - 4ab \text{ must be} = \text{or} > a^2$$
$$\text{or } b = \text{or} > a \times \frac{1+\sqrt{5}}{8}.$$

589. Generally let θ be the inclination of the cycloid to the horizon; then the axis being a, the semi-base is $\frac{1}{2}$. circumference of the generating circle $= \frac{\pi a}{2}$, and the chord of the semi-cycloid is therefore
$$\sqrt{a^2 + a^2 \times \frac{\pi^2}{4}} = \frac{a}{2}\sqrt{4+\pi^2}.$$

Again, if φ denote the inclination of this chord to the horizon, it may easily be shewn that
$$\frac{a}{2}\sqrt{(4+\pi^2)} \,.\, \sin.\varphi = a \sin.\theta$$
$$\text{and } \therefore \sin.\varphi = \frac{2}{\sqrt{(4+\pi^2)}} \times \sin.\theta.$$

Hence the time required is
$$t = \sqrt{\left\{\frac{2}{g \sin.\varphi} \times \frac{a}{2} \times \sqrt{4+\pi^2}\right\}}$$
$$= \sqrt{\frac{a}{2g \sin.\theta}} \times \sqrt{(4+\pi^2)}$$

590. Let W denote the weight of the given inclined plane, θ its required elevation; then since the moving force acting $\perp$ plane is
$$P \times \cos.\theta$$
that parallel to the horizon is
$$P \,.\, \sin.\theta \,.\, \cos.\theta.$$

Hence the accelerating force is

$$F = \frac{P \sin.\theta \cos.\theta}{P+W} = \frac{P}{2(P+W)} \times \sin. 2\theta$$

and since F = max. when the velocity = max. therefore

$$\sin. 2\theta = \max. = 1 = \sin. \frac{\pi}{2}$$

$$\therefore \theta = \frac{\pi}{4} = 45^\circ.$$

591. In any system of bodies P, P′, &c., acted upon by the accelerating forces, F, F′, &c., at the end of the time t, let u, u' &c., be the velocities impressed by these forces, or such as the bodies would have moved with at the end of the time t, had the system been free; also let v, v', &c., be their actual velocities, and φ, φ', &c., the forces which really accelerate them in their general courses. Then since by the connexion of the parts of the system there is occasioned no loss of *vis viva*, we have

$$P.\varphi.v + P'.\varphi'.v' + \&c. = P.Fu + P'F'u' + \&c$$

$$\text{But } \varphi = \frac{dv}{dt}, \varphi' = \frac{dv'}{dt}, \&c.$$

$$F = \frac{du}{dt}, F' = \frac{du'}{dt}, \&c.$$

$$\therefore P.\frac{vdv}{dt} + P'\frac{v'dv'}{dt} + \&c. = P.\frac{udu}{dt} + \frac{P'u'du'}{dt} + \&c.$$

$$\therefore Pv^2 + P'v'^2 + \&c. = Pu^2 + P'u'^2 + \&c.$$

there being no correction since the bodies are supposed to begin to move at the same instant. This expression will serve to resolve every dynamical problem that can be proposed.

In such as relate to systems situated on the surface of the earth, or as are impressed by the force of gravity, we have

$$u^2 = 2gx, u'^2 = 2gx', \&c.$$

x, x' being the spaces described at the end of the time t, the spaces descended through being positive, and those ascended through negative. Hence

$$Pv^2 + P'v'^2 + \&c. = 2g(Px + P'x' + \&c.) \ldots\ldots (1)$$

x, x' &c. being positive or negative according as the body that describes it is descending or ascending.

To apply this formula in the resolution of the problem, we have

$$Pv^2 + Wv'^2 = 2g\,(Px - Wx')$$

$$\text{But } v = v' \cdot \frac{ds}{ds'}$$

s and s' being the spaces described by P and W in the vertical and horizontal directions respectively in the time t. Hence

$$v'^2 = \frac{2g \cdot (Px - Wx')}{W + P\,\dfrac{ds^2}{ds'^2}}.$$

$$\text{But } x' = 0 \text{ and } ds = ds' \times \cos.\,\theta$$

when θ is the inclination of the string to the horizon. Hence

$$v'^2 = \frac{2gPx}{W + P\cos.^2\theta}$$

$$\begin{aligned}\text{But } x &= AB - AW \text{ (see fig. to Prob.)}\\ &= AB \cdot \left(1 - \frac{\sin.\,B}{\sin.\,\theta}\right)\\ &= a\left(1 - \frac{\sin.\,B}{\sin.\,\theta}\right)\end{aligned}$$

$$\therefore\ v'^2 = \frac{2ga\,P \cdot (\sin.\,\theta - \sin.\,B)}{\sin.\,\theta\,(W + P \cdot \cos.^2\theta)} \ldots\ldots\ldots\ldots (a)$$

Let $\theta = \dfrac{\pi}{2}$. Then

$$v'^2 = \frac{2ga\,P \cdot (1 - \sin.\,B)}{W}$$

which gives the velocity of W generated from B to C.

Hence also

$$\begin{aligned}v &= v' \times \frac{ds}{ds'} = \cos.\,\theta \times v'\\ &= \cos.\,\theta \times \sqrt{2ga\,P}\,\sqrt{\frac{\sin.\,\theta - \sin.\,B}{\sin.\,\theta\,(W + P\cos^2\theta)}}\end{aligned}$$

Hence when $\theta = \dfrac{\pi}{2}$, the velocity of P is 0.

This problem may be generalized thus:

Let two bodies P, P′ *connected by a string passing over a fixed pulley at* D, (Fig. 99,) *move along the inclined planes* CA, C′A′; P *ascending and* P′ *descending; then having given the inclinations of the planes to the horizon* β, β'; *the distance of* D *from the horizon, viz.* m; *the altitudes of the planes, viz.* a, a'; *and the lengths of the parts of the string between* P *and* D, P′ *and* D′, *viz.* l, l', *when the motion commences; required the velocities acquired* v, v' *in the time* t, *or when these lengths shall be* λ, λ'.

First we have

$$\lambda + \lambda' = l + l' \dots\dots\dots\dots\dots (2)$$

Moreover by (1)

$$Pv^2 + P'v'^2 = 2g\,(Px - P'x') \dots\dots (3)$$

and s, s' being the spaces described in the time t, viz. IP, I′P′, we have

$$v : v' :: ds : ds'$$

$$\therefore v = v'\,\frac{ds}{ds'}$$

$$\text{and } Pv'^2 \cdot \frac{ds^2}{ds'^2} + P'v'^2 = 2g\,(Px - P'x'.)$$

$$\therefore v'^2 = \frac{2g\,(Px - P'x')}{P' + P\,\dfrac{ds^2}{ds'^2}} \dots\dots\dots\dots\dots (4)$$

Again, it is easily seen from the figure that

$$\left.\begin{array}{l} x = s \times \sin.\beta \\ x' = s' \times \sin.\beta' \end{array}\right\}$$

and that

$$s = \sqrt{l^2 - (m-a)^2 \cos.^2\beta} - \sqrt{\lambda^2 - (m-a)^2 \cos.^2\beta}$$

$$\text{and } s' = \sqrt{l'^2 - (m-a')^2 \cos.^2\beta'} = \sqrt{\lambda'^2 - (m-a')^2 \cos.^2\beta'}$$

$$\therefore \frac{ds}{ds'} = \frac{\lambda d\lambda}{\lambda' d\lambda'} \times \sqrt{\frac{\lambda'^2 - (m-a')^2 \cos.^2\beta'}{\lambda^2 - (m-a)^2 \cos.^2\beta}}$$

But $\lambda + \lambda' = l + l'$

and λ' increases as λ decreases

$$\therefore d\lambda = -d\lambda'$$

Hence

$$\frac{ds}{ds'} = \frac{\lambda}{l+l'-\lambda} \times \sqrt{\frac{(l+l'-\lambda)^2-(m-a')^2\cos.^2\beta'}{\lambda^2-(m-a)^2\cos.^2\beta}}.$$

Also

$$x = \sin.\beta\,.\,\{\sqrt{l^2-(m-a)^2\cos.^2\beta} - \sqrt{\lambda^2-(m-a)^2\cos.^2\beta}\}$$

and $x' = -\sin.\beta'\{\sqrt{l'^2-(m-a')^2\cos.^2\beta'} - \sqrt{(l+l'-\lambda)^2-(m-a')^2\cos.^2\beta'}\}$

Hence substituting in equat. 8 we get

$$v'^2 = \frac{2g}{P' + \frac{P\lambda^2}{(l+l'-\lambda)^2} \times \frac{(l+l'-\lambda)^2(m-a')^2\cos.^2\beta'}{\lambda^2-(m-a)^2\cos.^2\beta}} \times$$

$$\{P\sin.\beta\,(\sqrt{l^2-(m-a)^2\cos^2\beta} - \sqrt{\lambda^2-(m-a)^2\cos.^2\beta}) +$$

$$P'\sin.\beta'(\sqrt{l'^2-(m-a')^2\cos.^2\beta'} - \sqrt{(l+l'-\lambda)^2-(m-a')^2\cos.^2\beta'}\}$$

which gives the velocity of P′; and this being found, that of P is obtained from

$$v = \frac{v'ds}{ds'},$$

$$= v' \times \frac{\lambda}{l+l'-\lambda} \times \sqrt{\frac{(l+l'-\lambda)^2-(m-a')^2\cos.\beta'}{\lambda^2-(m-a)^2\cos.^2\beta}}.$$

From these results a multitude of particular consequences may be deduced.

(1). Let $a = a' = m$. Then

$$v'^2 = \frac{2g}{P'+P} \times \{P(l-\lambda)\sin.\beta - P'(l-\lambda)\sin.\beta'\}$$

$$= \frac{2g\,.(l-\lambda)}{P+P'}\,.\,(P\sin.\beta - P'\sin.\beta')$$

$$\text{and } v^2 = v'^2 \times \frac{\lambda^2}{(l+l'-\lambda)^2} \times \frac{(l+l'-\lambda)^2}{\lambda^2} = v'^2$$

as it ought to be; *being the common velocity of two bodies along two inclined planes of the same altitude.*

(2.) Let $a = a' = m$, $\beta = \frac{\pi}{2}$, $\beta' = 0$.

Then

$$v'^2 = \frac{2gP(l-\lambda)}{P+P'}$$

which gives the common velocity of two bodies when one of them, hanging vertically from the edge of a table, or any other horizontal plane, causes the other to move in a straight line along the table or plane.

In this case if $P = P'$; then

$$v'^2 = 2g \cdot \frac{l-\lambda}{2}$$

or the velocity is such as would be acquired in falling freely through $\frac{l-\lambda}{2}$

(3.) Let $a = a' = m, \beta = \frac{\pi}{2} = \beta'$. Then

$$v'^2 = 2g\,(l-\lambda)\frac{P-P'}{P+P'}$$

which gives the velocities of two bodies hanging freely over a fixed pulley, after having moved through $l-\lambda$.

(4). Let $a = m, a' = 0, \beta = \frac{\pi}{2}, \beta' = 0$.

Then

$$v'^2 = \frac{2gP(l-\lambda)}{P' + \frac{P}{(l+l'-\lambda)^2} \times \{l+l'-\lambda)^2 - m^2 \cos.^2 \beta'\}}$$

which is easily made to coincide with the expression marked (a).

The problem may be still farther generalized, by *supposing* AC, A′C′ (Fig. 100,) *any curve lines in the same plane with the pulley.*

In this case we also have

$$v'^2 = \frac{2g\,(Px - P'x')}{P' + P \cdot \frac{ds^2}{ds'^2}} \ldots\ldots\ldots\ldots\ldots (b)$$

s, s' being the arcs described, and the other symbols retaining their former signification.

Now to resolve this case after a simple manner, let the coordinates at P, P′ of the curves originating in C, C′ be

X, Y; X′, Y′.

Also supposing the motions to commence at the points I, I′, let the co-ordinates at those points be

$$\alpha, \beta; \ \alpha', \beta';$$

and at the points A, A′

$$a, b; \ a', b'.$$

Moreover let the equations of the curves be expressed by

$$Y = f.(X), \text{ and } Y' = f'(X');$$

and make

$$ID = l, I'D = l'$$
$$PD = \lambda, P'D = \lambda'.$$

This being premised, we have

$$l + l' = \lambda + \lambda' \ldots\ldots\ldots\ldots\ldots (5)$$

Also

$$\left.\begin{array}{l} x = Y - \beta \\ x' = \beta' - Y' \end{array}\right\} \ldots\ldots\ldots\ldots\ldots (6)$$

$$\frac{ds^2}{ds'^2} = \frac{dX^2 + dY^2}{dX'^2 + dY'^2}$$

$$= \frac{dX^2 + d(fX)^2}{dX'^2 + d(f'X'^2)} \ldots\ldots (7)$$

Again,

$$\lambda = \sqrt{(m-Y)^2 + (a - X)^2}$$
$$\lambda' = \sqrt{(m-Y')^2 + (a'-X')^2}$$
$$\therefore \lambda = \sqrt{(m-fX)^2 + (a-X)^2}$$
$$\text{and } l + l' - \lambda = \sqrt{(m-f'X')^2 + (a'-X')^2}$$
$$\therefore \sqrt{(m-f'X')^2 + (a'-X')^2} = l + l' - \sqrt{(m-fX)^2 + (a-X)^2} \ldots\ldots (8)$$

Hence X′ may be expressed in terms of X, and ∴ dX' and $d(f'X')$ in terms of X. Substituting these functions of X in (6) and (7), and their resulting values in (5), we shall have v'^2 in terms of X and constants. Moreover v will be found from

$$v = v' \cdot \frac{ds}{ds'}.$$

Ex. 1. Let the curves be two parabolas, C, C′ being their vertexes, and p, p', their principal parameters; then

$$X^2 = pY,\ X'^2 = p'Y'$$

$$\therefore fX = \frac{X^2}{p}, f'X' = \frac{X'^2}{p'}.$$

$$\text{Also } d\,.\,(fX) = \frac{2XdX}{p}, \text{ and } d(f'X') = \frac{2X'dX'}{p'}$$

Hence equation (7) becomes

$$\frac{ds^2}{ds'^2} = \frac{1 + \frac{4X^2}{p^2}}{1 + \frac{4X'^2}{p'^2}} \times \frac{dX^2}{dX'^2}$$

and equation (8) is transformable to

$$\sqrt{\left(m - \frac{X'^2}{p'}\right)^2 + (a' - X')^2} = l + l' - \sqrt{\left(m - \frac{X^2}{p}\right)^2 + (a - X)^2}$$

which being developed gives

$$X'^4 - (2m - p')p'.X'^2 - 2a'p'^2 \times X' + (m^2 + a'^2)p'^2 =$$
$$p'^2 . \left\{l + l' - \sqrt{\left(m - \frac{X^2}{p}\right)^2 + (a - X)^2}\right\}^2$$

a biquadratic equation wanting its second term. This equation being resolved will give X' in terms of X, and therefore $\frac{dX'}{dX}$ in terms of X. Hence $\frac{ds^2}{ds'^2}$, x and x', and therefore v^2 may be expressed in terms of X, which will give the velocity required.

From this example it is perceptible how very difficult it is to investigate the velocity practically of one body which preponderates over another when they both describe the most simple of curves. If their paths be both ellipses, or even circles in the general sense, the difficulty will be still greater, and so on for all curves whose equations are more complex.

This difficulty is much diminished when the path of one of the bodies is a straight line, as in

Ex. 2. Let A′C′ (Fig. 100,) be a straight line whose equation is

$$Y' = AX'$$

A being a constant quantity, and AC any curve whose equation, as before, is expressed by

$$X = fX.$$

In this case equation (8) becomes

$$\sqrt{(m - A.X')^2 + (a' - X')^2} = l + l' - \sqrt{(m - fX)^2 + (a - X)^2}$$
$$= l + l' - \lambda$$

$$\therefore (m - AX')^2 + (a' - X')^2 = (l + l' - \lambda)^2.$$

Hence

$$X'^2 - 2.\frac{mA + a'}{A^2 + 1}.X' = \frac{(l + l' - \lambda)^2 - m^2 - a'^2}{A^2 + 1}$$

$$\text{and } X' = \frac{mA + a'}{A^2 + 1} + \sqrt{\left\{\frac{(mA + a')^2}{(A^2 + 1)^2} + \frac{m^2 + a'^2 - (l + l')^2}{A^2 + 1} + \frac{\lambda^2 - 2(l + l')\lambda}{A^2 + 1}\right\}}$$

and by way of abridgment if we put

$$\left.\begin{array}{l}\dfrac{mA + a'}{A^2 + 1} = B \\[2ex] \text{and } \left(\dfrac{mA + a'}{A^2 + 1}\right)^2 - \dfrac{m^2 + a'^2 - (l + l')^2}{A^2 + 1} = D\end{array}\right\}$$

we get

$$X' = B + \sqrt{\left(D + \frac{\lambda^2 - 2(l + l')\lambda}{A^2 + 1}\right)} \quad \ldots\ldots (9)$$

Hence

$$\frac{dX'}{dX} = \frac{d\lambda}{dX} \times \frac{\lambda - (l + l')}{\sqrt{\left(D + \dfrac{\lambda^2 - 2(l + l')\lambda)}{A^2 + 1}\right.}} \quad \ldots\ldots (10)$$

which is a function of X alone.

Now

$$\frac{ds^2}{ds'^2} = \frac{(dX)^2 + (dfX)^2}{(dX')^2 + A^2(dX')^2}$$

$$= \frac{dX^2}{d\lambda^2}.\left(D + \frac{\lambda^2 - 2(l + l')\lambda}{A^2 + 1}\right). \times \frac{1 + \left(\dfrac{dfX}{dX}\right)^2}{(\lambda - \overline{l + l'})^2(A^2 + 1)} \quad \ldots (11)$$

which is a function of X.

Also

$$x = fX - \beta,$$

$$\text{and } x' = \beta' - AB - A\sqrt{\left(D + \frac{\lambda^2 - 2(l + l')\lambda}{A^2 + 1}\right)}$$

and substituting in equation (b) we get v'^2 in terms of X, which gives the velocity of the body moving along the straight line or inclined plane. The velocity along the curve is

$$v = v' \times \frac{ds}{ds'}$$

which is therefore known.

As a particular case, let CA be the common parabola, its vertex being C and axis parallel to DB. Then the equation to the curve being

$$X^2 = pY$$

we have

$$\lambda = \sqrt{\left(m - \frac{X^2}{p}\right)^2 + (a - X)^2}$$

$$\therefore \frac{d\lambda}{dX} = \frac{-\frac{X}{p}.\left(m - \frac{X^2}{p}\right) - (a - X)}{\lambda},$$

$$\text{and } \frac{(dfX)^2}{(dX)^2} = \frac{X^2}{p^2},$$

and equation (11) becomes

$$\frac{ds^2}{ds'^2} = \frac{\lambda^2}{\left\{\frac{X}{p}\left(m - \frac{X^2}{p}\right) + (a - X)\right\}^2} \times \left(D + \frac{\lambda^2 - 2(l + l')\lambda}{A^2 + 1}\right)$$

$$\times \frac{p^2 + X^2}{p^2(A^2 + 1)(\lambda - \overline{l + l'})^2}$$

which together with x, x' would still give a very complex expression for v'^2.

Let us farther particularize by making $m = b$, or by supposing the pulley fixed in the curve, and consequently b not $< b'$. In this case

$$m - \frac{X^2}{p} = \frac{pb - X^2}{p} = \frac{a^2 - X^2}{p}$$

$$\text{and } \lambda = \frac{a - X}{p} \times \sqrt{a + p^2 + (X + a)^2}$$

$$\therefore \frac{ds^2}{ds'^2} = \frac{(X + a)^2 + p^2}{X^2 + aX + p^2} \times$$

$$\left(D + \frac{\lambda^2 - 2(l + l')\lambda}{A^2 + 1}\right) \times \frac{p^2 + X^2}{p^2(A^2 + 1)(\lambda - \overline{l + l'})^2},$$

which will also produce a very complicated result. These several computations being rather laborious than difficult, we leave to the student, to whom such exercises may be useful.

The question may be still farther simplified by supposing one of the paths a straight line perpendicular to the horizon, and the other any curve whatever.

In this case, $A = \tan \frac{\pi}{2} = \infty$.

and $X' = 0$ and $a' = 0$. Also equation (8) gives

$$fX' = m - (l + l' - \lambda)$$

$$\therefore \frac{d(fX')}{d\lambda} = 1$$

and $dX' = 0$.

Hence equation (7) becomes

$$\frac{ds^2}{ds'^2} = \frac{dX^2 + (dfX)^2}{d\lambda^2} \ldots\ldots\ldots (12)$$

Also

$$x = fX - \beta$$

$$\text{and } x' = \beta' - Y' = \beta' - (\beta' + \lambda' - l')$$

$$= l' - \lambda' = l' - \overline{l + l' - \lambda}$$

$$= \lambda - l.$$

Hence equation (b) gives

$$v'^2 = 2g\, \frac{\{P\overline{fX - \beta} - P'(\lambda - l)\}}{P' + P \times \frac{dX^2 + (d.fX)^2}{d\lambda^2}} \ldots\ldots\ldots (13)$$

$$\text{and } v = \frac{ds}{ds'} \times v'.$$

which give the velocities of P *and* P′ *when* P *is at that point of the curve whose abscissa is* X.

Let the curve which P describes be the common parabola whose equation is

$$X^2 = pY.$$

Then

$$fX = \frac{X^2}{p}$$

$$\text{and } \lambda = \sqrt{\left(m - \frac{X^2}{p}\right)^2 + (a - X)^2}$$

$$\therefore \frac{d.fX}{dX} = \frac{2X}{p}$$

$$\text{and } \frac{d\lambda}{dX} = -\frac{\frac{X}{p}.\left(m - \frac{X^2}{p}\right) + (a - X)}{\lambda}$$

and substituting in (18) we get

$$v'^2 = 2g\,\frac{\left\{P.\left(\frac{X^2}{p} - \beta\right) - P'(\lambda - l)\right\}}{P' + P \times \frac{\lambda^2.(4X^2 + 1)}{\{X.pm - X^2 + p^2(a - X)^2\}^2}}$$

which gives the velocity of P′ in terms of X.

If the curve and pulley coincide, we have

$$m = b,\ pm = pb = a^2 \text{ and } \therefore$$

$$\lambda = \frac{a - X}{p}\sqrt{(p^2 + \overline{a + X}|^2)}$$

and we get

$$v'^2 = \frac{2g}{p}.\frac{P'.(X^2 - a^2) - P'.(\lambda - l)}{P' + \frac{P}{(a - X)^2}\frac{\times \lambda^2.(4X^2 + 1)}{(aX + X^2 + p^2)^2}}.$$

$$= 2gp\,\frac{P.(X^2 - a^2) - P'.(\lambda - l)}{p^2P' + P \times (4X^2 + 1)\frac{p^2 + (a + X)^2}{(aX + X^2 + p^2)^2}}$$

Let $X = a$; then $\lambda = l$ and $v' = 0$, as it ought from the hypothesis.

Again, let $X = a$; then $\lambda = 0$, and we have

$$v'^2 = 2gp.\frac{P(a^2 - a^2) + P'l}{p^2P' + P.(4a^2 + 1)\frac{\times(p^2 + 4a^2)}{(p^2 + 2a^2)^2}}.$$

which gives the velocity of P′ when P arrives at the pulley.

592. Let $AD = x$, $AP = z$, $BC = AC = a$; then resolving the vertical tendency of P, *viz.*

$$\frac{a + x}{a}P$$

into directions tangential and normal to the curve, and then again the normal part into vertical and horizontal components, we get the pressure sustained by the curve in a vertical direction expressed by

$$p = \frac{a + x}{a}P \times \frac{dx^2}{dz^2}.$$

But since by the question

$$z^2 = 2ax + x^2$$

$$\therefore \frac{dx^2}{dz^2} = \frac{2ax + x^2}{(a+x)^2}$$

$$\text{and } p = \frac{a+x}{a} . \text{P} \frac{2ax + x^2}{(a+x)^2} = \frac{x}{a} . \frac{2a+x}{a+x} \text{P}.$$

Hence that part of P which remains unsupported by the re action of the curve is

$$\text{P}' = \text{P} . \frac{a+x}{a} \left(1 - \frac{2ax + x^2}{(a+x)^2}\right) = \text{P} . \frac{a}{a+x}$$

$$\text{or } \text{P}' . (a+x) = \text{P}a = \text{W}a,$$

that is P′ is supported by W. Hence the weight P is supported in all positions.

593. Let a be the height of the given cone, r the radius of its base; then since the moving force is P, and the momentum of inertia is (see *Venturoli*)

$$\text{S} = \frac{1}{10} \pi r^4 a = \frac{3}{10} \text{M}r^2$$

M being its whole mass; therefore the accelerating force is

$$\text{F} = \frac{\text{P}r^2}{\frac{3}{10} \text{M}r^2 + \text{P}r^2} = \frac{10\text{P}}{3\text{M} + 10\text{P}}$$

$$\text{and } \therefore v = g\text{F}t = \frac{10g\,\text{P}t}{3\text{M} + 10\text{P}}$$

the velocity required.

594. Let R, r, be the radii of the wheel and axle respectively; then the moving force is

$$p - \frac{qr}{\text{R}},$$

and the mass moved including the inertia

$$p + q \frac{r^2}{\text{R}^2} + \frac{\text{M}}{2} . \frac{\text{R}^2}{\text{R}^2} + \frac{mr^2}{2\text{R}^2}$$

M and m being the quantities of matter in the rims of the wheel and axle, $\therefore$ the accelerating force is

$$F = \frac{p - q \cdot \frac{r}{R}}{p + q\frac{r^2}{R^2} + \frac{MR^2}{2R^2} + \frac{mr^2}{R^2}}.$$

$$= \frac{2R^2 - qrR}{2R^2p + 2qr^2 + MR^2 + mr^2}$$

and $v = gFt =$ &c.

595. Let x be that portion of the chain which has descended in the time t; then the moving force in the direction of gravity is

$$x + \frac{n}{m}(l - x) = \frac{nl + \overline{m-n} \,.\, x}{m}$$

$$\therefore F = \frac{nl + \overline{m-n} \,.\, x}{ml} = \frac{d^2x}{dt^2}$$

$$\therefore \frac{2d^2xdx}{dt^2} = \frac{2nldx + 2(m-n)xdx}{ml}$$

and integrating

$$\frac{dx^2}{dt^2} = \frac{2nlx + \overline{m-n}x^2}{ml}$$

$$\therefore dt = \sqrt{ml} \times \frac{dx}{\sqrt{2nlx + \overline{m-n} \,.\, x^2}}$$

$$= \sqrt{\frac{ml}{m-n}} \cdot \frac{dx}{\sqrt{\left(\frac{2nl}{m-n} \,.\, x + x^2\right)}}$$

$$\therefore t = \sqrt{\frac{ml}{m-n}} \,.\, l \,.\, (x + \frac{nl}{m-n} + \sqrt{\left(\frac{2nb}{m-n} \,.\, x + x^2\right)} + C.$$

Let $x = 0$. Then

$$C = -\sqrt{\frac{ml}{m-n}} \,.\, l\, \frac{nl}{m-n}. \quad \text{Hence}$$

$$t = \sqrt{\frac{ml}{m-n}} \times l \,.\, \frac{x + \frac{nl}{m-n} + \sqrt{\left(\frac{2nl}{m-n}\, x + x^2\right)}}{\frac{nl}{m-n}}$$

Let $x = l$; then

$$t = \sqrt{\frac{ml}{m-n}} \cdot l \cdot \frac{m + \sqrt{(m^2 - n^2}}{n}.$$

596. By *Venturoli*, p. 140, we have

$$L = \frac{S'}{Mk} = \frac{S + Mk^2}{Mk}$$

where k is GS the distance of the centre of gravity from the axis of rotation, M the mass of the body, S the moment of inertia referred to an axis passing through the centre of gravity, and L the distance of the centre of oscillation from the axis.

Hence since S and M are the same for all values of k, L is constant for the same values of k, which proves the first part of the proposition.

Again, when the axis of rotation is any where in the circumference of the circle whose radius is GO, we have

$$\begin{aligned} S'' &= S' - M\,(SG^2 - GO^2) \\ &= S' - M \times SO \times SG + M \times SO \,.\, GO \end{aligned}$$

S'' being the momentum of inertia referred to this new axis.

$$\text{But } S' = M \,.\, SO \,.\, SG$$

$$\therefore\ S'' = M \,.\, SO \,.\, GO$$

$$\text{and } \therefore L' = \frac{S''}{Mk'} = \frac{M \,.\, SO \,.\, GO}{M \,.\, GO} = SO = L;$$

or the pendulum will oscillate in the same time as before.

597. Let r be the radius of the circle, m the distance of either body from the axis of rotation; then

$$L = \frac{S}{Mk} = \frac{m^2}{\frac{m^2}{2r}} = 2r.$$

598. Let a be the given length of the lever, M the weight required,

Then (see *Venturoli*)

$$L = \frac{S}{Mk} = \frac{\int x^2 dM'}{M' . \frac{\int x dM'}{M'}}$$

$$= \frac{\int x^2 dM'}{\int x dM'}.$$

But $M : M' :: a^n : x^n$.

$$\therefore M' = \frac{M}{a^n} . x^n$$

$$\therefore dM' = \frac{nM}{a^n} . x^{n-1} dx.$$

$$\therefore L = \frac{n+1}{n+2} . \frac{x^{n+2}}{x^{n+1}} = \frac{n+1}{n+2} x.$$

Let $x = a$; then

$$L = \frac{n+1}{n+2} . a$$

$$\text{and } T = \pi \sqrt{\frac{L}{g}} = \pi \sqrt{\frac{n+1}{n+2} . \frac{a}{g}} = \frac{t}{m}$$

by the question.

Hence

$$(n+1)\, a = \frac{t^2}{m^2 \pi^2} . (n+2)\, g$$

$$\therefore n = \frac{2gt^2 - am^2\pi^2}{am^2\pi^2 - gt^2}$$

the value of the index.

Again,

$$P \times a = M \times \frac{a}{n+1}$$

$$\therefore M = (n+1)\, P$$

$$= P . \frac{gt^2}{am^2\pi^2 - gt^2}$$

the weight required.

599. Let a be the length of the lever, x the distance of the

fulcrum from the end to which P is attached; then the moving force of P is

$$P - Q \cdot \frac{a-x}{x}$$

and the mass moved reckoning the inertia is

$$P + Q \cdot \frac{(a-x)^2}{x^2}.$$

Hence the accelerating force is

$$F = \frac{Px^2 - Q(a-x) \cdot x}{Px^2 + Q(a-x)^2}.$$

Again, the moving force which generates P's velocity is

$$\frac{Px^2 - Q(a-x)x}{Px^2 + Q(a-x)^2} \times P,$$

and that part of P which is sustained, or the tension of the string is

$$P\left(1 - \frac{Px^2 - Q\overline{a-x} \cdot x}{Px^2 + Q(a-x)^2}\right) = \frac{PQa(a-x)}{Px^2 + Q(a-x)^2} =$$

maximum by the question. Hence

$$\frac{Px^2 + Q(a-x)^2}{a-x} = \text{minimum},$$

$$\therefore \frac{2xP - 2(a-x)Q}{a-x} + \frac{Px^2 + Q(a-x)^2}{(a-x)^2} = 0$$

which gives

$$x^2 - 2ax = -\frac{a^2Q}{P+Q}$$

$$\text{and } x = a\left(1 \pm \sqrt{\frac{P}{P+Q}}\right)$$

which will give the position required.

600. Generally let two bodies P, P′ connected by a string passing over a fixed pulley move by the action of gravity along two given curves, as in 590. Then adopting the notation of that article, the moving force upon P′ along the curve which causes it to move with the velocity $\frac{ds'}{dt}$ is evidently

$$P' \times \left(g\frac{dx'}{ds'} - \frac{d^2s'}{dt^2}\right).$$

Hence the tension of the string, or the moving force along the string is

$$\left.\begin{aligned} \sigma &= P' \times \left(\frac{gdx'}{ds'} - \frac{d^2s'}{dt^2}\right)\frac{ds'}{d\lambda'} \\ \text{or } &= P \times \left(\frac{gdx}{ds} + \frac{d^2s}{dt^2}\right)\frac{ds}{d\lambda} \end{aligned}\right\} \dots\dots\dots(1)$$

Hence also the pressure on the pulley being the resultant of these equal tensions, is (see Fig. 100)

$$\left.\begin{aligned} \varpi &= 2P \times \left(\frac{gdx}{ds} + \frac{d^2s}{dt^2}\right)\frac{ds}{d\lambda}.\cos.\frac{PSP'}{2} \\ \text{or } &= 2P' \left(\frac{gdx'}{ds'} - \frac{d^2s}{dt^2}\right)\frac{ds'}{d\lambda'}.\cos.\frac{PSP'}{2} \end{aligned}\right\} \dots\dots(2)$$

The equations (1) will give the general result (a) of art. 590.

Let us apply them in the investigation of the tension and pressure for different systems. *As the simplest case, suppose both* P *and* P′ *to descend or ascend vertically.*

In this case

$$ds' = d\lambda' = dx' \text{ and } \cos.\frac{PSP'}{2} = \cos. 0 = 1.$$

$$\therefore \sigma = P'.\left(g - \frac{d^2s'}{dt^2}\right)$$

$$\text{and } \varpi = 2P'\left(g - \frac{d^2s'}{dt^2}\right)$$

$$\text{But } \frac{d^2s'}{dt^2} = \phi' = g\,\frac{P'-P}{P+P'}$$

$$\left.\begin{aligned} \therefore \sigma &= \frac{2gPP'}{P+P'} \\ \text{and } \varpi &= \frac{4gPP'}{P+P'} \end{aligned}\right\}$$

Let P = P′. Then

$$\left.\begin{aligned} \sigma &= gP \\ \text{and } \varpi &= 2gP \end{aligned}\right\}$$

which is known to be true from statics.

Again, let P′ descend vertically, draw P up an inclined plane whose equation is

$$Y = AX.$$

In this case, which is that of the problem to be resolved, we have also

$$ds' = d\lambda' = dx',$$

and therefore

$$\left.\begin{aligned} \sigma &= P'(g-\phi') \\ \text{and } \varpi &= 2P'.(g-\phi')\cos.\frac{PSP'}{2} \end{aligned}\right\}$$

But $\phi' = -\dfrac{v'dv'}{ds'}$ and by (13) art. 590, we easily get

$$v'^2 = 2g\,\frac{P(AX-\beta)+P'(\lambda-l)}{P'+P.\dfrac{(A^2+1)\lambda^2}{\{(A^2+1)X-Am+a\}^2}} \dots\dots\dots (2)$$

since $\lambda = \sqrt{(m-Y)^2+(a-X)^2}$.

Also $\cos.PSP' = \dfrac{m-AX}{\lambda} = 2\cos.^2\dfrac{PSP'}{2} - 1$;

$$\therefore \cos.^2\frac{PSP'}{2} = \frac{1}{2}.\frac{m-AX+\lambda}{\lambda}$$

whence in making the substitutions and necessary reductions we shall obtain

$$\sigma \text{ and } \varpi.$$

If the pulley be at the top of the inclined plane, the calculation becomes much less tedious. For in that case we have

$$d\lambda' = -d\lambda = ds' = -ds$$

Also $m = b = Aa$, and $\therefore$

$$\lambda = (a-X).\sqrt{(A^2+1)}$$

$$\therefore \{(A^2+1)X - a(A+1)\}^2 = (a-X)^2.(A^2+1)^2$$
$$= (A^2+1)\lambda^2$$

$$\therefore v'^2 = \frac{2g}{P+P'}\{-(AX-\beta)P+P'.(\overline{a-X}.\sqrt{A^2+1}-l)\}$$

which gives the velocity at any given point of the descent.

Moreover

$$\frac{v'dv'}{dX} = \frac{-g}{P+P'}(PA + P'\sqrt{A^2+1})$$

$$\text{and } \frac{d\lambda}{dX} = \frac{ds'}{dx} = \sqrt{A^2+1}$$

$$\therefore \frac{v'dv'}{ds'} = \frac{g}{P+P'} \cdot \frac{PA+P'\sqrt{A^2+1}}{\sqrt{A^2+1}}.$$

Also

$$\cos.^2 \frac{PSP'}{2} = \frac{1}{2\lambda} \cdot \{(a-X)A+\lambda\}$$

$$= \frac{1}{2\sqrt{A^2+1}} \times (A+\sqrt{A^2+1}).$$

Hence

$$\sigma = \frac{PP'g}{(P+P')(A^2+1)^{\frac{1}{2}}} \times (\sqrt{1+A^2}+A)$$

the actual tension,

$$\text{and } \varpi \ \frac{\sqrt{2}\,.\,PP'g}{(P+P')(A^2+1)^{\frac{3}{4}}} \cdot (\sqrt{1+A^2}+A)^{\frac{3}{2}}$$

the pressure on the pulley.

If θ be the inclination of the plane to the horizon, we have $A = \tan.\theta$, and $\sqrt{1+A^2} = \sec.\theta$.

Hence

$$\sigma = \frac{gPP'}{(P+P')\sec.\theta} \cdot (\sec.\theta + \tan.\theta)$$

$$\left.\begin{aligned} \text{or } \sigma &= \frac{gPP'}{P+P'} \cdot (1+\sin.\theta) \\ \text{and } \varpi &= \frac{gPP'\sqrt{2}}{P+P'} \times (1+\sin.\theta)^{\frac{3}{2}} \end{aligned}\right\}$$

which somewhat simplifies the expressions.

Let $\theta = \frac{\pi}{2}$ or let both bodies move vertically; then

$$\left.\begin{aligned} \sigma &= \frac{2gPP'}{P+P'} \\ \varpi &= \frac{4gPP'}{P+P'} \end{aligned}\right\}$$

as before.

If $\theta = 0$. Then we get

$$\left.\begin{aligned} \sigma &= \frac{gPP'}{P+P'} \\ \text{and } \varpi &= \frac{gPP'\sqrt{2}}{P+P'} \end{aligned}\right\} \ldots\ldots\ldots (a)$$

for the tension and pressure when the body P is drawn along an horizontal table by P′ descending from its edge.

Again, if P′ draw P along an horizontal plane situated below the pulley, then we have

$$Y = 0, A = 0 \text{ and } \beta = 0$$

and equation (2) becomes

$$v'^2 = \frac{2gP'(\lambda - l)}{P' + P.\dfrac{\lambda^2}{(X-a)^2}}$$

Also

$$\lambda = \sqrt{m^2 + (a - X)^2}$$

$$\text{and } \cos.^2 \frac{PSP'}{2} = \frac{1}{2} \cdot \frac{m+\lambda}{\lambda}$$

$$\text{Hence } (X - a)^2 = \lambda^2 - m^2$$

$$\therefore v'^2 = 2g \frac{P.'(\lambda - l).(\lambda^2 - m^2)}{(P'+P)\lambda^2 - P'm^2}$$

Hence

$$-\frac{v'dv'}{gd\lambda} = \frac{1}{\{(P+P')\lambda^2 - P'm^2\}^2} \times$$

$$\{P'(P+P')\lambda^4 + m^2(PP' - 2P'^2)\lambda^2 - 2PP'lm^2\lambda + P'^2m^4\}$$

$$\text{Also } ds' = -d\lambda.$$

Hence

$$\sigma = P'g \times \frac{(P^2 + PP')\lambda^4 - 4m^2PP'\lambda^2 + 2PP'lm^2\lambda}{\{(P+P')\lambda^2 - P'm^2\}^2}$$

$$\text{and } \varpi = \sqrt{2} \times \sqrt{\frac{m+\lambda}{\lambda}} \times \sigma$$

which give the tension of the string and pressure on the pulley in this case.

Let $m = 0$. Then

$$\left.\begin{aligned}\sigma &= \frac{gPP'}{P + P'} \\ \text{and } \varpi &= \frac{gPP'\sqrt{2}}{P + P'}\end{aligned}\right\}$$

which confirm the results marked (e).

601. Let r be the radius of the given circle, t the given time, x the altitude of the required diameter; then by the question

$$2r = \frac{g}{2} \cdot Ft^2 = \frac{g}{2} \cdot \frac{x}{2r} t^2$$

$$\text{and } x = \frac{8r^2}{gt^2}$$

which determines the position of the required diameter.

602. Let $2b$ be the length of the string to which W is attached, and x the space descended through in the time t; then the length of the pendulum being $2b - x$, the time of one oscillation is

$$\pi \sqrt{\frac{2b - x}{g}}$$

and the number of oscillations in a second is

$$\frac{1}{\pi} \sqrt{\frac{g}{2b - x}}.$$

Hence N being the number of oscillations swung in the interval t, we have

$$dN = \frac{dt}{\pi} \cdot \sqrt{\frac{g}{2b - x}}$$

$$\text{But } dt = \frac{dx}{v} = \frac{dx}{\sqrt{2g \cdot Fx}}$$

F being the accelerating force on P

$$\text{and } \therefore = \frac{P - W}{P + W}.$$

Hence

$$dN = \frac{1}{\pi\sqrt{2F}} \times \frac{dx}{\sqrt{(2bx - x^2)}}$$

$$\text{and } N = \frac{1}{b\pi\sqrt{2F}} \times \text{vers.}^{-1}x.$$

Let $x = 2b$; then $\text{vers.}^{-1}\, 2b = 180° = b\pi$.

$$\text{and } N = \frac{1}{\sqrt{2F}} = \sqrt{\frac{P+W}{2(P-W)}}$$

the number of vibrations required.

It is evident that N is independent of the length of the string.

Let $P = \infty$. Then

$N = \frac{1}{\sqrt{2}}$, or the number of oscillations performed in the time through $2b$ is $\frac{1}{\sqrt{2}}$, however great may be the power.

603. Let x denote the ratio required; then W and w being the weights of the wheel and axle R, r, their radii, we have

$$F = \frac{pRr - qr^2}{pR^2 + qr^2 + \frac{wr^2}{2} + \frac{WR^2}{2}}$$

$$= \frac{p - q\frac{r}{R}}{p\cdot\frac{R}{r} + q\cdot\frac{r}{R} + \frac{w}{2}\cdot\frac{r}{R} + \frac{W}{2}\cdot\frac{R}{r}}$$

$$= \frac{p - qx}{\frac{p}{x} + qx + \frac{w}{2}\cdot x + \frac{W}{2x}}$$

$$= \frac{2px - 2qx^2}{2p + 2qx^2 + wx^2 + W}$$

$$= \frac{2px - 2qx^2}{2p + W + (2q + W)x^2}.$$

Now s being the given space described in the time t, we get

$$s = \frac{g}{2}Ft^2$$

$$\therefore t^2 \propto \frac{1}{F} \propto \frac{2p+W+\overline{2q+w}.x^2}{px-qx^2} = a\ minimum = M.$$

Hence putting $\frac{dM}{dx} = 0$

we have

$$2(2q+w)x(px-qx^2) = (p-2qx)(2p+W+\overline{2q+w}x^2)$$

$$\therefore (2q+w)x^2 + 2q(2p+W)x = (2p+W)p$$

$$\therefore x^2 + 2q.\frac{2p+W}{2q+w}.x = \frac{2p+W}{2q+w}p$$

$$\therefore x = -q.\frac{2p+W}{2q+w} + \frac{\sqrt{q^2.(2p+W)^2 + p(2p+W)(2q+w)}}{2q+w}$$

the ratio required.

604. Let a be the height of the cone, r the radius of its base; then the distance of the centre of oscillation from the axis of rotation is (*Venturoli,*)

$$L = \frac{S}{Mk} = \frac{\int x^2 dM}{M\int\frac{xdM}{M}}$$

$$= \frac{\int x^2 dM}{\int xdM}$$

x being the distance of the molecule dM from that axis.

But $dM = 2\pi ydz$

$$= \frac{2\pi xr}{a} \times \frac{dx\sqrt{r^2+a^2}}{a}$$

$$= \frac{2\pi r.\sqrt{r^2+a^2}.xdx}{a^2}$$

$$\therefore L = \frac{\int x^2.xdx}{\int x.xdx} = \frac{3}{4}x.$$

Let $x = a$. Then

$$L = \frac{3}{4}a.$$

The centre of oscillation of a *conical surface* cannot possibly be in the base.

605. Generally let R, R′, &c., be the radii of the wheels, r, r', &c., those of the axles; then x denoting the weight to be raised each time, and P the power, the moving force of P is

$$P - x \times \frac{rr' \ldots\ldots}{RR' \ldots.}$$

and the mass moved reckoning the inertia is

$$P + \frac{x + wd^2 + w'd'^2R^2 + w''d''^2R^2R'^2 + \&c.}{R^2R'^2R''^2 \&c.}$$

wherein w, w', w'' &c. are the weights of the several pairs of wheels and axles, and d, d', d'' &c., the distances of their centres of gyration from the respective axes of rotation. Hence the accelerating force is

$$F = \frac{P - x.\dfrac{rr' \ldots.}{R.R' \ldots}}{P + x + wd^2 + w'd'^2R^2 + \&c.} \times (RR'R'' \ldots.)^2$$

$$= \frac{P.RR' \ldots.. - x.rr' \ldots..}{P.R^2R'^2 x + wd^2 + wd'^2R^2 + \&c.} \times RR'R'' \&c.$$

and the force which accelerates the ascent of x is ∴

$$F' = \frac{F}{RR'R'' \&c.} = \frac{PM - xm}{PM^2 + x + wd^2 + w'd'^2R^2 + \&c.}$$

where

$$M = RR' \&c, \text{ and } m = rr' \&c.$$

Now n being the number of ascents,

Q the whole quantity raised through the space s in the whole time T, we have

$$nx = Q \text{ and } \therefore n = \frac{Q}{x}.$$

Also

$$t^2 = \frac{2}{gF'} s$$

$$\therefore T^2 = n^2t^2 = \frac{Q^2}{x^2} \cdot \frac{2s}{gF'}$$

$$\propto \frac{1}{x^2F'} = \text{max. by question.}$$

$$\therefore \frac{PMx^2 - x^3m}{PM^2 + x + D} = \text{min.}$$

D being substituted for $wd^2 + w'd'^2R^2 + \&c.$

$$\therefore \frac{2PMx - 3mx^2}{PM^2+D+x} = \frac{PMx^2 - x^3m}{(PM^2+D+x)^2}$$

and we finally obtain

$$x^2 + \frac{3mPM^2+3mD-PM}{2m} \cdot x = \frac{2PM(PM^2+D)}{2m}$$

which being resolved will give x, and therefore the quantity required, when for M, m and D we substitute their values, *viz.*,

$$RR', rr', wd^2 + w'd'^2$$

606. Let either weight be denoted by P and the additional one by p, and put $P + p = Q$, also let l be the length of the whole string, a, b the lengths of the parts of it to which P, p are attached previously to the motion, and suppose the weight Q to have descended x feet. Then since the weight of the strings may be represented by their lengths the moving force is

$$Q - P + b + x - (a - x) = Q - P + b - a + 2x$$

and the mass moved is $P + Q + l$

$\therefore$ the accelerating force is

$$F = \frac{Q-P+b-a+2x}{P+Q+b-a} = \frac{M+2x}{N} \text{ by hypothesis}$$

Hence

$vdv = Fdx$ gives]

$$v^2 = \frac{2gM}{N}x + \frac{2g}{N}x^2$$

$$\text{and } v = \sqrt{\frac{2g}{N}} \times \sqrt{(Mx + x^2)}$$

$$= \sqrt{\frac{2g}{P+Q+b+a}} \times \sqrt{(Q-P+b-a)x + x^2}$$

Let $b = a = 0$. Then the velocity resulting from the actions of P, Q alone is

$$v' = \sqrt{\frac{2g}{P+Q}} \sqrt{(Q-P)x + x^2}$$

and therefore that which is due to the weight of the string is

$v - v'$, which being put $= 0$

according to the question gives

$$\frac{(Q-P+b-a).x+x^2}{P+Q+b+a} = \frac{Q-Px+x^2}{P+Q}$$

$$\therefore \frac{Q-P+b-a}{Q+P+b+a} - \frac{Q-P}{Q+P} = x\left(\frac{1}{P+Q} - \frac{1}{P+Q+b+a}\right.$$

$$\therefore x = \frac{2(Pb-Qa)}{a+b}$$

which gives the part of the descent required.

607. The distance required (d) is that of the centre of oscillation from the axis of suspension, that is

$$d = \frac{S}{Mk} = \frac{\int x^2 dM}{\int x dM}$$

M being the mass (see *Venturoli.*)

$$\text{But } M = x \times x \tan. \frac{\theta}{2},$$

θ being the angle at the vertex.

$$\therefore dM = 2x dx. \tan. \frac{\theta}{2}; \text{ and}$$

$$d = \frac{3x^4}{4x^3} = \frac{3}{4}x.$$

Let $x = a$, the altitude of the triangle; then

$$d = \frac{3}{4}a.$$

608. Let r be the radius of the base of the paraboloid, W its weight, and l the length of the chain, (whose weight is represented by its length), a that part of it which is unwound at the commencement of the motion, and $a + x$ that part unwound after the body has moved t seconds; then since the accelerating force is

$$F = \frac{(a+x)r^2}{(a+x)r^2 + W\frac{r^2}{3}}$$

$$= \frac{a+x}{a+x+\frac{W}{3}}$$

$$\therefore v^2 = 2\int F dx$$

$$= 2gx - \frac{2}{3} g W l.(a + \frac{W}{3} + x) + C$$

Let $x = 0$. Then $v = 0$

and $C = \frac{2}{3} g W . l(a + \frac{W}{3})$.

Hence

$$v^2 = 2gx + \frac{2}{3} g W \, .l. \frac{a + \frac{W}{3}}{a + \frac{W}{3} + x}$$

But

$$dt = \frac{dx}{v}$$

$$= \frac{1}{\sqrt{2g}} \cdot \frac{dx}{\sqrt{\frac{W}{3} l.(a + \frac{W}{3}) + x - \frac{W}{3} l.(a + \frac{W}{3} + x)}}$$

which being integrated between $t = 0$ and $t = t''$, will give t'' in terms of x, and therefore x in terms of t'', the value required.

609. Let $r_1, r_2 \ldots\ldots r_p$ denote the ratios of the number of teeth in the pairs of wheels and pinions, and $N_1, N_2 \ldots\ldots N_p$ their respective numbers of revolutions in any time t; then since

$$N_1 = r_1 N$$

$$N_2 = r_2 N_1 = r_1 r_2 N$$

$$\&c. = \&c.$$

$$\therefore N_n = r_1 r_2 \ldots r_n \times m.$$

610. Let W be the weight of the given cylinder, r the

radius of its base, θ the required inclination of the plane to the horizon, and l its length, then the moving force is

$$\mathrm{W}\sin.\theta$$

and the mass is

$$\frac{1}{2}\mathrm{W}+\mathrm{W}$$

$$\therefore \mathrm{F}=\frac{2}{3}\sin.\theta$$

$$\therefore l=\frac{g}{2}\mathrm{F}t^2=\frac{g}{3}\sin.\theta\times t^2$$

$$\therefore \sin.\theta=\frac{3l}{gt^2}$$

which determines the inclination required.

611. Let r be the radius of the circle; then D being the distance of the required centre from the axis, we have

$$\mathrm{D}=\sqrt{\frac{\int x^2 d\mathrm{M}}{\mathrm{M}}}$$

M being the mass, and x the distance of the particle $d\mathrm{M}$ from the axis of gyration. But

$$d\mathrm{M}=2ydx=2dx\sqrt{r^2-x^2}$$

$$\therefore \mathrm{D}^2=\frac{\int 2x^2dx\sqrt{r^2-x^2}}{\mathrm{M}}$$

Let $\mathrm{P}=x(r^2-x^2)^{\frac{3}{2}}$. Then

$$d\mathrm{P}=r^2dx\sqrt{r^2-x^2}-4x^2dx\sqrt{r^2-x^2}$$

$$\therefore \int x^2dx\sqrt{r^2-x^2}=\frac{r^2}{4}\mathrm{M}-x(r^2-x^2)^{\frac{3}{2}}$$

$$\text{and } \mathrm{D}^2=\frac{\frac{r^2}{2}\mathrm{M}-2x(r^2-x^2)^{\frac{3}{2}}}{\mathrm{M}}$$

Let $x=r$. Then

$$\mathrm{D}=\sqrt{\frac{r^2}{2}\times\frac{\mathrm{M}}{\mathrm{M}}}=\frac{r}{\sqrt{2}}.$$

612. Generally if S denote the moment of inertia, M the

mass of the oscillating body, k the distance between the centre of gravity and suspension; then the length of the pendulum is

$$L = \frac{S}{Mk}$$

But in this case

$$k = r - \frac{rc}{l}$$

r being the radius of the circle, and c the chord, and l the length of the arc.

Also

$$S = l \times 2r^2\left(1 - \frac{c}{l}\right)$$

see *Whewell's Dynamics*, p. 238.

$$\therefore L = \frac{2r^2(l - c)}{r(l - c)} = 2r.$$

Hence the time of an oscillation

$$t = \pi\sqrt{\frac{L}{g}} = \pi\sqrt{\frac{2r}{g}}$$

is independent of the length of the arc.

613. Generally for n pulleys, let Q_1 Q_2 ..... be their weights; then, the weight sustained is

$$\frac{W}{2^n - 1}$$

and the moving force is

$$P' - \frac{W}{2^n - 1}$$

where $P' = P + Q_1 + Q_2 + \dots Q_n$

Now the inertia of $W = \dfrac{W}{(2^n - 1)^2}$

$$\text{of } Q_1 = \frac{Q_1}{2(2^n - 1)^2}$$

$$\text{of } Q_2 = \frac{(2^2 - 1)^2 . Q_2}{2(2^n - 1)^2}$$

$$\text{of } Q_3 = \frac{(2^3 - 1)^2}{2.(2^n - 1)}$$

&c. = &c.

Hence, supposing $Q_1 = Q_2 =$ &c., the inertia arising from the rotation of the pulleys, is

$$\frac{Q_1}{2(2^n-1)^2} \times 1 + 3^2 + 7^2 + 15 \times \text{\&c.})$$

$$= \frac{Q_1}{2(2^n-1)^2} \times \frac{2^{2n+2} - 3 \times 2^{n+2} + 3n + 8}{3}$$

Also the inertia from the descent of Q_2, Q_3, &c. is

$$\frac{Q_1}{(2^n-1)^2} \times \frac{2^{2n} - 3.\ 2^{n+1} + 3n + 5}{3}$$

$\therefore$ the whole inertia is

$$P' + \frac{W}{(2^n-1)^2} + \frac{Q}{6(2^n-1)^2} \times (2^{2n+2} + 2^{2n+1} - 6 \times 2^{n+2} + 9n$$

$$+ 18) = \frac{M}{6(2^n-1)^2} \text{ by supposition.}$$

Hence the accelerating force of P is

$$F = \frac{6P'(2^n-1) - 6W(2^n-1)}{M},$$

and that on W is

$$F' = \frac{6P'(2^n-1) - 6W}{M}.$$

Let $n = 2$. Then it is found that the inertia $= P + Q \times \frac{5}{3} + \frac{W}{9}$; that the moving force is

$$P + Q - \frac{W}{3}$$

and that

$$F = \frac{P + Q - \frac{W}{3}}{P + \frac{5}{3}Q + \frac{W}{9}}$$

$$F' = \frac{1}{3} \cdot \frac{P + Q - \frac{W}{3}}{P + \frac{5}{3} + \frac{W}{9}}.$$

Again, let T_1, T_2 . . . . T_n be the several tensions required, and I_1 I_2 . . . . I_n the corresponding inertias; then we have

$$P - T_1 = I_1 F = PF, \quad \therefore T_1 = P \times (1 - F)$$
$$T_1 - T_2 = I_2 F, \qquad \therefore T_2 = T_1 - I_2 F$$
$$T_2 - T_3 = I_3 F, \qquad \therefore T_3 = T_2 - I_3 F$$

and so on.

The pressure on each axis is double the tension of the string that goes round it.

614. Let θ denote the angular distance required, measured from the West, then by the composition of forces, it readily appears that

$$\frac{a}{b} = \sin.\ \theta$$

$$\text{and } \therefore \theta = \sin.^{-1}.\frac{a}{b}$$

Hence it seems that at sea, when the ship is sailing in a direction oblique to the wind, the position of the vane is no certain criterion as to the quarter from which it blows.

It would also appear, at first sight, that the magnet is affected in like manner by the motion of the vessel; but a moment's reflection is sufficient to be convinced of the contrary. The magnet consisting of two equal arms, the action of the ship's motion upon the one is counteracted by its opposite action upon the other.

615. Generally, w being the weight of the cylinder, and p the power or weight which puts it in motion, ϱ the radius of the base, and r the distance, then the accelerating force on p is

$$F = \frac{pr^2}{pr^2 + \frac{w}{2}\varrho^2};$$

$$\therefore v = gFt = \frac{gpr^2}{pr^2 + \frac{w}{2}\varrho^2} \times t$$

$$\therefore t = \frac{pr^2 + \frac{w}{2}\varrho^2}{gpr^2} \times v.$$

Now by the question

$p = 20, w = 133 . 6, v = 1, \varrho = 1, r = 10$, and it will be found, after substituting and reducing, that

$$t = .03214, \text{ \&c. seconds.}$$

616. *Generally, required the length of a pendulum that would oscillate seconds at the distance of* n *radii of the Earth from its centre.*

If F be the force which accelerates the pendulum, whose length is L, then the time of an oscillation is got from (see *Bridge*, vol. II.)

$$T \propto \sqrt{\frac{L}{F}}.$$

Hence, if l be the length of a second's pendulum at the surface of the earth, where g is the accelerating force, we have

$$1'' : T :: \sqrt{\frac{l}{g}} : \sqrt{\frac{L}{F}}$$

But by the question $T = 1''$, and we have

$$F : g :: \frac{1}{(nR)^2} : \frac{1}{R^2} :: 1 : n^2$$

$$\therefore \frac{Ln^2}{g} = \frac{l}{g}.$$

$$\text{or } L = \frac{l}{n^2}.$$

If $n = 2$. Then

$$L = \frac{l}{4} = \frac{39.2}{4} \text{ inches} = 9.8 \text{ inches.}$$

Within the surface $F = g \cdot \frac{\varrho}{R}$ whence ϱ is easily found.

617. Let x be the distance of the particle dM from the axis of suspension, a the length of the rod, D its density at the lower extremity, L the distance between the point of suspension and centre of oscillation; then

$$L = \frac{\int x^2 dM}{\int x dM} \text{ (see } Vince\text{, p. 127.)}$$

But the density at the distance x being $= \frac{x}{a} \text{D}$

$$\therefore d\text{M} = \frac{x}{a} \text{D} \times dx$$

$$\therefore \text{L} = \frac{\frac{\text{D}}{a} \cdot \frac{x^4}{4}}{\frac{\text{D}}{a} \cdot \frac{x^3}{3}} = \frac{3}{4} x$$

and for the whole length a

$$\text{L} = \frac{3}{4} a.$$

Hence

$$\text{T} = \pi \sqrt{\frac{\text{L}}{g}} = \frac{\pi}{2} \sqrt{\frac{3a}{g}}.$$

618. *Generally let* AC (Fig. 102,) *be any curve whatever revolving uniformly round the vertical axis* AB, *and suppose the body* P *descending along the curve by the force of gravity; required the velocity of the body at any given epoch* t.

Let the $\perp$ PM $= y$, AM $= x$, BC $= \beta$, AB $= a$, and V the known velocity of the point C; then resolving the centrifugal force RP into the tangential and normal ones PQ and RQ, we have

$$\left.\begin{aligned} \text{PQ} &= \text{PR} \times \cos.\ \text{RPQ} = \text{PR} \times \frac{dy}{ds} \\ \text{and RQ} &= \text{PR} \times \sin.\ \text{RPQ} = \text{PR} \times \frac{dx}{ds} \end{aligned}\right\}$$

Again, the velocity at C being V, that at P is

$$\frac{y}{\beta} . \text{V}.$$

Hence the centrifugal force is

$$\text{PR} = \frac{y^2}{\beta^2} \text{V}^2 \times \frac{1}{y} = \frac{\text{V}^2}{\beta^2} \times y$$

$$\left.\begin{aligned} \therefore\ F = PQ &= \frac{V^2}{\beta^2} \cdot \frac{ydy}{ds} \\ P = RQ' &= \frac{V^2}{\beta^2} \cdot \frac{xdy}{ds} \end{aligned}\right\} \dots\dots\dots\dots (1)$$

which are the accelerative force and pressure due to the angular velocity respectively.

Again, the accelerating force down the curve due to gravity is

$$F' = g \cdot \frac{dx}{ds} \dots\dots\dots\dots\dots\dots (2)$$

and the pressure arising from gravity is

$$P' = y \cdot \frac{dy}{ds} \dots\dots\dots\dots\dots\dots (3)$$

Hence the whole accelerating force is

$$F + F' = \frac{V^2}{\beta^2} \cdot \frac{ydy}{ds} + g\,\frac{dx}{ds} \dots\dots\dots (4)$$

and the whole pressure is

$$-P + P' = -\frac{V^2}{\beta^2} \cdot \frac{xdy}{ds} + \frac{gdy}{ds} \dots\dots (5)$$

Hence

$$vdv = \frac{V^2}{\beta^2}\, ydy + gdx$$

which gives

$$v^2 = \frac{V^2}{\beta^2}\, y^2 + 2gx \dots\dots\dots\dots\dots\dots (6)$$

Ex. Let the curve be the quadrant of a circle convex to the axis of rotation, which also touches it,

Then

$$(\beta - y)^2 = \beta^2 - x^2$$

$$\text{or } x^2 = \beta^2 - \overline{\beta - y}\,|^2$$

$$= 2\beta y - y^2$$

$$\therefore\ v^2 = \frac{V^2}{\beta^2}\, y^2 + 2g\,\sqrt{(2\beta y - y^2)}$$

$$\text{But } V = \frac{2\pi\beta}{t} \dots$$

$$\therefore v^2 = \frac{4\pi^2}{t^2} y^2 + 2g \sqrt{(2\beta y - y^2)}$$

which gives the velocity at any point of the descent, and solves the problem.

To find the pressure we have

$$\frac{dy}{ds} = -\frac{\sqrt{\beta^2-(\beta-y)^2}}{\beta}$$

$$= -\frac{\sqrt{2\beta y - y^2}}{\beta} = -\frac{x}{\beta}.$$

$$\therefore -P + P' = \frac{V^2}{\beta^3} x^2 - g\frac{x}{\beta}$$

$$= \frac{4\pi^2}{t^2\beta} x^2 - \frac{gx}{\beta} \quad \text{.........} (7)$$

the pressure required.

To find when the ring would fly off from the circle if unrestrained, or when the pressure changes its direction, we have

$$\frac{4\pi^2}{t^2} x - g = 0$$

$$\text{or } x = \frac{gt^2}{4\pi^2}.$$

The student will find no difficulty in making other applications.

619. Let y denote the ordinate of the generating parabola, on the radius of the circle in which the body revolves; then since the force of gravity is counteracted by the centrifugal force and the reaction of the surface, we get by the triangle of forces the centrifugal force, which in a circle is equal to the centripetal one, expressed by

$$F = g \times \frac{y}{\text{subnormal}}$$

$$= g \times \frac{y}{\left(\frac{p}{2}\right)}$$

p being the parameter of the parabola. Hence the time of revolution is (see 440)

$$T = 2\pi \sqrt{\frac{y}{F}} = 2\pi \sqrt{\frac{\left(\frac{p}{2}\right)}{g}}$$

But the time of oscillation of a pendulum whose length is $\frac{p}{2}$ is

$$T' = \pi \sqrt{\frac{\left(\frac{p}{2}\right)}{g}}$$

Hence $T = 2T'$. Q. E. D.

The pressure against the surface is easily found to be

$$P = g \times \sqrt{\frac{\left(\frac{p}{2}\right)^2 + y^2}{\left(\frac{p}{2}\right)^2}}$$

$$= g \times \sqrt{\frac{\frac{p^2}{4} + px}{\frac{p^2}{4}}}$$

$$= g \sqrt{\frac{p + 4x}{p}}$$

620. Let $a - x$ be the distance fallen through to acquire the velocity, a being the altitude of the cone; then if generally the equation of the generating Δ be

$$y = x \tan. \alpha$$

α being the inclination of the slant side to the axis, we have

$$v = \sqrt{2g \cdot (a - x)} = \sqrt{Fy},$$

F being the centrifugal force. But since gravity is counteracted by F and the pressure against the surface,

$$F = g \cdot \frac{x}{y}.$$

Hence

$$2g \cdot (a - x) = gx$$

which gives

$$x = \frac{2}{3} a,$$

a remarkable result, inasmuch as it shews that the altitude due to the velocity with which the body is whirled is the same for all cones of the same altitude. Hence

$$F = \frac{g}{\tan. \alpha}$$

and the pressure against the surface is

$$P = \frac{g}{\sin. \alpha}.$$

621. Let R be the radius of the earth, λ any latitude, t the time of the earth's rotation, and W, W′ the weight of the same body at the equator, and in latitude λ respectively; then at the equator the centrifugal force is

$$\varphi = \frac{v^2}{R} = \frac{4\pi^2}{t^2} \cdot R$$

and in latitude λ it is

$$\varphi' = \frac{4\pi^2}{t^2} R \cdot \cos. \lambda$$

and the resolved part of φ' which counteracts gravity is therefore

$$\varphi' \times \cos. \lambda = \frac{4\pi^2}{t^2} R \cos.^2 \lambda.$$

Hence

$$W : W' :: 1 - \frac{4\pi^2}{t^2} R : 1 - \frac{4\pi^2}{t^2} R \cos.^2 \lambda.$$

622. If L denote the length of the pendulum, F the force, and F the time of an oscillation, then (see *Venturoli*, or *Bridge*,)

$$L \propto F \times T^2.$$

Hence $T^2 \propto \frac{1}{F}$.

Hence, and by 621,

$$T^{\prime} : T^{\prime\prime} : 1 :: 1 - \frac{4\pi^2}{t^2} R.$$

623. If $r =$ altitude of the point descended from, and h' that of the point descended to, r being also the radius of the generating circle; then the time of descent is

$$t = \sqrt{\frac{r}{g}} \times \cos.^{-1} \frac{2h - r}{r}.$$

see *Venturoli*, p. 103.

$$\therefore \sqrt{r} \times \cos.^{-1} \frac{2h - r}{r} = \text{minimum, by the question.}$$

$$\therefore \frac{1}{2\sqrt{r}} dr \times \cos.^{-1} \frac{2h - r}{r} + \sqrt{r}. \frac{d . \left(\frac{2h - r}{r}\right)}{\sqrt{\left(1 - \left(\frac{2h - r}{r}\right)^2\right)}} = 0$$

$$\text{Hence } \cos.^{-1} \frac{2h - r}{r} = \frac{2\sqrt{h}}{\sqrt{(r - h)}}.$$

whence $4r$, the length of the pendulum, may be found by approximation.

624. Let P and p be the lengths of a degree at the pole and equator, m, n, the lengths in latitudes, λ, λ'; then since the length of a degree $\propto$ radius of curvature, we have

$$P : p :: \frac{a^2}{b} : \frac{b^2}{a} :: a^3 : b^3.$$

Also it is easily demonstrable that

$$m - p : n - p :: \sin.^2\lambda : \sin.^2\lambda'$$

$$\text{and } \therefore m - n : n - p :: \sin.^2\lambda - \sin.^2\lambda' : \sin.^2\lambda'$$

$$\therefore n - p = \frac{\overline{m - n} . \sin.^2\lambda'}{\sin.^2\lambda - \sin.^2\lambda'}.$$

$$\text{But } P - p : n - p :: 1^2 : \sin.^2\lambda'$$

$$\therefore P - p = \frac{m - n}{\sin.^2\lambda - \sin.^2\lambda'}$$

$$\text{and } P = p + \frac{m-n}{\sin.^2\lambda - \sin.^2\lambda'}.$$

$$\text{But } p = n - \frac{\overline{m-n}\,.\,\sin.^2\lambda'}{\sin.^2\lambda - \sin.^2\lambda'}$$

$$= \frac{n\sin.^2\lambda - m\sin.^2\lambda'}{\sin.^2\lambda - \sin.^2\lambda'}$$

$$\therefore P = \frac{n\sin.^2\lambda - m\sin.^2\lambda' + m - n}{\sin.^2\lambda - \sin.^2\lambda'}$$

$$= \frac{m\cos.^2\lambda' - n\cos.^2\lambda}{\sin.^2\lambda - \sin.^2\lambda'}$$

$$\therefore P : p :: m\cos.^2\lambda' - n\cos.^2\lambda : n\sin.^2\lambda - m\sin.^2\lambda'$$

$$\text{or } a : b :: (m\cos.^2\lambda' - n\cos.^2\lambda) : (n\sin.^2\lambda - m\sin.^2\lambda')$$

Q. E. I.

Again, let L′ and l' denote the given lengths of two pendulums vibrating seconds in the latitudes λ, λ'; then since the length of the pendulum $\propto$ force, and by *Simpson's Fluxions*, Vol. II. Art. 388, it is shewn that the attraction upon a spheroid $\propto$ that part of *normal to the generating ellipse* passing through the body attracted, which is intercepted between the ellipse and axis-minor, therefore

$$L : l :: \text{force at equat.} : f.\text{ at pole}$$
$$:: b : a.$$

But

$$L' - L : l' - L :: \sin.^2\lambda : \sin.^2\lambda'$$

$$\therefore L'\sin.^2\lambda' - L\sin.^2\lambda' = l'\sin.^2\lambda - L\sin.^2\lambda$$

$$\therefore L = \frac{l'\sin.^2\lambda - L'\sin.\lambda'}{\sin.^2\lambda - \sin.^2\lambda'}.$$

Also

$$L' - L : l - L :: \sin.^2\lambda : 1$$

$$\therefore l = \frac{L' - L\cos.^2\lambda}{\sin.^2\lambda}$$

$$= \frac{L'}{\sin.^2\lambda} - \frac{(l'\sin.^2\lambda - L'\sin.^2\lambda')\cos.^2\lambda}{\sin.^2\lambda\,(\sin.^2\lambda - \sin.^2\lambda')}$$

$$= \frac{L'\sin.^2\lambda - L'\sin.^2\lambda' - l'\sin.^2\lambda + l\sin.^4 + L\sin.^2\lambda' - L'\sin.^2\lambda\sin.^2\lambda'}{\sin.^2\lambda\,(\sin.^2\lambda - \sin.^2\lambda')}$$

and which finally reduces to

$$l = \frac{L' \cos.^2 \lambda' - l' \cos.^2 \lambda}{\sin.^2 \lambda - \sin.^2 \lambda'}.$$

Hence then

$$b : a :: L : l :: l' \sin.^2 \lambda - L' \sin.^2 \lambda' : L' \cos.^2 \lambda' - l' \cos.^2 \lambda$$

the ratio required.

625. (Generally we have (see Jesuit's edition of Newton, note to cor. 2. prop. LIII.)

$$\frac{b\,F^2}{2f\int F d\rho} = \frac{\rho^2}{\rho^2 - p^2}$$

where F denotes the required force, $f =$ the force corresponding to a given arc to be described, *viz.*, b, p the $\perp$ on the tangent, and ρ the radius vector. Now in the logarithmic spiral,

$$p = \rho \sin. \alpha,$$

α being the constant angle between the radius vector and curve,

$$\therefore \frac{b\,F^2}{2f\int F d\rho} = \frac{\rho^2}{\rho^2 - \rho^2 \sin.^2 \alpha} = \frac{1}{\cos.^2 \alpha}$$

$$\therefore b\,F dF = \frac{f}{\cos.^2 \alpha} \times F d\rho$$

$$\therefore dF = \frac{f}{b \cos.^2 \alpha} \cdot d\rho$$

$$\text{and } F = \frac{f}{b \cos.^2 \alpha} \cdot (\rho + c)$$

but when $F = f$, let $\rho = R$; then $c = b \cos.^2 \alpha - R$

$$\therefore F = \frac{f}{b \cos.^2 \alpha} (\rho - R + b \cos.^2 \alpha)$$

$$\therefore F \propto \rho - R + b \cos.^2 \alpha$$

the law required.

626. We know that the time of an oscillation is generally

$$T = \pi \sqrt{\frac{L}{F}} \propto \frac{1}{\sqrt{F}} \propto \rho$$

But since the pendulum when on the top of the mountain loses n seconds in a day, the time of one vibration is

$$T' = T \times \frac{24 \times 60 \times 60 - n}{24 \times 60 \times 60}$$

$$\therefore \frac{1}{\sqrt{g}} : \frac{1}{\sqrt{F}} :: T : T'$$

$$\text{or } R : \rho :: 24 \times 60 \times 60 - n : 24 \times 60 \times 60$$

R being the radius of the earth.

$$\therefore \rho - R = R \times \left(\frac{24 \times 60 \times 60}{24 \times 60 \times 60 - n} - 1\right)$$

$$= \frac{Rn}{24 \times 60 \times 60 - n}$$

the altitude required.

627. Let T be the time of vibration at the pole; then the time of vibration at the other place is

$$T' = \frac{99}{100} T.$$

$$\therefore \pi \sqrt{\frac{L}{F}} = \frac{99}{100} \times T$$

F being the accelerating force, and L the given length of the pendulum. But by 621

$$F = \frac{4\pi^2}{t^2} R \cos.^2 \lambda$$

R being the radius of the sphere, λ the given latitude, and t the time of the rotation.

$$\therefore \sqrt{\frac{Lt^2}{4R \cos.^2 \lambda}} = \frac{99}{100} \times T$$

$$\text{and } t = \frac{99 \cos. \lambda \sqrt{4R}}{\sqrt{L}} \times T$$

the time required for any latitude.

628. The times of oscillation are

$1''$ and $\frac{1''}{3}$. Hence

$$\frac{1}{3} = \pi\sqrt{\frac{L}{F}} = \pi\sqrt{\frac{Lt^2}{4\pi^2 R \cos.^2\lambda}}$$

$$= \frac{t}{2\cos.\lambda}\sqrt{\frac{L}{R}}$$

$$\therefore t = \frac{2}{3}\cos.\lambda\sqrt{\frac{R}{L}}.$$

By the problem, $\lambda = 60^\circ$. Therefore

$$t = \frac{1}{3}\sqrt{\frac{R}{L}}.$$

629. $\quad T = \pi\sqrt{\frac{L}{F}}$ and T is given

$$\therefore F \propto L.$$

But by 621, $F = \dfrac{4\pi^2}{t^2} R \cos.^2\lambda$

$$\propto \cos.^2\lambda$$

$$\therefore L \propto \cos.^2\lambda$$

$$\therefore a : L :: \cos.^2 60^\circ : \cos.^2 0 :: \frac{1}{4} : 1$$

$$\therefore L = 4a.$$

630. This curve is called the Tractrix, and its equation is

$$ydx = dy\sqrt{(a^2 - y^2)}$$

(see *Whewell's Dynamics*, p. 127,) y, x, being the ordinate and abscissa originating in B, and $a =$ AB.

Hence its area is

$$S = \int ydx = \int dy\sqrt{(a^2 - y^2)}$$

$$= \frac{y\sqrt{a^2 - y^2}}{2} - \frac{a^2}{2}\int\frac{dy}{\sqrt{a^2 - y^2}}$$

$$= \frac{y\sqrt{(a^2-y^2)}}{2} - \frac{a^2}{5}\sin.^{-1}\frac{y}{a} + \frac{a^2}{2}\cdot\frac{\pi}{2}$$

Let $y = 0$. Then

$$S = \frac{a^2}{4}\pi \dots\dots\dots\dots\dots\dots (a)$$

Again, the surface is

$$\sigma = \int 2\pi y ds = -2\pi \int y \times \frac{ady}{y}$$
$$= -2\pi a \int dy$$
$$= -2\pi ay + C$$
$$= 2\pi (a^2 - ay)$$

and when $y = 0$

$\sigma = 2\pi a^2 = 8S.$ Q. E. D.

631. Let a be the length of the chain, w its weight, b the part unwound at the commencement of the motion, r the radius of the quadrant, and u the part unwound at any time during the descent; then since the accelerating force upon every particle $du = ds$ of the chain in contact with the curve, by the theory of the inclined plane, is (y and s being the ordinate and arc of the quadrant,)

$$\frac{dy}{ds}$$

and the weight of that particle is

$$\frac{ds}{a} \times w$$

therefore the moving force of each particle is

$$\frac{dy}{ds} \times \frac{ds}{a} \times w = \frac{w}{a} \cdot dy.$$

Hence the whole moving force of $a - u$, or of the part in contact is

$$M = \frac{w}{a}\int dy = \frac{w}{a} y + C$$

But when $y = r$, $M = w$, and then

$$w = \frac{w}{a} r + C;$$

$$\therefore\ C = w.\left(1 - \frac{r}{a}\right) = w\ .\ \frac{a-r}{a}.$$

$$\therefore\ M = \frac{w}{a}(y + a - r.$$

Again the moving force of w is

$$\frac{w}{a}\ .\ u.$$

$\therefore$ the whole moving force or *tension of the string* is

$$\frac{w}{a}(y + u + a - r).$$

Hence the accelerating force is

$$F = \frac{y + u + a - r}{a}$$

and $\therefore\ vdv = Fdu$
produces

$$v^2 = \frac{2}{a}\int ydu + \frac{u^2}{a} + \frac{2}{a}(a-r)\,u$$

But

$$du = ds = \frac{rdy}{\sqrt{r^2 - y^2}}$$

$$\therefore\ v^2 = C - \frac{2r}{a}\sqrt{r^2 - y^2} + \frac{u^2}{a} + 2\ .\ \frac{a-r}{a}\,u.$$

Now when $v = 0$, $y = 0$, and $u = b$,

$$\therefore\ C = \frac{2r^2}{a} - \frac{b^2}{a} - 2\,\frac{a-r}{a}\,b$$

$$\therefore\ v^2 = \frac{2r}{a}\left(r - \sqrt{r^2 - y^2}\right) + \frac{u-b}{a}\left(u - b + 2\ \overline{b + a - r}\right)$$

$$= \frac{2r}{a}\ .\ \text{vers.}\ z + \frac{z^2}{a}\left(+ \frac{2.\overline{b + a - r}}{a}\right).\ z$$

z being the arc quitted by the chain.

Make $z = 90^\circ = \frac{\pi r}{2}$; and

the velocity of the chain, when quitting the quadrant, is given by

$$V^2 = \frac{r}{4a}\left\{8r + 4\,(2a - r)\,\pi - r\pi^2\right\}.$$

632. Let y', y, be the radii of the required annulus, and x the distance of its centre from the pole; also let a, b, be the semi-axis of the generating ellipse; then it may easily be shewn that

$$y' = y \times \frac{a}{b}$$

and therefore the breadth of the annulus is

$$y' - y = y \cdot \frac{a-b}{b}.$$

Again, if u denote the radius of any of the concentric circles which compose the ring, the attraction of any particle in its circumference upon the pole is

$$\frac{1}{u^2+x^2}$$

and by the resolution of forces the attraction in the direction of the axis

$$\frac{1}{a^2+x^2} \times \frac{x}{\sqrt{u^2+x^2}} = \frac{x}{(u^2+x^2)^{\frac{3}{2}}}.$$

Hence the whole force of the circle in this direction is

$$\frac{2\pi ux}{(u^2+x^2)^{\frac{3}{2}}}$$

and that of the annulus is $\therefore$

$$2\pi x \int \frac{udu}{(u^2+x^2)^{\frac{3}{2}}}$$

taken between $u = y'$ and $u = y$; that is

$$2\pi x \left\{\frac{1}{\sqrt{y^2 + x^2}} - \frac{1}{\sqrt{y'^2 + x^2}}\right\}$$

$$\text{or } 2\pi \left\{\frac{x}{\sqrt{2bx}} - \frac{x}{\sqrt{\frac{a^2}{b^2} \cdot (2bx - x^2) + x^2}}\right\}$$

or it is

$$2\pi \left\{\frac{\sqrt{x}}{\sqrt{2b}} - \frac{b\sqrt{x}}{\sqrt{\{2ba^2 - (a^2 - b^2 x\}}}\right\}.$$

Hence, and by the question,

$$\frac{\sqrt{x}}{\sqrt{2b}} - \frac{b\sqrt{x}}{\sqrt{(2ba^2 - \overline{a^2 - b^2} \, . \, x)}} = \text{max.} = m$$

$$\therefore \frac{dm}{dx} = \frac{1}{2\sqrt{2b}} \times \frac{1}{\sqrt{x}} - \frac{b}{2\sqrt{x}} \cdot \frac{1}{\sqrt{(2ba^2 - \overline{a^2 - b^2} \, . \, x)}}$$

$$- \frac{b}{2} \cdot (a^2 - b^2) \frac{\sqrt{x}}{\overline{2ba^2 - \overline{a^2 - b^2} \, . \, x})^{\frac{3}{2}}} = 0$$

$$\therefore (2ba^2 - (a^2 - b^2)\, x)^{\frac{3}{2}} = 2b^2 a^2 \sqrt{2b}$$

$$\text{and } x = \frac{1}{a^2 - b^2} \times 2ba^2 - (8b^5 a^4)^{\frac{1}{3}}$$

$$= \frac{2\,(ba^2 - b^{\frac{5}{3}}\, a^{\frac{4}{3}})}{a^2 - b^2}$$

which gives the position of the annulus for spheroids of all eccentricities.

633. Let the distance of the particle attracted from the centre of the sphere be a, r the radius of the sphere, and suppose any circular section whose radius is y to be made by a plane $\perp$ to a, and distant from the point by the interval x; then if y' denote the radius of any $\odot$ concentric with the former, the attraction of any particle in the circumference of this circle is

$$\frac{1}{(y'^2 + x^2)^{\frac{3}{2}}}$$

and this resolved into the direction of x is

$$\frac{1}{(y'^2 + x^2)^{\frac{3}{2}}} \times \frac{x}{(y'^2 + x^2)^{\frac{1}{2}}} = \frac{x}{(y'^2 + x^2)^{\frac{5}{2}}}.$$

Hence the attraction of the whole circumference is

$$\frac{2\pi x y'}{(y'^2 + x^2)^{\frac{5}{2}}}$$

and that of the area of the section is

$$2\pi x \int \frac{y' dy'}{(y'^2 + x^2)^{\frac{5}{2}}}$$

taken between $y' = y$, and $y' = 0$; that is

$$\frac{2}{3}\pi x\left\{\frac{1}{x^3} - \frac{1}{(y^2+x^2)^{\frac{3}{2}}}\right\}.$$

Hence the attraction of the whole sphere is

$$\frac{2}{3}\pi\int\frac{dx}{x^3} - \frac{2}{3}\pi\int\frac{dx}{(y^2+x^2)^{\frac{3}{2}}}.$$

taken between the limits of $x = a + r$, $x = a - r$.

But by the equation to the circle

$$y^2 = r^2 - (a - x)^2$$

$$\therefore \int\frac{dx}{(y^2+x^2)^{\frac{3}{2}}} = \int\frac{dx}{(r^2 - a^2 + 2ax)^{\frac{3}{2}}} = -\frac{1}{a}\times$$

$\dfrac{1}{(r^2 - a^2 + 2ax)^{\frac{1}{2}}}$; and we have for the whole attraction

$$\frac{\pi}{3}\cdot\left\{\frac{1}{(a-r)^2} - \frac{1}{(a+r)^2}\right\} - \frac{2\pi}{3a}\cdot\left\{\frac{1}{a-r} - \frac{1}{a+r}\right\}$$

$$\text{or } \frac{4\pi}{3}\cdot\frac{ar}{(a^2 - r^2)^2} - \frac{4\pi}{3a}\cdot\frac{r}{a^2 - r^2}$$

$$\text{or } \frac{4}{3}\cdot\frac{\pi}{a}\cdot\frac{r^3}{a^2 - r^2}.$$

634. If x be the altitude of the cone, and s its slant side, the whole attraction is (*Vince*, p. 142)

$$x - \frac{x^2}{s} = u.$$

Again, supposing the given quantity of matter to be a^3, and the density to be constant, we have

$$a = \frac{2}{3}\pi.(s^2 - x^2)\,x$$

$$\therefore s^2 = \frac{3a^3}{2\pi x} + x^2$$

But by the question

$$\frac{du}{dx} = 0, \text{ which gives}$$

$$\frac{x^3}{s^3} - \frac{8a^3}{4\pi} \times \frac{1}{s^3} - \frac{2x}{s} = -1$$

whence by substituting for x &c. &c., x may be found exactly or by approximation.

635. Let c be the distance of P from the surface of the sphere, r the radius of the sphere, then $\frac{c+r}{2}$ is the radius of the generated sphere. Let any circular section of the sphere be made by a plane $\perp$ to c or to the axis, and let y be the radius of that section, and y' that of any circle concentric with it; also let x be the distance of P from the plane of this section; then the attraction of any particle in the circumference of the circle whose radius is y', is

$$\frac{1}{y'^2 + x^2}$$

and the attraction of the whole circumference is

$$\frac{2\pi y'}{y'^2 + x^2}.$$

Hence the attraction of the whole section is

$$\pi \int \frac{2y'dy'}{y'^2 + x^2} = \pi\, l.\, (y'^2 + x^2) + \text{C}$$

taken between $y' = 0$ and $y' = y$; it is $\therefore$

$$\pi\, l.\, \frac{y^2 + x^2}{x^2}.$$

Hence the attraction of the sphere is

$$\text{F} = \pi . \int dx l. \frac{y^2 + x^2}{x^2}$$

$$= \pi \int dx.l.\, \frac{2(c+r)x - c.(c+2r)}{x^2}$$

since $y^2 = r^2 - (c + r - x)^2$.

$$\therefore\ \frac{\text{F}}{\pi} = xl.\, \frac{2(c+r)x - c.(c+2r)}{x^2}$$

$$+ 2x - 2.(c+r) \int \frac{xdx}{2(c+r)x - c(c+2r)}$$

and we finally obtain, after finding the correction on the supposi-

tion that $F=0$, when $x=c$, and putting $2.(c+r)x-c.(c+2r=X$,
$\frac{F}{\pi}=x-c+x.l.\frac{X}{x^2}-\frac{c}{2}.\frac{c+2r}{c+r}l.\frac{X}{c^2}\ldots\ldots(1)$ *which gives the sum of the direct attractions, of every particle in the portion cut off by the section which is distant from* P, *by the interval* x.

Again, when $c=0$, P is in the surface, and then we have

$$\frac{F'}{\pi}=x+xl.\frac{2r}{x}\ldots\ldots\ldots(1')$$

for

$$clc=l.c^c=l.o^0=l1=0, \&c.$$

Hence it follows that the sum of the attractions of that part of the sphere, whose radius is $\frac{c+r}{2}$, which is between P and the same section is

$$f=\pi x+\pi xl.\frac{c+r}{x}\ldots\ldots(2).$$

Again, for the section which passes through the intersection of the spherical surfaces, it is easily shewn that

$$x=c.\frac{c+2r}{c+r}.$$

Hence

$$x-c=\frac{cr}{c+r}.$$

$$\frac{X}{x^2}=\frac{(c+r)^2}{c(c+2r)}$$

$$\frac{X}{c^2}=\frac{c+2r}{c}$$

and equations (1) and (2) become

$$\frac{F}{\pi}=\frac{cr}{c+r}+c.\frac{c+2r}{c+r}.l.\frac{(c+r)^2}{c(c+2r)}-\frac{c}{2}.\frac{c+2r}{c+r}l.\frac{c+2r}{c}$$

and $\frac{f}{\pi}=\frac{c.(c+2r)}{c+r}+\frac{c.(c+2r)}{c+r}l.\frac{(c+r)^2}{c(c+2r)}$.

Also, when $x=c+r$, equation (2) becomes

$$f'=\pi\,(c+r)$$

and hence the attraction of the part scooped out of the given sphere is

$$F + f' - f = f' + F - f = \pi \left\{c + r + \frac{cr}{c+r} - c.\frac{c+2r}{c+r}\right.$$

$$\left. - \frac{c}{2}.\ \frac{c+2r}{c+r} l.\frac{c+2r}{c}\right\} = \pi\left\{r - \frac{c}{2}.\ \frac{c+2r}{c+r} l.\frac{c+2r}{c}\right\}.$$

But by equation (1), when $x = c + 2r$ we get, for the whole attraction of the sphere

$$2 \times \pi \left\{r - \frac{c}{2}.\ \frac{c+2r}{c+r} l.\frac{c+2r}{c}\right\}$$

which being twice as great as that of the part scooped out, resolves the problem.

636. Let a be the distance of the two centres of force, l the length of the bar; also let x be the distance of any point in the bar from one centre; then the attraction of this centre at the distance a being A, we have

$$Aa \times \int \frac{dx}{x}$$

for the attraction of the particle: and if y, $y + l$, be the distances of the extremities of the rod from this centre of force, when in the required position, the attraction upon the whole rod is the above integral taken between $x = y + l$ and $x = y$; it is $\therefore$

$$Aa\ \{l.(y+l) - l.y\}$$

$$\text{or } Aa.\ l.\frac{y+l}{y}.$$

Hence, and by the question, we easily get, for the whole attraction of the other force

$$2Aa\ l.\frac{a-y}{a-y-l}.$$

But to effect the equilibrium, these attractions must be equal;

$$\therefore\ l.\frac{y+l}{y} = 2l.\frac{a-y}{a-y+l}$$

$$\text{and } \frac{y+l}{y} = \frac{(a-y)^2}{(a-y+l)^2}$$

whence y is easily found, and with it the required position of the rod.

637. The spheroid revolves, of course, round its minor axis. Now

$$y^2 = PC^2 \times \sin.^2 \text{(latitude)}$$
$$= PC^2 \times \frac{1}{3}, \text{ by hypothesis}$$
$$= \frac{b^2}{a^2}.(a^2 - x^2) = \text{also } PC^2 - x^2$$

Hence

$$PC^2 = \frac{3a^2}{\frac{a^2}{b^2} + 2} = \frac{a^2.(1-e^2)}{1 + \frac{2e^2}{3}}$$

where $e = \frac{\sqrt{a^2 - b^2}}{a}$ or the ratio of the eccentricity to the semi-axis major.

$$\therefore PC = a(1 - e^2)^{\frac{1}{2}} \times \left(1 - \frac{2e^2}{3}\right)^{-\frac{1}{2}}$$
$$= a\ (1 - \frac{1}{2}e^2 + \&c.) \times (1 + \frac{e^2}{3} + \&c.)$$
$$= a\left\{1 - \frac{e^2}{2} + \frac{e^2}{3}\right\} \text{ nearly.}$$
$$= a\left(1 - \frac{e^2}{6}\right) \text{ nearly.}$$

the terms involving e^4, e^6, &c. being neglected because of their smallness.

Again, since the solid content of the spheroid, (see *Vince* p. 95) is

$$\frac{4\pi a^2 b}{3}$$

to find the sphere of the same volume with the spheroid, suppose its radius u, and we have

$$\frac{4\pi}{3}a^2b = \frac{4\pi}{3}u^3$$

$$\therefore u^3 = a^2b = a^2a\sqrt{1 - e^2}$$
$$\therefore u = a \times (1 - e^2)^{\frac{1}{6}}$$
$$= a \times (1 - \frac{1}{6}e^2) \text{ nearly.}$$

$\therefore$ PC $= u$. Q. E. D.

Again, if PG be the normal at P, meeting the axis b in G, and

$$\frac{a^2}{b^2} = 1 + \mathrm{B}$$

the attraction of the spheroid is (see *Simpson*, Vol. II. p. 387).

$$4\pi\left(\frac{1}{3} - \frac{\mathrm{B}}{3.5} + \frac{\mathrm{B}^2}{5.7} - \&c.\right) \times \mathrm{PG}.$$

But

$$\mathrm{PG} = x. \sqrt{\left(1 + \frac{dx^2}{dy^2}\right)}$$

and $x^2 = a^2 - \frac{a^2}{b^2} y^2$

Hence

$$\mathrm{PG} = \sqrt{\frac{a^4 - \overline{a^2 - b^2}.x^2}{b^2}}$$

$$= \frac{a}{b}\sqrt{(a^2 - e^2x^2)}$$

Hence since the eccentricity is small, and at P, $x^2 = \frac{2}{3}\,\mathrm{PC}^2$

$$= \frac{2}{3}.\ a^2.\frac{1-e^2}{1-\frac{2e^2}{3}}.$$

$$\therefore \mathrm{PG} = \frac{a^2}{b}\sqrt{\frac{1-\frac{4}{3}e^2}{1-\frac{2e^2}{3}}} \text{ nearly.}$$

$$= a \times \left(1 - \frac{4}{3}e^2\right)^{\frac{1}{2}} \times \left(1 - \frac{2e^2}{3}\right)^{-\frac{1}{2}} \times (1 - e^2)^{-\frac{1}{2}}$$

$$= a\left(1 - \frac{2}{3}e^2\right) \times \left(1 + \frac{e^2}{3}\right)\left(1 + \frac{e^2}{2}\right) \text{ nearly.}$$

$$= a\left(1 + \frac{e^2}{6}\right) \text{ nearly.}$$

Also $\mathrm{B} = \frac{a^2 - b^2}{b^2} = \frac{a^2e}{a^2(1-e^2)} = e^2$ nearly.

$\therefore$ the whole attraction of the spheroid as

$$4\pi \times \left(\frac{1}{3} - \frac{e^2}{3.5}\right) \times a.\left(1+\frac{e^2}{6}\right) \text{ nearly,}$$

or $\frac{4\pi a}{3}\left(1 - \frac{e^2}{30}\right)$, which differs so little from the attraction of the sphere whose radius is $a\left(1-\frac{e^2}{6}\right)$ viz. $\frac{4\pi a}{3}\left(1-\frac{e^2}{6}\right)$ see *Simpson's Fluxions*, Art. 381, that they may be considered as equal. Q. E. D.

638. Since the accelerating forces are a, b, c and the masses A, B, C the moving forces are Aa, Bb, Cc. Let A, B, C be joined and in the $\triangle$ ABC assume P for the point required; then through P let a line be drawn parallel to the direction of the forces, and let x, y, z be the distances of A, B, C from this line. Then when there is an equilibrium, by the property of the lever,

$$\mathrm{A}a\,x + \mathrm{B}by + \mathrm{C}cz = 0.$$

The remainder of the question is merely geometrical and consists in finding from the given sides and angles of the $\triangle$ ABC, and the given inclination of the line passing through P, the quantities x, y, z.

639. The body will evidently move along the diameter of the sphere. Moreover it will be accelerated through the first half of the diameter and retarded through the remaining half. Suppose the body has descended through x; then the attraction of any circle whose plane, $\perp x$, is distant from the body by the interval u, and whose radius is w, is measured by (*Vince*, Art. 69.)

$$1 - \frac{u}{\sqrt{u^2+y^2}}$$

But by the equation to the circle

$$\begin{aligned} w^2 &= 2r\,(x \pm u) - (x \pm u)^2 \\ &= 2rx - x^2 \pm 2ru - u^2 \mp 2xu \\ &= 2rx - x^2 - u^2 \pm 2\,(r - x)\,u \end{aligned}$$

$\therefore$ the attraction of the circle is

$$1 - \frac{u}{\sqrt{2rx - x^2 \pm 2(r-x)u}}$$

Hence the attraction of the shell is

$$\alpha = \int \left\{ du - \frac{udu}{\sqrt{\{2rx - x^2 \pm 2(r-x)u\}}} \right\}$$

$$= u - \int \frac{udu}{\sqrt{2rx - x^2 \pm 2(r-x)u}}$$

$$= u - \left\{ \frac{x^2}{3} - \frac{2}{3} rx \pm \frac{1}{3}(r-x)u \right\} \frac{\sqrt{\{2rx - x^2 \pm 2(r-x)u\}}}{(r-x)^2} + \text{C}.$$

see *Hirsch's Tables*, p. 94.

Let $u = 0$. Then $\alpha = 0$ and $\text{C} = \left(\frac{x^2}{3} - \frac{2}{3} rx\right) \frac{\sqrt{2rx - x^2}}{(r-x)^2}$

and if the ordinate corresponding to x be y, we get

$$\alpha = u - \{y^2 \pm \frac{1}{3}(r-x)u\} \frac{\sqrt{\{y^2 \pm 2(r-x)u\}}}{3(r-x)^2} + \frac{y^{\frac{3}{2}}}{3(r-x)^2}.$$

Again, let $u = 2r - x$; then

$$\alpha = 2r - x - \frac{1}{3} \left\{\overline{2r - x}.\overline{r + 2x}\right\} \times \frac{2r - x}{3(r-x)^2} + \frac{y^{\frac{3}{2}}}{3(r-x)^2}$$

Let $u = x$. Then

$$\alpha' = x - \left(\frac{5}{3} rx - \frac{2}{3} x^2\right) \frac{x}{3(r-x)^2} + \frac{y^{\frac{3}{2}}}{3(r-x)^2}$$

$$\therefore \text{F} = \alpha - \alpha' = 2r - 2x - \frac{4}{9} \cdot \frac{r^3 + r^2x - 3rx^2 + x^3}{(r-x)^3}$$

which is the force which accelerates the body at any point of the descent.

Hence by means of the formula

$$vdv = \text{F}dx, \text{ and } dt = \frac{dx}{v}$$

the velocity at that point may be found, and also the time in reaching it.

640. Owing to the diurnal rotation of the earth, the body will have a centrifugal force in the direction of that ⊥ to the axis which passes through the top of the tower; this will consequently counteract, in some degree, the effect of gravity, and cause the body, during its descent, continually to recede from the tower in the meridian towards the equator. So that in northern latitudes

the body will strike the ground southward of the tower, and *vice versâ*. For a full discussion of this subject see *Laplaca Bulletin de Sciences*, No. 75. Also see *Emerson's Algebra*, prob. 198.

641. Let y be the radius of any section parallel to the plane passing through the centre, x the distance of its centre from the centre of the sphere, and y' the radius of any circle concentric with the former; then since the attraction to the centre of any particle within a sphere $\propto$ distance, that attraction will be duly measured by the distance, and when resolved into two directions, one $\perp$ to, and the other parallel to the plane, we have the pressure of this particle upon the plane, measured by x. Hence the pressure of the whole circle whose radius is y is

$$2\pi x \int y'dy'$$

taken between $y' = 0$, and $y' = y$; or

$$\pi xy^2 = \pi x \cdot (r^2 - x^2)$$

r being the radius of the earth.

Again, the pressure of the whole hemisphere is

$$\pi \int xdx\,(r^2 - x^2)$$

taken between $x = 0$, and $x = r$; that is

$$P = \pi \cdot \frac{r^4}{4}.$$

But since the density of the earth is supposed uniform, and consequently its weight at the surface proportional to its magnitude $\times\, r$, that weight is measured by

$$W = \frac{4}{3}\pi r^4$$

$$\therefore P : W :: 3 : 16.$$

HYDROSTATICS.

642. The pressure against the curve is measured by the product of the part pressed, and the depth of its centre of gravity in the fluid, (*Vince*, Prop. VII.)

Let x be the abscissa required, and y the corresponding ordinate, and α, β, the whole abscissa and corresponding ordinate of the parabola; then since the areas of the lower and upper parts of the parabola are respectively

$$\frac{4}{3}\,yx \text{ and } \frac{4}{3}\,(\alpha\beta - yx)$$

and the distances of their centres of gravity from the vertex are

$$\frac{\int yxdx}{\int ydx} = \frac{\int x^{\frac{3}{2}}dx}{\int x^{\frac{1}{2}}dx}$$

integrated between $x = 0$ and x, and $x = x$ and α respectively; or they are

$$\frac{3}{5}\,x, \text{ and } \frac{3}{5}\,(\alpha - x)$$

$\therefore$ the depths of these centres in the fluid are

$$\alpha - \frac{3}{5}\,x, \text{ and } \alpha - \frac{3}{5}\,(\alpha - x)$$

Hence

$$\frac{4}{3}\,yx \times \left(\alpha - \frac{3}{5}x\right) : \frac{4}{3}\,(\alpha\beta - yx) \times \left(\frac{2}{5}\,\alpha + \frac{3}{5}x\right) :: n : m$$

But $y^2 = px$, and $\beta^2 = p\alpha$

$$\therefore \frac{y^3}{p} \times \left(\frac{\beta^2}{p} - \frac{3}{5}\,\frac{y^2}{p}\right) : \left(\frac{\beta^3}{p} - \frac{y^3}{p}\right) + \left(\frac{2\beta^2}{5p} + \frac{3y^2}{5p}\right) :: n : m$$

$$\therefore y^3 \times (\beta^3 - \frac{3}{5}y^3) : (\beta^3 - y^3) \times \frac{2\beta^2 - 3y^2}{5} :: n : m$$

which gives the equation

$$y^5 - \frac{\beta^2}{m+n} \times \frac{5m+2n}{3} y^3 - \frac{n\beta^3}{m+n} y^2 + \frac{2n\beta^5}{3(m+n)} = 0.$$

whereby to determine y and x.

643. Let x be the depth of the required section, and a the whole depth or length of the cylinder; then the depths of the centres of gravity of the upper and lower portions are

$$\frac{x}{2}, \text{ and } x + \frac{a-x}{2}.$$

Also the surfaces pressed are as

$$x \text{ and } a - x$$

$\therefore$ by *Vince*, Prop. VII., and the question, we have

$$\frac{x}{2} \times x : \frac{x+a}{2} \times (a-x) :: 1 : 1$$

$$\text{or } a^2 - x^2 = x^2$$

$$\text{or } x = \frac{a}{\sqrt{2}},$$

which gives the section required.

644. Let x be the depth of the heavier fluid, $a - x$ that of the lighter; and suppose y the depth of that column of the heavier fluid which has the same weight as the column of the lighter fluid; then since the pressure against the surface $\propto$ surface pressed $\times$ depth of centre of gravity of the surface $\times$ the specific gravity of the fluid, we have

$$P : P' :: (a-x)\frac{a-x}{2} \times 1 : x \times \left(\frac{x}{2} + y\right) \times n$$

P and P′ being the pressures on the upper and lower surfaces.

Hence, by the question

$$(a-x)^2 = x \cdot (x + 2y)\, n$$

But since the weight of a fluid $\propto$ volume $\times$ specific gravity,

$$y = \frac{a-x}{n}$$

$$\therefore (a-x)^2 = x.(nx + 2 . \overline{a-x})$$

$$= x\,(2a + \overline{n-1} . x)$$

$$\therefore x^2 + \frac{4a}{n-3}\,x = \frac{a^2}{n-3}$$

which gives x and determines the quantities required.

645. Let x be that part of the axis a which is immersed, and suppose r the radius of the base of the cone; then since the base of the part immersed is

$$\pi \times \left(\frac{xr}{a}\right)^2$$

the part itself is in volume

$$\pi \times \frac{x^2r^2}{a^2} \times \frac{x}{3} = \frac{\pi r^2}{3a^2} \times x^3.$$

Also the volume of the whole cone is

$$\frac{\pi a^3 r^2}{3a^2} = \frac{\pi r^2 a}{3}.$$

Hence, and by *Vince*, Prop. XVI.,

$$\frac{\pi r^2}{3a^2}\,x^3 : \frac{\pi r^2 a}{3} :: 1 : 8$$

$$\therefore x^3 = \frac{a^3}{8}$$

$$\text{and } x = \frac{a}{2}.$$

646. Let a be the axis of the parabola, x the depth of the required ordinate y; then since the pressure $\propto$ surface pressed $\times$ depth of its centre of gravity $\times$ density of the fluid, we have

$$2y \times x \times x = \text{max.}$$

$$\text{But } y \propto \sqrt{a-x}$$

$$\therefore\ x^2 \times \sqrt{a-x} = \text{max.}$$

$$\text{or } x^4 a - x^5 = \text{max.}$$

$$\therefore\ 4ax^3 - 5x^4 = 0$$

$$\text{or } x = \frac{4a}{5}$$

which will give the required ordinate.

647. When the bodies are weighed in vacuo, the equilibrium, which before subsisted, will be destroyed, and the wood will preponderate; because when bodies are taken out of a medium and weighed in vacuo they become heavier in proportion to the weights of the fluid they before displaced.

Moreover, to restore the equilibrium, let s, s', be the specific gravities of the wood and iron, B, B′, their bulks in cubic feet, α the weight of a cubic foot of air, and x the magnitude of the wood to be added in order to effect the equilibrium in vacuo; then the weight of the wood in vacuo is

$$(B + x)\, s$$

and that of the iron is

$$B's'$$

and when in equilibrio, we have

$$(B + x)\, s = B's'$$

$$\text{and } \therefore\ x = \frac{s'}{s} B' - B.$$

Also, since in a fluid, a body loses as much of its weight as is equal to the weight of the fluid displaced,

$$Bs - B\alpha = B's' - B'\alpha = W$$

the given weight of either body in the air. Hence

$$B = \frac{W}{s-\alpha},\ B' = \frac{W}{s'-\alpha}$$

$$\therefore\ x = \frac{s'}{s} \times \frac{W}{s'-\alpha} - \frac{W}{s-\alpha}$$

$$= \frac{\alpha \times (s - s')}{s\,(s-\alpha)\,(s'-\alpha)} \times W.$$

If the bulk of wood weighed in air is given, the part to be added is

$$x = \frac{\alpha.(s - s')}{s(s' - \alpha)} \times W.$$

648. By *Vince*, Prop. XVI., the part immersed of the sphere is $\frac{3}{4}$ of the whole sphere; and by *Vince's Fluxions*, p. 95, the volume of that part of a sphere which is cut off by a plane $\perp$ to the corresponding segment of a diameter x is

$$\pi\left(rx^2 - \frac{x^3}{3}\right)$$

and the whole sphere is

$$\frac{4\pi r^3}{3}$$

$\therefore$ by the question, we have

$$\pi\left(rx^2 - \frac{x^3}{3}\right) = \frac{3}{4} \times \frac{4\pi r^3}{3}$$

$$\therefore x^3 - 3rx^2 = -3r^3$$

which gives x the space required.

649. Let y be the radius of the horizontal section or $2y =$ AB, and x the depth of its centre of gravity; then (*Vince*, p. VII.)

$\pi y^2 \times x =$ pressure $=$ constant quantity, by the question; so that the equation to the generating curve of the vessel is

$$y^2 = \frac{c}{x}.$$

EF, and the axis, are asymptotes.

650. Let s be the specific gravity of the fluid, s' that of the cones. Also let $2y$ be the diameter of the base of any one cone, and x its altitude, and V the given content; then

$$\frac{\pi y^2 \times x}{3} = V$$

and by *Vince*, Prop. VII., the part immersed is

$$\frac{s'}{s} \times \mathrm{V}.$$

Hence

$$\frac{s'}{s} \times \mathrm{V} : \mathrm{V} :: y'^2 x' : y^2 x$$

where y' and x' are the radius of the base, and altitude of the part immersed.

$$\text{But } \frac{y'}{y} = \frac{x'}{x}$$

$$\therefore \frac{s'}{s} : 1 :: x'^3 : x^3$$

$$\therefore x' = x \left(\frac{s'}{s}\right)^{\frac{1}{3}}$$

Hence the surface of the part immersed is

$$\frac{2\pi y}{\sqrt{r^2 + y^2}} \int z dz = \frac{\pi y}{\sqrt{x^2 + y^2}} \times z^2$$

z being the slant side.

$$\text{But } z^2 = x'^2 + y'^2 = x'^2 + x'^2 \frac{y^2}{x^2}$$

$$= (x^2 + y^2) \left(\frac{s'}{s}\right)^{\frac{2}{3}}$$

Hence by the question

$$y \sqrt{x^2 + y^2} = \text{minimum}$$

$$\therefore y^2 (x^2 + y^2) = y^2 \left(\frac{g\mathrm{V}^2}{\pi^2 y^4} + y^2\right)$$

$$= \frac{g\mathrm{V}^2}{\pi^2 y^2} + y^4 = \text{minimum}$$

$$\therefore 4y^3 - \frac{18\mathrm{V}^2}{\pi^2 y^3} = 0$$

$$\text{and } y^6 = \frac{9\mathrm{V}^2}{2\pi^2}$$

$$\therefore\ y = \left(\frac{9V^2}{2\pi^2}\right)^{\frac{1}{6}}.$$

Hence $2y$ and x and $\therefore\ \frac{2y}{x}$ the ratio required may be found.

651. Let a be a side of the cube; then since the pressure $\propto$ surface pressed $\times$ depth of its centre of gravity $\times$ specific gravity of the fluid; the pressure arising from the upper fluid upon the sides of the upper half of the vessel is

$$P = \frac{4a^2}{2} \times \frac{a}{4} \times s = \frac{a^3 s}{2}$$

and that arising from both the fluids upon the sides of the lower part, (which is the same as would be due to a column of mercury of the altitude $\frac{a}{2} + \frac{s}{s'} \times \frac{a}{2}$, s and s' being the specific gravities of water and mercury) is

$$P' = 4 \times \frac{a^2}{2} \times \frac{a}{4} \times \left(1 + \frac{s}{s'}\right) \times s' = \frac{a^3.(s+s')}{2}.$$

The pressure upon the bottom is

$$Q = a^2 \times \left(\frac{a}{2} + \frac{a}{2} \cdot \frac{s}{s'}\right) \times s' = \frac{a^3}{2} \times (s+s')$$

$$\therefore\ P : Q :: s : s+s'$$

$$\text{and } P' : Q :: 1 : 1.$$

652. Let ABC (Fig. 103,) be the generating parabola, and AD its axis. Also let Qq be the intersection of this parabola with the surface of the fluid. Draw PV parallel to AD, bisecting Qq in V and meeting the $\perp$ qm in m. Join S (the focus) and P; draw the tangent PT and the ordinate PN.

Now when the paraboloid is in equilibrio, it is evident that the axis will be in a vertical plane, for the solid being symmetrical with regard to that axis, there is no reason why the axis should be otherwise disposed.

Again, by hydrostatics, if s denote the specific gravity of the body, and s' that of the fluid; Q the volume of the part immersed, and V that of the whole paraboloid; then, in the case of an equilibrium, we have

$$Q : V :: s : s'.$$

$$\therefore Q = \frac{sV}{s'} \dots\dots\dots\dots\dots\dots (1)$$

Another condition of the equilibrium of the floating body is that the line Gq joining the centres of gravity of the whole body and the part immersed, be vertical. These conditions expressed symbolically will afford the required solution.

Let AD $= \alpha$, BD $= \beta$, PN $= y$, AN $= x$, QV $= y'$, PV $= x'$, and $p =$ the principal parameter of the generating parabola. Then it is easily shewn (see the small treatise of Conics, published at Cambridge,) that the section Qq is an ellipse whose semi-axes are y' and $\sqrt{px'}$.

Hence its area is

$$\pi y' \sqrt{px'}$$

and the volume of the part immersed is

$$Q = \pi \int y' \sqrt{px'} \times d\,.\,PM.$$

But PM $= y' \sin. V$, and by the property of the parabola

$$y'^2 = 4SP \times x'$$

$$\therefore d\,.\,PM = dy' \times \sin. V$$

$$\text{and } Q = \frac{\pi \sqrt{p} \sin. V}{2\sqrt{SP}} \int y'^2 dy'$$

$$= \frac{\pi}{6} \sin. V \sqrt{\frac{p}{SP}} \,.\, y'^3 = \frac{sV}{s'}$$

$$\therefore \frac{y'^3}{\sqrt{SP}} = \frac{6sV}{\pi s' \sin. V \times \sqrt{.p}} \dots\dots\dots (2)$$

Again the distance gn of the centre of gravity of Q from the tangent is

$$\frac{\int PMdQ}{\int dQ} = \frac{\int y' \times \sin. V \times \frac{\pi \sin. V \sqrt{p}}{2\sqrt{SP}} y'^2 dy'}{Q}$$

$$= \frac{\pi \sin.^2 V \times \sqrt{p}}{8\sqrt{SP}} \times y'^4 \times \frac{6\sqrt{SP}}{\pi \sin. V \times \sqrt{p} \times y'^3} = \tfrac{3}{4} \sin. V \times y'.$$

Also (see Vince's Fluxions) $AG = \frac{3}{5} a$, and $AT = x$, and $nG = (\frac{2}{5} a + x)$ sin, $V = \frac{3}{4}$ sin. $V \times y' + \frac{y}{\cos. V.}$

Since ng, gG are in the same straight line. Hence

$$(\frac{3}{5} a + \frac{y^2}{p} - \frac{3}{4} y') \sin. V \cos. V = y \ldots\ldots (3)$$

But since $\angle T = \angle V$, and $y = 2x \tan. T = 2x \tan. V$, we easily get $(y^2 = px)$

$$\sin. V = \frac{p}{\sqrt{(p^2 + 4y^2)}} \text{ and}$$

$$\cos. V = \frac{2y}{\sqrt{(p^2 + 4y^2)}}.$$

Moreover

$$SP = \sqrt{\{y^2 + \left(\frac{p}{4} - x\right)^2\}}$$

$$= \sqrt{\left\{\frac{y^4}{p^2} + \frac{y^2}{2} + \frac{p^2}{16}\right\}} = \frac{y^2}{p} + \frac{p}{4}.$$

Hence equations (2) and (3) become

$$y'^2 = 8\ (p^2 + 4y^2)\frac{sV}{s'p^2} \ldots\ldots\ldots (2')$$

$$y'^2 = \frac{64}{27}\left(\frac{3}{5} a - \frac{p}{2} - \frac{y^2}{p}\right)^3 \ldots\ldots\ldots (3')$$

which being equated, and the resulting equation resolved according to y, will give y, and therefore the position of the tangent PT, &c. &c.

653. Let $2r$ be the diameter of the sphere, s the specific gravity of the sphere, and s' that of the fluid; then since the volume of that segment of the sphere, which is cut off by a plane $\perp$ to the part of the diameter x, is (*Vince*)

$$\pi\ (rx^2 - \tfrac{1}{3}x^3)$$

that part of it which corresponds to $x = \frac{3}{4}(2r) = \frac{3}{2}r$, is

$$\frac{9\pi r^3}{8}.$$

Hence (Vince's Hyd. p. 28.)

$$\frac{9\pi r^3}{8} : \frac{4\pi r^3}{3} :: s : s'$$

$$\therefore s = s' \times \tfrac{27}{32}$$

the specific gravity,

Now, when the air is admitted, let P, Q, be the parts in the upper and lower fluids; then (*Vince's* Hyd. p. 35,)

$$P : Q :: s' - s : s - 0.00122$$

$$\therefore P + Q : Q :: s' - 0.00122 : s - 0.00122$$

$$\therefore Q = \frac{s - 0.00122}{s' - 0.0'0122} \times \frac{4\pi r^3}{3}$$

which being put $= \pi\left(rx^2 - \frac{x^3}{3}\right)$ will give x the depth required.

If the fluid be common water $s' = 1$.

654. Let r be the exterior sphere, x that of the interior one, and suppose s, s' the specific gravities of iron and water respectively. Then since the body loses just its weight, or the weight of the quantity of water it displaces, we have

$$\frac{4\pi}{3}(r^3 - x^3) \times s = \frac{4\pi}{3} r^3 \times s'$$

$$\therefore x^3 = r^3 \times \frac{s'}{s} - r^3 = r^3 . \frac{s' - s}{s}$$

$$\text{and } x = \left(\frac{s' - s}{s}\right)^{\frac{1}{3}} \times r,$$

whence the proportion required.

655. Let s, s', s'', denote the given specific gravities of the bodies and of the fluid; also let Q, Q' be the magnitudes of the bodies; then the absolute weights are Qs, $Q's'$, and they lose in the fluids the weights Qs'', $Q's''$, consequently by the question, we have

$$Qs - Qs'' = Q's' - Q's''$$

$$\therefore \frac{Q}{Q'} = \frac{s' - s''}{s - s''}$$

the ratio required.

656. Let s denote the specific gravity of the solid; then if P, Q be the parts in the upper and lower fluids, we have (*Vince*, p. 35),

$$P : Q :: 6 - s : s - 3$$
$$\therefore P + Q : Q :: 3 : s - 3$$
$$\therefore s = \frac{3Q}{P+Q} + 3$$

Now since the volume of the paraboloid is

$$\pi\int y^2 dx = \frac{4\pi}{a}\int y^5 dy$$
$$= \frac{2\pi}{3a} y^6 = \frac{2\pi}{3a} . (ax)^{\frac{3}{2}}$$
$$= \frac{2\pi\sqrt{a}}{3} x^{\frac{3}{2}}.$$
$$\therefore Q = \frac{1}{2} \frac{\pi\sqrt{a}}{3} a^{\frac{3}{2}}$$
$$\text{and } P + Q = \frac{2\pi\sqrt{a}}{3} a^{\frac{3}{2}}$$

where a is the length of the axis.

Hence

$$s = \tfrac{3}{4} + 3$$

the specific gravity required.

657. The pressure on the top of the vessel is the same as it would be on the other side of it, if a column of fluid equal in height to the tube, were supported by it. Hence, the pressure required is

$$a^2 \times m a = m a^3$$

a being the side of the cube.

658. The pressure of the fluid against any section parallel to the horizon $\propto$ depth of that section in the fluid. Hence the thickness of the cylinder must increase proportionally with the depth, and the exterior form of the vessel will be that of the frustum of a cone.

659. Let s be the specific gravity required, and P, Q the magnitudes of the parts immersed in the upper and lower fluids; then (*Vince*, p. 35)

$$P : Q :: 7 - s : s -$$

$$\therefore P + Q : Q :: 4 : s - 3$$

$$\therefore s = \frac{4Q}{P+Q} + 3$$

But the volume of a paraboloid whose axis is α and radius of the base β is ($\frac{1}{2}$ its circumscribing cylinder)

$$\frac{\pi\beta^2\alpha}{2} = \frac{\pi p\alpha^2}{2}$$

p being the parameter.

$$\therefore P + Q = \frac{\pi p\alpha^2}{2}$$

and $Q = \frac{\pi p}{2} \cdot \frac{\alpha^2}{16}$

$$\therefore s = \tfrac{1}{4} + 3.$$

660. Let M, M′ be the magnitudes of the bodies, s, s' their specific gravities, and S, S′ the specific gravities of water and air; then the absolute weights of the bodies are Ms, $M's'$; and since a body, when weighed in a fluid, loses in weight that of the fluid it displaces, ∴

$$6 = Ms - MS',\ 2 = Ms - MS$$

$$7 = M's' - M'S',\ 4 = M's' - M'S$$

$$\therefore M = \frac{6}{s - S'} = \frac{2}{s - S}$$

$$M' = \frac{7}{s' - S'} = \frac{4}{s' - S}$$

$$\therefore s = \frac{3S - S'}{2},\ \text{and } s' = \frac{7S - 4S'}{3}$$

the specific gravities required.

661. Let r be the radius of the base of the cylinder, x the height of the fluid, and y the height of the section; then the pressure on the base is

$$\pi r^2 \times x;$$

the pressure on the upper surface (*Vince*, Prop. VII.)

$$2\pi r\,(x - y) \times \frac{x - y}{2}$$

and that on the lower surface is

$$2\pi r y \times \left(\frac{y}{2} + x - y\right);$$

and these pressures are all equal by the question.

$$\therefore rx = (x-y)^2 = 2y\left(x - \frac{y}{2}\right)$$

$$\text{or } rx = x^2 - 2xy + y^2 = 2xy - y$$

Hence

$$y = x\left(1 - \frac{1}{\sqrt{2}}\right)$$

$$\text{and } \therefore rx = (x-y)^2 = \frac{x^2}{2}$$

$$\therefore x = 2r, \text{ and } y = 2r\left(1 - \frac{1}{\sqrt{2}}\right).$$

662. If a be a side of the squarè, the pressures on the upper and lower halves are (*Vince*, Prop. VII.)

$$P = \frac{a^2}{2} \times \frac{a}{4} \text{ and } P' = \frac{a^2}{2} \times \left(\frac{a}{4} + \frac{a}{2}\right)$$

$$\therefore P : P' :: \tfrac{1}{8} : \tfrac{3}{8} :: 1 : 3.$$

663. Let x be the specific gravity required; then by *Vince*, Prop. XXI. and the question, we have

$$\frac{1}{n} : \frac{n-1}{n} :: b - x : x - a$$

$$\therefore n : n-1 :: b - a : x - a$$

$$\therefore x = \frac{n-1}{n} \cdot (b-a) + a = \frac{n-1}{n} \cdot b + \frac{1}{n} a.$$

664. Let M, M′ be the magnitudes of the bodies, s, s' their specific gravities, and S the specific gravity of water; then their absolute weights are

$$14 = Ms,\ 8 = M's'$$

also the weights lost in the water are

$$MS,\ M'S.$$

$$\therefore 9 = M \times (s - S) \text{ and } 7 = M' \times (s' - S)$$

$$\therefore M = \frac{14}{s} = \frac{9}{s - S}, \; M' = \frac{8}{s'} = \frac{7}{s' - S},$$

whence s and s'.

665. Let M be the magnitude of the brass when weighed in vacuo, $M + x$ when weighed in water; and let s, s' be the specific gravities of brass and gold, and S that of water; then the magnitude of the gold is

$$M' = \frac{s}{s'} \times M$$

and (*Vince*, Prop. XVII.)

$$(M + x)\,s - (M + x)\,S = M's' - M'S$$

$$\therefore M + x = M' \times \frac{s' - S}{s - S}$$

which gives

$$x = \frac{S}{s} \cdot \frac{s' - s}{s - S} \times M'.$$

666. The altitudes are inversely as the specific gravities of the fluids. Therefore

$\frac{1}{14} : \frac{1}{1} :: 10 : 140$ inches the altitude required.

667. If a be the side of the cube, the pressures upon the base and four faces of the cube are respectively

$$a^2 \times a, \text{ and } 4a^2 \times \frac{a}{2}$$

or as 1 : 2.

668. Let M be the magnitude of the globe, s' the specific gravity; then its absolute weight is

$$Ms'$$

and its weight in air is

$$W = Ms' - Ms = M.\,(s' - s)$$

and in water

$$w = M.(s' - S)$$

$$\therefore \frac{W}{w} = \frac{s' - s}{s' - S}$$

which gives

$$s' = \frac{WS - ws}{W - w}.$$

$$\text{Also } M = \frac{W}{s' - s} = \frac{W - w}{S - s}.$$

Hence if $2r$ be the diameter required, we have

$$\frac{4\pi r^3}{3} = \frac{W - w}{S - s}$$

$$2r = \left\{\frac{6}{\pi} \cdot \frac{W - w}{S - s}\right\}^{\frac{1}{3}}.$$

669. Let a be the length of the cylinder, r the radius of its base, s its specific gravity, and S that of the fluid; also let x be the depth required; then (*Vince*, Prop. XVI.)

$$\pi r^2 \times x : \pi r^2 a :: s : S$$

$$\therefore x = a \times \frac{s}{S}.$$

If s be $>$ S, as in the enunciation, the cylinder will not be at rest.

670. Let r be the radius of the base of the cone, a its altitude, and x the distance required; then since the area of the section is

$$\pi.\left(\frac{x}{a}r\right)^2 = \frac{\pi r^2}{a^2}x^2$$

the pressure upon it is

$$\frac{\pi r^2}{a^2}x^2 \times (a - x)$$

$\therefore$ by the question

$$ax^2 - x^3 = \max.$$

$$\therefore 2ax - 3x^2 = 0$$

or $x = \frac{2}{3}a$.

671. Let a, b be the vertical and horizontal sides of the rectangle, and suppose the required line to divide the base into the two parts x and $b - x$; then the areas of the two segments of the rectangle are

$$\frac{a \times x}{2}, \text{ and } ab - \frac{ax}{2};$$

the depths of their centres of gravity are also

$$\tfrac{2}{3}\, a, \text{ and } \frac{3b - 2x}{2b - x} \cdot \frac{a}{3}$$

$\therefore$ the pressures are (*Vince*, Prop. VII.)

$$\frac{a^2 x}{3} \text{ and } \frac{a^2}{6} \cdot (3b - 2x)$$

$\therefore$ by the question

$$x = \frac{3b - 2x}{2}$$

$$\therefore x = \tfrac{3}{4}\, b,$$

which determines the position of the dividing line.

672. If x denote the depth of the part immersed; then its volume or magnitude is (*Vince's* Fluxions)

$$\pi \cdot \left(rx^2 - \frac{x^3}{3}\right)$$

But by the question $x = \frac{2}{3}\, r$.

$\therefore$ the part immersed is

$$\frac{28}{81}\, \pi r^3.$$

And the whole hemisphere is

$$\frac{2\pi}{3}\, r^3.$$

$\therefore$ by *Vince*, Prop. XVI.

$$\frac{28}{81} : \tfrac{2}{3} :: s : S :: 14 : 27$$

s and S being the specific gravities of the body and the fluid.

673. Let w be the given weight of the vessel, r its radius; then since the volume of any segment of a sphere whose axis is a, is

$$\pi \left(ra^2 - \frac{a^3}{3}\right)$$

the volume of the fluid displaced is

$$\pi\left\{r.\left(\frac{r}{3}\right)^2-\tfrac{1}{3}.\left(\frac{r}{3}\right)^3\right\}$$

$$=\frac{8\pi r^3}{81}.$$

Hence if s be the specific gravity of the fluid, we have

$$w=\frac{8\pi r^3}{81}s\ (\textit{Vince}, \text{Prop. XIV.})$$

$$\therefore s=\frac{81w}{8\pi r^3}.$$

Again, let x be additional weight required; then since by the question this, together with the vessel, is to have the effect of displacing

$$\pi.\left\{r.\left(\frac{2r}{3}\right)^2-\tfrac{1}{3}.\left(\frac{2r}{3}\right)^3\right\}$$

$$\text{or } \frac{28\pi}{81}r^3$$

of the fluid.

$$\therefore x=\frac{28\pi}{81}r^3\times s=\frac{28\pi}{81}r^3\times\frac{81w}{8\pi r^3}$$

$$=\frac{28}{8}w=\tfrac{7}{2}w.$$

the weight required.

674. Let a be the altitude of the cylinder, r the radius of its base; then the pressure upon the base is measured by

$$\pi r^2a.$$

Again, let x be the breadth of the first annulus; then the pressure upon it is

$$2\pi rx\times\frac{x}{2}$$

which by the question gives

$$\pi rx^2=\pi r^2a$$

$$\therefore x=\sqrt{ra}.$$

Again, let x' be the breadth of the second annulus; then the pressure upon it is

$$2\pi r x' \times \left(\frac{x'}{2} + \sqrt{ra}\right)$$

which gives

$$x'^2 + 2\sqrt{ra}\, x' = ra$$
$$\therefore x' = -\sqrt{ra} + \sqrt{2ra}$$
$$= \sqrt{ra} \times (\sqrt{2} - 1).$$

Again, let x'' be the breadth of the third annulus; then we

$$2\pi r x'' \times \left(\frac{x''}{2} + x' + x\right) = \pi r^2 a$$
$$\text{or } 2x'' \left(\frac{x''}{2} + \sqrt{2ra}\right) = ra$$
$$\therefore x''^2 + 2\sqrt{2ra}\, x'' = ra$$
$$\therefore x'' = -\sqrt{2ra} + \sqrt{3ra}$$
$$= \sqrt{ra}\, (\sqrt{3} - \sqrt{2})$$

Similarly the breadth of the fourth annulus is

$$\sqrt{ra}\, (\sqrt{4} - \sqrt{3})$$

and so on.

Hence the breadth of the p^{th}. annulus is

$$\sqrt{ra} \times (\sqrt{p} - \sqrt{p-1}).$$

Now the height is

$$a = \sqrt{ra} \times \{1 + (\sqrt{2} - 1) + (\sqrt{3} - \sqrt{2}) + \&c.$$
$$(\sqrt{n} - \sqrt{n-1})\} = \sqrt{ra} \times \sqrt{n}$$
$$\therefore a = rn$$

and consequently the breadth of the p^{th}. annulus becomes

$$r\sqrt{n} \times (\sqrt{p} - \sqrt{p-1}).$$

675. Let S be the specific gravity of the fluid, s of the atmosphere, and s' that of the paraboloid; then if P be the part in air, and Q the part in the fluid, we have (*Vince*, Prop. XXI.)

$$P : Q :: S - s' : s' - s$$
$$\therefore P + Q : Q :: S - s : s' - s$$
$$\therefore s' = \frac{Q}{P+Q} \cdot (S - s) + s.$$

But if α be the axis of the paraboloid, and β the radius of its base,

$P + Q = \frac{1}{2} \times \frac{3}{4}$ circumscribing cylinder of the whole paraboloid

$$= \frac{\pi\beta^2\alpha}{2} \times \frac{3}{4} = \frac{3\pi p\alpha^2}{8}$$

and since $\frac{\pi y^2 x}{2} = \frac{\pi p x^2}{2}$ is the volume of any part of the paraboloid corresponding to the axis x, it is easily shewn that the axis of the frustum is

$$\alpha - \frac{\alpha}{2} = \frac{\alpha}{2}.$$

and

$$Q = \frac{\pi p}{2} . \left\{ \left(\frac{3}{4} \alpha \right)^2 - \left(\frac{1}{2} \alpha\right)^2 \right\}$$

$$= \frac{5\pi p}{32} \alpha^2$$

Hence

$$s' = \frac{5}{12} \times (S - s) + s$$

$$= \frac{5S + 7s}{12}.$$

676. Let s, s' be the specific gravities of the fluids; M, M′ their magnitudes; also let S be the specific gravity required; then by the question, the magnitude of the mixture is

$$(M + M')\, n$$

Again, the sum of the weights of the ingredients = weight of the mixture;

or

$$MS + M'S' = (M + M')\, n \times S$$

$$\therefore\ S = \frac{MS + M'S'}{n(M + M')}$$

the specific gravity required.

677. Let s, s', S be the specific gravities of iron, the fluid and water respectively, s', as the problem requires, being supposed

less than S. Let also r be the radius of the sphere, a the thickness of the shell, and x the required depth (we'll call it) of the shell.

Then (*Vince's* Fluxions) the volume of the exterior segment is

$$M = \pi\left(rx^2 - \frac{x^3}{3}\right) = \frac{\pi . x^2}{3} . (3r - x)$$

and that of the interior segment is

$$m' = \pi \left\{\overline{r - a}. (x - a)^2 - \frac{(x-a)^3}{3}\right\} = \frac{\pi(x-a)^2}{3} . (3r - 2a - x)$$

Hence the bulk of the iron is

$$m = \frac{\pi}{3}\{x^2 . \overline{3r - x} - (x - a)^2 . (3r - 2a - x)\}$$

$$= \frac{\pi}{3}\{x^3 - 3rx^2 + (6ar - 4a^2)\, x + 2a^3\}.$$

Now, since the vessel just floats the weight of it and its contents = weight of the water it displaces (*Vince*, Prop. XIV.)

$$\therefore\ m s + m' s' = MS$$

which gives by substitution

$$\{x^2(3r - x) - (x - a)^2 . (3r - 2a - x)\}\, s +$$
$$(x - a)^2 . (3r - 2a - x)\, s' = x^2 . (3r - x)\, S$$

which reduces to

$$x^3 - 3r\, x^2 + 3a . (2r - a)\frac{s - s'}{S - s'}\, x - a^2 . (3r - 2a)\frac{s - s'}{S - s'} = 0.$$

whence by approximation, or otherwise, the required value of x may be found.

678. The centres of gravity of the three surfaces pressed will be in the same point, so that the pressures will be as the surfaces, or as

$$\pi r^2, \frac{r^2}{2}, r^2$$

or as

$$\pi,\ \tfrac{1}{2}, \text{ and } 1.$$

679. By *Vince*, Prop. XVI.

$$Q : P + Q :: 2 : 3.$$

But if x be the depth required, and α the axis of the paraboloid,

$$Q = \frac{1}{2}\,\pi y^2 x = \frac{\pi p x^2}{2} \text{ (p being the parameter),}$$

$$\text{and } P + Q = \frac{\pi p \alpha^2}{2}$$

$$\therefore x^2 = \frac{2}{3}\,\alpha^2$$

$$\text{and } x = \alpha \sqrt{\frac{2}{3}}.$$

680. Let a be a side of the cube; then the distance of the centre of percussion or centre of pressure of the face which is loose from the surface of the fluid, is (*Vince's Flux.* p. 130,)

$$\frac{3}{4}\,a.$$

Hence the locus of this centre is the straight line in the face parallel to the surface, and distant from it by

$$\frac{3}{4}\,a.$$

But there is no reason, from the symmetry of the vessel, why the centre should be on one side of the middle of this line, rather than on the other. It is, therefore, in the middle of this line; hence the point of application of the required force. Moreover the pressure $\perp$ to the face is

$$a^2 \times \frac{a}{2} \times s$$

s being the specific gravity of the fluid. Hence the force to be applied in a direction $\perp$ to the face, and at the point, as found above, is

$$\frac{a^3 s}{2}.$$

681. Let r, r', be the radii of any two of these circles, d, d' the depths of their centres of gravity; then the pressures are as

$$\pi r^2 \times d,\ \pi r'^2 d'.$$

But by the question, these pressures are as

$$r^3,\ r'^3,$$

$$\therefore r^2 d : r'^2 d' :: r^3 : r'^3$$

$$\text{or } d : d' :: r : r',$$

and so on.

682. Let α be the axis, β the radius of the base of the hollow part of the paraboloid; then its volume is

$$\frac{\pi\beta^2\alpha}{2} = \frac{\pi p\alpha^2}{2}$$

p being the parameter of the generating parabola.

Hence the quantity of fluid put into it is

$$\frac{\pi p\alpha^2}{2} \times \frac{1}{n} \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

and if x be the depth of that fluid, we have

$$\frac{\pi p x^2}{2} = \frac{\pi p \alpha^2}{2n}$$

$$\therefore x = \frac{\alpha}{\sqrt{n}} \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Again, let r be the radius of the sphere, s its specific gravity, and S the specific gravity of the fluid; then if Q be the part immersed, and P the other part, we have

$$Q : P + Q :: s : S$$

$$\therefore Q = \frac{s}{S} . (P + Q) = \frac{4\pi r^3}{3} . \frac{s}{S}$$

Hence the portion of the paraboloid occupied up to the surface of the water is now

$$\frac{\pi p \alpha^2}{2n} + \frac{4\pi r^3 s}{3S} = \frac{\pi p y^2}{2}$$

if y denote the depth of the water; which gives y, and $\therefore y - x$, or the height required.

683. Let R, r, be the radii of the generating circles of

the unequal cycloids; then since the equation to the latter referred to its vertex is

$$y = \sqrt{(2rx - x^2)} + \text{vers.}^{-1}\, x$$

we have

$$ds = dx\sqrt{\left(1 + \frac{dy^2}{dx^2}\right)} = \sqrt{\frac{2r}{x}} \times dx.$$

Hence the distance of its centre of gravity from the vertex is

$$\frac{\int x ds}{s} = \frac{\sqrt{2r}\int\sqrt{x}dx}{s}$$

$$= \frac{2\sqrt{2r}}{3} \cdot \frac{x^{\frac{3}{2}}}{s} = \frac{2\sqrt{2r}}{3} \times \frac{(2r)^{\frac{3}{2}}}{2 \cdot (2r)}$$

$$= \frac{2}{3} r \text{ for the whole cycloid,}$$

and the distance of the centre of gravity of the former curve from its vertex is, in like manner,

$$\frac{2}{3} \text{R}.$$

Now, when at rest, the condition is that the line joining the centres of gravity of the part out of the fluid, and of the part immersed, shall be vertical, that is, $\perp$ to the surface of the fluid. Hence we may find the inclination of the line joining the extremities of the rod to the surface of the water, and therefore the position required.

For the distance between the centres of the bases of the component cycloids is

$$\pi r + \pi \text{R} = \pi \cdot (r + \text{R}).$$

$\therefore$ if the line joining the centres of gravity cut this distance at an angle θ, and divide it into the two portions

$$x \text{ and } \pi\, (r + \text{R}) - x$$

we easily get

$$\frac{4}{3} r = x \tan. \theta$$

$$\frac{4}{3} \text{R} = (\pi \cdot \overline{\text{R}+r} - x) \tan. \theta$$

$$\therefore \frac{4}{3} R = \pi . (R + r) \tan. \theta - \frac{4r}{3}$$

$$\therefore \tan. \theta = \frac{4}{3\pi}$$

which gives θ, and therefore the complement of θ, or the inclination sought.

684. Let r be the radius of the sphere; then the pressure on the base is

$$P = \pi r^2 \times r = \pi r^3.$$

Again, since the surface of the hemisphere is

$$2\pi r^2$$

and the depth of its centre of gravity (*Vince's Flux*. p. 119,)

$$\frac{r}{2}$$

the pressure upon the curved surface is

$$P' = 2\pi r^2 \times \frac{r}{2} = \pi r^3$$

$$\therefore P : P' :: 1 : 1.$$

$$\text{or } P = P'.$$

685. Let r be the radius of the interior; then the surface is

$$4\pi r^2$$

and the depth of the centre of gravity is r. Therefore if s be the specific gravity of the fluid, the internal pressure of

$$P = 4\pi r^2 \times r \times s = 4\pi r^3 s.$$

Also the weight of the fluid is

$$W = \frac{4\pi r^3}{3} \times s$$

$$\therefore P : W :: 1 : \frac{1}{3} :: 3 : 1.$$

686. Let r be the radius of the inner sphere; then if y be the radius of any section parallel to the base, and $a-x$ (a being the altitude of the fluid,) its depth in the fluid, the pressure upon its circumference $\perp$ surface is measured by (*Vince*, Prop. VII.)

$$2\pi y\,(a-x)$$

and resolving this, the pressure vertically upon this section is

$$2\pi y\,(a-x) \times \frac{y}{r} \times s = \frac{2\pi}{r}(r^2-x^2)\,(a-x)$$

Hence the whole vertical pressure is

$$P = \frac{2\pi}{r}\int(r^2-x^2)\,(a-x)\,ds = 2\pi\int\sqrt{(r^2-x^2)}\,(a-x)\,dx$$

$$\therefore\ P = a\pi x\sqrt{(r^2-x^2)} + \frac{2\pi}{3}(r^2-x^2)^{\frac{3}{2}} + ra\pi\ \sin.^{-r}x + C$$

Let $x=a$; then $P=0$, and

$$C = -\left\{a\pi a\,.\,\sqrt{(r^2-a^2)} + \frac{2\pi}{3}.(r^2-a^2)^{\frac{3}{2}} + ra\pi\ \sin.^{-r}a\right\}$$

Hence when $x=0$; the whole vertical pressure of the fluid whose altitude is a (S being the specific gravity of the fluid) is

$$Q = \left\{\frac{2\pi}{3}.r^3 - \pi a^2.\sqrt{r^2-a^2} - \frac{2\pi}{3}.(r^2-a^2)^{\frac{3}{2}} - ra\pi\ \sin.^{-r}a\right\} \times S.$$

Again, if S′ denote the specific gravity of the vessel, its solid part being in magnitude

$$\frac{2\pi(r+t)^3}{3} - \frac{2\pi}{3}r^3 = \frac{2\pi}{3}.(\overline{r+t}|^3 - r^3)$$

its weight is

$$W = \frac{2\pi}{3}.\{(r+t)^3 - r^3\} \times S';$$

which, by the question, being equated to Q, will give

$\frac{S}{S'}$, as required.

In the problem $a = \frac{r}{2}$

$$\therefore Q = \frac{\pi r^3}{12} \times (8 - 6\sqrt{3} - \pi) \times S = \frac{2\pi}{3}\left\{(r+t)^3 - r^3\right\} \times S'$$

$$\therefore \frac{S'}{S} = \frac{r^3}{8} \times \frac{8 - 6\sqrt{3} - \pi}{(r+t)^3 - r^3}.$$

687. The surface pressed is the same in both cases; ∴ the pressures will be as the depths of the centres of gravity. Now the distance of the centre of gravity of the surface from the vertex is

$$\frac{\int x d\,(\text{surf.})}{\text{surf.}} = \frac{2}{3} \text{ altitude.}$$

Hence the pressures are as

$$\frac{2}{3} \text{ and } \frac{1}{3} \text{ or as 2 to 1.}$$

688. Let α be the axis of the paraboloid, and β the radius of its base, s its specific gravity, and s' that of the fluid; then the part immersed is (see *Vince*, Prop. XVI.)

$$Q = (P + Q) \cdot \frac{s}{s'} = \frac{\pi \beta^2 \alpha}{2} \times \frac{s}{s'}.$$

Hence, if x denote the depth of the axis immersed, and p the parameter of the generating parabola, we have

$$\frac{\pi y^2 x}{2} = \frac{\pi p x^2}{2} = \frac{\pi p \alpha^2}{2} \cdot \frac{s}{s'}$$

$$\therefore x = \alpha \sqrt{\frac{s}{s'}}.$$

Again, the distance of the centre of gravity of the whole solid, and of the part immersed from the vertex, are respectively (*Vince's Fluxions*, p. 117,)

$$\frac{2}{3} \alpha \text{ and } \frac{2}{3} x = \frac{2}{3} \alpha \sqrt{\frac{s}{s'}}.$$

Hence the distance between these two centres is

$$d = \frac{2}{3} a \times \left(1 - \sqrt{\frac{s}{s'}}\right) \ldots\ldots\ldots\ldots (1)$$

Again, the area of the section made by the surface of the water is

$$\pi y^2 = \pi p x = \pi p a \sqrt{\frac{s}{s'}}$$

and if du denote an element of this area, and x denote the distance of this particle from the diameter about which the solid will revolve (if at all); then by *Poisson's Mech.* vol. II. p. 416, the body will be *stable, unstable, or in a state of indifference,* according as

$$\int x^2 du - d \times Q$$

is positive, negative, or zero, $\int x^2 du$ being taken through the whole extent of the area $\pi p a \sqrt{\frac{s}{s'}}$.

Now if generally R be the radius of this section, it is easily found that

$$\int x^2 du = \frac{R^4 \pi}{4}.$$

Hence the equilibrium is *stable, unstable,* or *indifferent,* according as

$$\frac{R^4 \pi}{4} - d \times Q \ldots\ldots\ldots\ldots\ldots (2)$$

is *positive, negative,* or *zero.*

In the present case we have

$$R^2 = px = pa \sqrt{\frac{s}{s'}}$$

$$d = \frac{2}{3} a \left(1 - \sqrt{\frac{s}{s'}}\right)$$

$$\text{and } Q = \frac{\pi p a^2}{2} \cdot \frac{s}{s'}$$

and substituting, we get

$$pa^2 \frac{s}{s'} \pi \times \left\{\frac{p}{4} - \frac{a}{3} \cdot \left(1 - \sqrt{\frac{s}{s'}}\right)\right\}.$$

which will shew the state of the equilibrium when p, α and $\frac{s}{s'}$ are given in numbers.

When the paraboloid tends to fall, α must be increased to α', so that

$$\frac{p}{4} - \frac{\alpha'}{3} \cdot \left(1 - \sqrt{\frac{s}{s'}}\right) = 0$$

$$\text{or } \alpha' = \frac{3}{4} \cdot \frac{p}{1 - \sqrt{\frac{s}{s'}}}.$$

689. Let ABC (Fig. 104) be a vertical section, parallel to the base of the horizin, and the $\angle$B being immersed, let DE be the surface of the water, and suppose E the given point which meets the water; also if AC, DE, be bisected in F, H, and BF, BH be trisected in G, g; G, g, will be the centres of gravity of the whole section ABC, and of the part immersed DBE. Now the condition of equilibrium of the prism is evidently the same for the prism as for this section; it is therefore *that* Gg *be vertical or* $\perp$ DE.

$$\text{Hence FG} : \text{H}g :: \frac{1}{3}\text{BF} : \frac{1}{3}\text{HB}$$

$$:: \text{BF} : \text{HB}$$

$\therefore$ Gg is parallel to FH

and FH is also $\perp$ DE.

Moreover DE is bisected in H.

Therefore

FD = FE,

which gives the required position.

Since in the same fluid the volume of the part immersed is constant, the area of DBE is constant, and the locus of the points H, G is an hyperbola (see Problem 44, vol. II.); to find whose equation, let

$$\left.\begin{array}{l} \text{BC} = a, \\ \text{BA} = c \\ \text{BF} = m \end{array}\right\} \quad \left.\begin{array}{l} \angle\,\text{FBC} = \alpha \\ \angle\,\text{FBA} = \beta \\ \text{BE} = x \\ \text{BD} = y \end{array}\right\}$$

and the specific gravities of the prism and fluid s and s'; then it is easily found that

$$x^4 - 2m\cos.\alpha \times x^3 + \frac{2msac\cos.\beta}{s'}x - \frac{s^2a^2c^2}{s'^2} = 0$$

whose roots will give all the positions of equilibrium which the prism can take.

690. Let a be the length of the straight line, and x, y, and $a - \overline{x+y}$ the three portions required; then the pressures on them are

$$x + \frac{x}{2},\ y \times \left(\frac{y}{2} + x\right), \text{ and } (a - \overline{x+y}) \times \left(\frac{a - \overline{x+y}}{2} + x + y\right)$$

and by the question

$$x^2 = y.\ (y + 2x)$$

$$x^2 = (a - \overline{x+y})\ (a + x + y).$$

$$= a^2 - (x+y)^2$$

Hence

$$y^2 + 2x.\ y = x^2$$

$$\therefore y = -x + \sqrt{2x^2} = x.\ (\sqrt{2} - 1)$$

$$\therefore x^2 = a^2 - 2x^2$$

$$\therefore x = \frac{a}{\sqrt{3}}$$

$$y = \frac{\sqrt{2} - 1}{\sqrt{3}}a$$

$$\text{and } a - \overline{x+y} = \frac{\sqrt{3} - \sqrt{2}}{\sqrt{3}}a$$

the parts required.

691. Let x be the side of the cube; then the depth of the part immersed is $x - 100$, and the volumes of the whole iceberg and of the part immersed are

$$x^3 \text{ and } x^2 \times (x - 100).$$

$\therefore$ (*Vince* prop. XVI)

$$x^3 : x^2 \times (x - 100) :: 1.0263 : .9214$$

which gives

$$x = 978.36 \text{ feet.}$$

The same process of reasoning as is used in (687,) will shew whether the position be stable or unstable.

692. Let s ounces be the specific gravity of the wood; that of water being 1000 ounces; then the specific gravity of lead is 11,000, and the magnitudes of the wood and lead are

$$\frac{12 \times 16}{s} \text{ cubic feet,}$$

$$\text{and } \frac{22 \times 16}{11000} \text{ cubic feet}$$

respectively. Hence, since the water displaced weighs 8 lb., we have

$$\left(\frac{12}{s} + \frac{2}{1000}\right) 16 + 1000 = 8 \times 16$$

which gives

$$s = 2000.$$

693. This is nothing more than "*Required the greatest segment of a sphere which can be described in a given cone.*"

If r denote the radius of the sphere, and x the depth of the segment in the cone, the volume of that segment is

$$\pi\left(rx^2 - \frac{x^3}{3}\right) = \frac{\pi x^2}{3}.(3r - x).$$

But it is easily shewn that

$$r = (a - x)\frac{b}{a}\left\{\frac{b}{a} + \sqrt{\frac{b^2}{a^2} + 1}\right\} \dots (1)$$

where $a =$ the altitude of the cone, and b the radius of its base.

$\therefore$ by the question

$$x^2\left\{3\frac{b}{a}.(b + \sqrt{a^2 + b^2}) - \left(\frac{3b}{a^2}.(b + \sqrt{a^2 + b^2}) + 1\right)x\right\}$$

$= \text{maximum}.$

Hence by the rule we get

$$x = \frac{2ab.(b + \sqrt{a^2 + b^2})}{3b(b + \sqrt{a^2 + b^2}) + a^2},$$

which, by substituting in (1), will give the radius of the required sphere.

HYDRODYNAMICS.

DISCHARGE OF FLUIDS.

694. THE velocity of the water issuing through the orifice $\propto \sqrt{}$ (depth of the orifice in the fluid). See *Vince* prop. XXXVIII; and the quantity of the fluid injected $\propto$ time $\times$ velocity. But the cubes being equal, these quantities are the same, and the time $\propto \frac{1}{\text{vel.}}$. Hence the times required are as

$$\frac{1}{\sqrt{4}} : \frac{1}{\sqrt{1}} :: 1 : 2.$$

695. Let z be the distance from the surface of the water at which the orifice must be made; then the velocity with which the fluid begins to describe the parabola is that acquired down z (*Vince* prop. XXXIX), and the parameter of the parabola is $4z$. Hence by the property of the parabola, and the question,

$$(32)^2 = 4z \times (32 - z)$$

$$\text{and } z = \frac{32}{2} = 16.$$

696. Generally, if A denote the area of the orifice, x the depth of the fluid to be discharged, and X the variable area of the descending surface; then the time of discharging the fluid is (*Vince's* Fluxion, p. 268)

$$t = \frac{1}{A\sqrt{g}} \times \int \frac{Xdx}{\sqrt{x}}$$

integrated between $x = x$, and $x = 0$.

Hence, if r be the radius of the base of the cylinder, when its axis is horizontal, we have

$$X = 2r \times 2(2rx - x^2)^{\frac{1}{2}} = 4r.(2rx - x^2)^{\frac{1}{2}}$$

and when the axis is vertical

$$X' = \pi r^2$$

$$\therefore t = \frac{1}{A\sqrt{g}} \times 4r \int \sqrt{2r - x} \times dx = \frac{8r}{3A\sqrt{g}} \{(2r)^{\frac{3}{2}} - (2r-x)^{\frac{3}{2}}\}$$

$$= \frac{8}{3A\sqrt{g}} . (2r)^{\frac{3}{2}}, \text{ when } x = 2r ; .$$

$$\text{and } t' = \frac{\pi r^2}{A\sqrt{g}} \int \frac{dx}{\sqrt{x}} = \frac{2\pi r^2}{A\sqrt{g}} . \sqrt{x}$$

$$= \frac{2\pi r^2}{A\sqrt{g}} \sqrt{2r}, \text{ when } x = 2r.$$

$$\therefore t : t' :: \frac{8}{3} : \pi r :: 8 : 3\pi r.$$

697. By the preceding problem it readily appears that the time of emptying any depth is

$$t = \frac{2\pi r^2}{A\sqrt{g}} \times \sqrt{x} \propto \sqrt{x}.$$

or $x \propto t^2$.

Hence

$4^2 : 16 :: 1$: space described in the last hour $= 1$ foot.

$4^2 : 16 :: 2^2$: space described in the two last hours $= 4$ feet.

Similarly $(3^2 = 9)$ feet is the space described in the three last hours; so that the spaces required are

$$16 - 9 = 7, \; 9 - 4 = 5 \; 4 - 1 = 3 \text{ and } 1.$$

698. By 696, the time of discharging the depth x is

$$t = \frac{1}{A\sqrt{g}} \int 2a \sqrt{(2r - x)}\, dx$$

$$= \frac{4a}{3A\sqrt{g}} \times \{(2r)^{\frac{3}{2}} - (2r - x)^{\frac{3}{2}}\}$$

a being the axis of the cylinder, and r the radius of its base.

Let $x = 2r$ and r; then we get $t : t' :: (2r)^{\frac{3}{2}} : r^{\frac{3}{2}} . (2^{\frac{3}{2}} - 1)$, and $t : t - t' :: 2^{\frac{3}{2}} : 1$

the proportion required.

699. Let p be the principal parameter of the generating parabola, P that corresponding to the diameter at the extremity of which the orifice is made.

Also let β be the inclination of the axis to the horizon or to the surface of the fluid; then it may easily be shewn that the descending surface for any depth of the fluid x is an ellipse, whose semiaxes are

$$a = \sqrt{\frac{Px}{\sin. \beta}}, \text{ and } b = \sqrt{\frac{px}{\sin. \beta}}.$$

Hence (see 696)

$$X = \pi ab = \frac{\pi x}{\sin. \beta} \sqrt{Pp}$$

$$\text{and } t = \frac{\pi \sqrt{Pp}}{A \sin. \beta \sqrt{g}} \times \int \sqrt{x} dx$$

$$= \frac{2\pi \sqrt{Pp}}{3A \sin. \beta \sqrt{g}} \times x^{\frac{3}{2}}$$

Again, if D be the given $\perp$ distance from the surface to the base, we have

$$2 \cos. \beta \sqrt{\frac{Px}{\sin. \beta}} = D$$

$$\therefore x = \frac{D^2}{4P} \frac{\sin. \beta}{\cos.^2 \beta}$$

the whole depth of the fluid; which being substituted in the above expression, will give the time required.

700. Let l, a, r, be the slant side, axis and radius of the base of the given cone, and x the altitude of the surface of the fluid filling any portion of the cone; then, since this surface is a parabola, whose axis and base are

$$\frac{l.}{2r}\left(2r - \frac{xl}{a}\right), \text{ and } 2\sqrt{\left(2r.\frac{xl}{a} - \frac{x^2 l^2}{a^2}\right)}$$

we have (see 696)

$$X = \frac{2}{3} \cdot \frac{l}{r} \cdot \left(2r - \frac{xl}{a}\right) \sqrt{\left(\frac{2rl}{a} x - \frac{l^2}{a^2} x^2\right)}$$

$$\text{and } t = \frac{2l^{\frac{3}{2}}}{3ra^2 A\sqrt{g}} \times \int (2ar - lx) \times \sqrt{(2ar - lx^2)}\, dx$$

$$= \frac{2l^{\frac{3}{2}}}{3ra^2 A \sqrt{g}} \times \{\sqrt{2ar - lx^2} \times (arx - \frac{l}{3}x^2 + \frac{2a}{3}r)$$

$$\perp \frac{a^2 r^2}{\sqrt{l}} \cdot \sin.^{-1} x \sqrt{\frac{l}{2ar} - \frac{1}{3}} \cdot (2ar)^{\frac{3}{2}}\}$$

See *Hirsch's* Integral Tables, pp. 130, and 122.

Let $x = \frac{2ar}{l}$; then the time of emptying the whole cone is obtained.

701. Let the sluice be (as sluices usually are,) rectangular, two of its sides, (a) being parallel to the surface of the fluid. Also, let H, and h be the distances of these sides from the surface of the fluid, and supposing this finite orifice $a \times (\text{H} - h)$ to be composed of innumerable indefinitely small ones, $a \times dx$, distant from the surface by x, the quantity of fluid run out in the time t will be denoted by

$$\begin{aligned} Q &= \int a dx \times t \times \text{velocity} \\ &= at \int dx \sqrt{gx} \\ &= \frac{2at\sqrt{g}}{3} \times (\text{H}^{\frac{3}{2}} - h^{\frac{3}{2}}) \end{aligned}$$

on the supposition that the velocity through each small orifice is due to half its depth in the fluid.

Now the reservoir being supplied at a *given* rate, suppose the supply to be the given quantity Q in the given time t; then H being also known by the question, we have

$$h = (\text{H}^{\frac{3}{2}} - \frac{3}{2at\sqrt{g}} \times Q)^{\frac{2}{3}}$$

and $\therefore$ H $-$ h, or the height required.

702. Let α be the axis of the paraboloid, β the radius of its base, x the distance of any horizontal section from the vertex; then

$$X = \pi y^2 = \pi p x$$

p being the parameter of the generating parabola. Hence (see 696)

$$t = \frac{1}{A\sqrt{g}} \int \frac{Xdx}{\sqrt{x}}$$

$$= \frac{2\pi p}{3A\sqrt{g}} x^{\frac{3}{2}} \propto x^{\frac{3}{2}}$$

Let $x = a$, $x = \frac{a}{2}$; then

$$t : t' :: a^{\frac{3}{2}} : \frac{a^{\frac{3}{2}}}{2^{\frac{3}{2}}} :: 2^{\frac{3}{2}} : 1$$

$$\therefore t - t' : t' :: 2^{\frac{3}{2}} - 1 : 1$$

the proportion required.

703. Let a be the common altitude of the cylinder and cone, and r the radius of their common base; then since (696)

$$t \propto \int \frac{Xdx}{\sqrt{x}}$$

$$\text{and } X : X' :: \pi r^2 : \pi\left(\frac{rx}{a}\right)^2$$

$$:: a^2 : x^2$$

$$\therefore t : t' :: \int \frac{a^2 dx}{\sqrt{x}} : \int x^{\frac{3}{2}} dx$$

$$:: 2a^2\sqrt{x} : \frac{2}{5} x^{\frac{5}{2}}$$

$$:: a^{\frac{5}{2}} : \frac{a^{\frac{5}{2}}}{5} :: 5 : 1$$

when $x = a$, or for the whole solids.

704. Let a be the altitude of the cylinder, $2r$ the diameter of its base, and A the area of the aperture. Also let x be any variable height of the surface of the water; then the quantity discharged in 1″ is

$$A\sqrt{gx}$$

and, if the constant supply in 1″ be $A\sqrt{gm}$ then the rate of evacuation is

$$A\sqrt{g} \times (\sqrt{x} - \sqrt{m})$$

Also by the evacuation the surface descends with the velocity

$$\sqrt{gx} \times \frac{A}{\pi r^2}$$

and by means of the supply it ascends at the same time with the velocity

$$\sqrt{gm} \times \frac{A}{\pi r^2}$$

so that upon the whole, the descending velocity is

$$\frac{A\sqrt{g}}{\pi r^2} \times (\sqrt{x} - \sqrt{m}).$$

Hence the time through x is

$$t = \frac{\pi r^2}{A\sqrt{g}} \times \int \frac{dx}{\sqrt{x} - \sqrt{m}}$$

$$= \frac{2\pi r^2}{A\sqrt{g}} \times \left\{\sqrt{x} + \sqrt{m}\, l. (\sqrt{x} - \sqrt{m}) + C\right\}$$

and putting $x = a$ and $x = m$, and taking the difference between the results, we have the time in which the influx is equal to the efflux, *viz.*

$$\frac{2\pi r^2}{A\sqrt{g}} \times \left\{\sqrt{a} - \sqrt{m} + \sqrt{m} \,.\, l. \frac{\sqrt{a} - \sqrt{m}}{\sqrt{m} - \sqrt{m}}\right\}$$

which, being infinite, shews that there can never exist an equality between the efflux and influx.

705. Let πy^2 be any circular section of the vessel whose altitude is x; and A the area of the orifice, then the velocity at the orifice is

$$\sqrt{gx}$$

and that of the descending surface is

$$\frac{A}{\pi y^2} \sqrt{gx} = \text{const.} = c$$

by the question.

$$\therefore\ y^2 = \frac{A}{c\pi} \sqrt{g} \times \sqrt{x}$$

the equation to a parabola of the fourth order.

If a be the given altitude, then the radius of the base is

$$\left\{\frac{A}{c\pi}\sqrt{ga}\right\}^{\frac{1}{2}}$$

which gives the required dimensions of the vessel.

706. Let r be the radius of the sphere, and x the altitude of any common section parallel to the horizon of the sphere and cylinder; then (see 696)

$$X = \pi r^2, \text{ and } X' = \pi y^2 = \pi(2rx - x^2)$$

and since $t \propto \int \frac{Xdx}{\sqrt{x}}$

$$\therefore t : t' :: r^2 \int \frac{dx}{\sqrt{x}} : \int (2r - x)\sqrt{x}dx$$

$$:: 2r^2\sqrt{x} : \frac{4}{3} r x^{\frac{3}{2}} - \frac{2}{5} x^{\frac{5}{2}}$$

Let $x = 2r$; then

$$t : t' :: 15 : 8,$$

the proportion required.

707. Here the area of the descending surface is ($2y$ being a side of the generating square when at a distance x)

$$X = (2y)^2 = 4y^2 = 4px^{\frac{3}{2}}$$

from the equation to the semicubical parabola. Hence (see 696)

$$t = \frac{1}{A\sqrt{g}} \int 4pxdx$$

$$= \frac{2p}{A\sqrt{g}} x^2 = \frac{2p}{A\sqrt{g}} \times a^2$$

a being the altitude of the vessel.

708. Let r be the radius of the sphere, and x the altitude of any horizontal section made by the water; then (696)

$$X = \pi y^2 = \pi(2rx - x^2)$$

$$\text{and } X' = \pi y^2 = \pi(r^2 - x'^2)$$

according as the vertex or base is downwards.

$$\therefore t : t' :: \int(2r - x)dx : \int \frac{(x^2 - x'^2)}{\sqrt{x}} r$$

$$:: \frac{4r}{3} x^{\frac{3}{2}} - 2x^{\frac{5}{2}} : 2r^2\sqrt{x'} - \frac{2}{5} x'^{\frac{5}{2}}$$

Let $x = x' = 2r$; then

$$t : t' :: 8 : 3.$$

709. Let α be the axis, and 2β the base of either parabola, and supposing them to consist of an indefinite number of orifices $2y \times dx$, (y being the ordinate, and x the abscissa of the parabola measured from the surface of the fluid,) the quantity of water discharged in the time t, when the vertex is upwards is

$$Q = \int 2ydx \times t \times \text{velocity at the orifice } 2ydx$$
$$= 2t\int ydx \sqrt{gx}$$
$$= 2t \sqrt{pg} \int xdx$$
$$= t \sqrt{pg}\ x^2$$

and for the whole parabola

$$Q = t \sqrt{pg} \times \alpha^2.$$

Again, the quantity discharged, when the vertex is downwards in the same time is

$$Q' = 2t\int ydx \sqrt{gx}$$

But in this case $y = \sqrt{p.(\alpha - x)}$

$$\therefore Q' = 2t \sqrt{pg}\int\sqrt{(\alpha x - x^2)}\, dx$$

$$= 2t\sqrt{pg} \times \left\{ \left(\frac{x}{2} - \frac{\alpha}{4}\right)\sqrt{(\alpha x - x^2)} + \frac{\alpha^2}{8} \times \text{vers.}^{-1} \frac{2}{\alpha} x \right\}$$ see *Hirsch's Tables*, p. 150.

Let $x = \alpha$; then

$$Q' = 2t\sqrt{pg} \times \frac{\alpha^2}{8} \text{vers.}^{-1} 2$$

$$= 2t\sqrt{pg} \times \frac{a^2}{8} \times \pi$$

$$\therefore Q : Q' :: 1 : \frac{\pi}{4} :: 4 : 3.14159.$$

710. Let u be the height of the vessel above the orifice, y the radius of the base; then the velocity with which the water is projected from the vessel is due to $\frac{1}{4}$ parameter of the parabola described, and also to the altitude u; $\therefore$ this parameter is $4u$.

Hence, by the question, and the equation to the parabola,

$$b^2 = 4u \times a$$

$$\text{and } u = \frac{b^2}{4a} \dots\dots\dots\dots\dots\dots (1)$$

is known.

Again, $c^2 = pa$, where p is the parameter of the parabola described at the end of the time t. Hence the velocity at the *vena-contracta* is then due to $\frac{p}{4}$ or to $\frac{c^2}{4a}$; so that the surface has descended through the space

$$\frac{b^2 - c^2}{4a}.$$

But (*Vince's Fluxions*, p. 268,)

$$t = \frac{2\pi y^2}{m\sqrt{g}} \times \left(\sqrt{\frac{b^2}{4a}} - \sqrt{\frac{c^2}{4a}}\right)$$

$$= \frac{\pi y^2}{m\sqrt{ag}} \times (b - c)$$

$$\therefore \pi y^2 = \frac{mt\sqrt{ag}}{b-c}$$

the area of the base of the cylinder is known; and its altitude is

$$u + a = \frac{b^2}{4a} + a = \frac{b^2 + 4a^2}{4a}$$

and the required content is

$$\frac{mt}{4}\sqrt{\frac{g}{a}} \times \frac{b^2+4a^2}{b-c}.$$

711. Let α denote the axis of the paraboloid, β the radius of its base, and p its parameter; then the section parallel to the axis and horizon is a parabola, whose axis is

$$\frac{x(2\beta - x)}{p}$$

and base is

$$2\sqrt{(2\beta x - x^2)}$$

x being the distance of the section from the lowest point. Hence (696,)

$$X = \frac{2}{3} \times \frac{2\beta x - x^2}{p} \times 2\sqrt{(2\beta x - x^2)}$$

$$= \frac{4}{3p} \times (2\beta x - x^2)^{\frac{3}{2}}$$

Therefore (696)

$$t = \frac{4}{3pA\sqrt{g}} \times \int x dx \sqrt{(2\beta - x)}$$

$$= \frac{4}{3pA\sqrt{g}} \times \left(\frac{2\beta - x}{5} - \frac{2\beta}{3}\right) 2(2\beta - x)^{\frac{3}{2}} + C$$

see *Hirsch's Tables*, p. 104.

When $x = 0$, $t = 0$, and $C = \frac{16\sqrt{2}}{15} \times \beta^{\frac{5}{2}}$

$$\therefore t = \frac{8}{45pA\sqrt{g}} \times \left\{ 8\sqrt{2}\beta^{\frac{5}{2}} - (4\beta + 3x)(2\beta - x)^{\frac{3}{2}} \right\}$$

Let $x = 2\beta$; then the time required is

$$t = \frac{64\sqrt{2}}{45pA\sqrt{g}} \times \beta^{\frac{5}{2}}.$$

712. Let AN (draw the figure) be the altitude of the orifice, NP the horizontal distance; produce NA to T making

AT = NA and join TP. Make the $\angle$ TPS = $\angle$ PTS; then since SP = ST, and TP is a tangent to the parabola described by the fluid, the point S is the focus of the parabola. But SA = $\frac{1}{4}$ of the parameter = space due to the velocity at the *vena-contracta* = altitude of the fluid above the orifice A. Make, therefore, AB = AS, and we have NB = NA + AB = the required altitude of the fluid.

713. Let y be the radius of any horizontal surface of water in the required vessel, distant from the bottom by the interval x; then if A be the area of the orifice, the velocity of the orifice is

$$\sqrt{gx}$$

and that of the descending surface is

$$\frac{A}{\pi y^2}\sqrt{gx} \propto x^n = cx^n$$

by the question

$$\therefore\ y^2 = \frac{A}{c\pi}\sqrt{g} \times x^{\frac{1}{2}-n}$$

the equation to the generating curve.

Again, the content is

$$\pi \int y^2 dx = \frac{A}{c}\sqrt{g}\int x^{\frac{1}{2}-n} dx$$

$$= \frac{2}{3-n} \times \frac{A}{c}\sqrt{g} \times x^{\frac{3-2n}{2}} + C.$$

Now the surface being supposed to begin its descent with the given velocity V, we have (α being the given value of x)

$$c\alpha^n = V$$

$$\therefore\ c = \frac{V}{\alpha^n} \text{ is known.}$$

Hence the content is

$$\frac{2}{3-n} \times \frac{\alpha^n A}{V}\sqrt{g} x^{\frac{3-2n}{2}} + C.$$

When $x = 0$, the content is 0, and C will be 0, $-\frac{4}{3} \times \frac{a^{\frac{3}{2}}A}{V}\sqrt{g}$, or ∞

according as n is $<$ = or $>$ than $\frac{3}{2}$.

714. Let A be the area of either orifice, A′ that of the descending surface; then the surface being distant from the upper orifice by the interval x, and from the lower by $a+x$, the velocities at these orifices are

$$\sqrt{(gx)} \text{ and } \sqrt{(g\,.\,\overline{a+x})}$$

respectively. Hence the rate of afflux is

$$\sqrt{g} \times (\sqrt{x} + \sqrt{a+x}).$$

and the velocity of the descending surface is ∴

$$v = \frac{A}{A'} \times \sqrt{g} \times (\sqrt{x} + \sqrt{a+x})$$

∴ the time of discharging the depth x is

$$t = \frac{A'}{A\sqrt{g}} \times \int \frac{dx}{\sqrt{x} + \sqrt{a+x}}$$

$$= \frac{A'}{A\sqrt{g}} \times \int \frac{\sqrt{a+x} - \sqrt{x}}{a}\, dx$$

$$= \frac{2A'}{3aA\sqrt{g}} \times \left\{ (a+x)^{\frac{3}{2}} - x^{\frac{3}{2}} + C \right\}$$

Let $x = 0$; then $t = 0$ and $C = -a^{\frac{3}{2}}$.

∴ when $x = a$, we get the time of emptying the first half of the prism, *viz.*

$$t = \frac{2A'}{3aA\sqrt{g}} \times \left\{ (2a)^{\frac{3}{2}} - 2a^{\frac{3}{2}} \right\}$$

$$= \frac{4A'\sqrt{a}}{3A\sqrt{g}} \times (\sqrt{2} - 1).$$

715. The semi-cube thus cut off may evidently be a right-angled triangular prism, whose ends are each equal to the semi-side of the cube. In this case there will be no difficulty in finding the

time of discharge. If, however, the bisection of the cube be effected by a plane passing through the diagonal of the cube, and equally inclined to the pairs of faces which meet at the extremities of that diagonal, then the semicube will be a kind of *non-descript* solid, having two faces rectangular trapeziums, two others equal right-angled $\triangle$, another side a square, (the face of the cube,) and the last a rhombus made by the cutting plane.

Moreover, all sections parallel to this rhombus will be segments of this same rhombus, being deficient by an isosceles $\triangle$. If the student cut a cubical piece of an apple, or any other substance, as above directed, he will more clearly understand the form of this solid, than by inspecting any diagram.

Now the semi-cube being filled with water, and its face (the rhombus) being placed parallel to the horizon, let an orifice, whose area is A, be made in the lowest point, *viz.*, in the solid $\angle$ subtended by the rhombus. Again, let a be the side of the cube, α the distance of the lowest point from the rhombus, and x the distance of that point from the surface of the water at any time during its descent; then since the diagonals of the rhombus are those of the cube and its base, they are

$$a\sqrt{3} \text{ and } a\sqrt{2}; \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

and $\therefore$ the sides of the rhombus are each equal to

$$\sqrt{\left(\frac{3a^2}{4} + \frac{2a^2}{4}\right)} = \frac{a}{2}\sqrt{5} \ldots\ldots\ldots \quad (2)$$

Also the area of this rhombus is

$$a\sqrt{3} \times a\sqrt{2} = a^2 \times \sqrt{6} \ldots\ldots\ldots\ldots\ldots \quad (3)$$

Moreover

$$\alpha : a :: a\sqrt{2} : a\sqrt{3} :: \sqrt{2} : \sqrt{3}$$

$$\therefore \alpha = a\sqrt{\frac{2}{3}} \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (4)$$

and the leg of either triangular face is

$$\sqrt{\left(\frac{5a^2}{4} - a^2\right)} = \frac{a}{2} \ldots\ldots\ldots\ldots\ldots \quad (5)$$

Also the part cut off from these legs by the water when descended to x, is

$$\frac{a}{\alpha}\cdot(\alpha - x)$$

and $\therefore$ by similar $\triangle$, we get

$$\frac{a}{2} : \left\{\frac{a}{2} \mp \frac{a}{\alpha} \cdot (\alpha - x)\right\} :: \frac{a}{2}\sqrt{5} : \frac{2x-\alpha}{2\alpha} a\sqrt{5}$$

the defective side of the rhombus; wherein $\pm$ is used according as the descending surface is $>$ or $<$ than half the rhombus. Hence this descending surface is

$$X = \frac{a^2\sqrt{6}}{2} \pm \frac{a^2\sqrt{6}}{2} \times \left\{1 - \left(1 \mp \frac{2x-\alpha}{\alpha}\right)^2\right\}$$

$$= a^2\sqrt{6} \times \left(\frac{4x}{\alpha} - \frac{2x^2}{\alpha^2} - 1\right) \ldots\ldots\ (a)$$

$$\text{or} = \frac{2a^2\sqrt{6}}{\alpha^2} \times x^2 \ldots\ldots\ldots\ldots\ldots\ldots\ (b)$$

according as $+$ or $-$ is taken,
the former value obtaining from

$x = \alpha$ to $x = \frac{\alpha}{2}$, and the latter from

$x = \frac{\alpha}{2}$ to $x = 0$.

Now supposing the section $\frac{a^2\sqrt{6}}{2}$, where the descending surface first changes its form from the segment of a rhombus to an isosceles $\triangle$ to be an orifice, the time of the descent of the fluid to that section is (696)

$$t = \frac{a^2\sqrt{6}}{A\sqrt{g}} \times \int\left(\frac{4\sqrt{x}dx}{\alpha} - \frac{2x^{\frac{3}{2}}dx}{\alpha^2} - \frac{dx}{\sqrt{x}}\right)$$

$$= \frac{a^2\sqrt{6}}{A\sqrt{g}} \times \left(\frac{8x^{\frac{3}{2}}}{3\alpha} - \frac{4}{5}\frac{x^{\frac{5}{2}}}{\alpha^2} - 2x^{\frac{1}{2}} + C\right)$$

Let $x = \alpha - \frac{\alpha}{2} = \frac{\alpha}{2}$; then $t = 0$ and $C = \frac{13}{15\sqrt{2}}\sqrt{\alpha}$

$\therefore$ when $x = \alpha$, the time of the descent from $x = \alpha$ to $x = \frac{\alpha}{2}$ is

$$T = \frac{a^2\sqrt{\alpha}}{5\sqrt{3}A\sqrt{g}} \times (13 - 2\sqrt{2}) \ldots\ldots\ (c)$$

Again, the time of the fluid's descent in the pyramidal portion of the vessel is (696 and (b))

$$t' = \frac{2a^2\sqrt{6}}{\alpha^2 A\sqrt{g}} \int x^{\frac{3}{2}} dx$$

$$= \frac{4a^2\sqrt{6}}{5\alpha^2 A\sqrt{g}} x^{\frac{5}{2}}$$

and down the whole of this part of the vessel, or for $x = \frac{\alpha}{2}$, it is

$$T' = \frac{4a^2\sqrt{6}}{2^{\frac{5}{2}} 5\alpha^2 A\sqrt{g}} \alpha^{\frac{5}{2}} = \frac{a^2\sqrt{3}\times\sqrt{\alpha}}{5A\sqrt{g}}$$

Hence the whole time required is

$$T + T' = \frac{a^2\sqrt{\alpha}}{5A\sqrt{g}} \times \left\{\frac{1}{\sqrt{3}} \times \overline{13 - 2\sqrt{2}} + \sqrt{3}\right\}$$

$$= \frac{2a^2\sqrt{\alpha}}{5A\sqrt{3g}} \times (8 - \sqrt{2}).$$

716. In the first place *let us seek the locus of the intersections of the parabolas described by the fluid issuing through the holes in any given vertical line of the surface.*

Let y be the variable ordinate of any of these parabolas, x the corresponding ordinate measured from the top of the vessel, and not from the vertex or hole, and p the variable parameter of all the parabolas; then their general equation is (*Vince*, *Hyd.* prop. XXXIX.)

$$y^2 = p \times \left(x - \frac{p}{4}\right)$$

Now differentiating this equation on the supposition that p alone is variable, and putting the result $= 0$, and thence deducing p in terms of x, substitute for p in the equation, and the transformed equation will belong to the required locus. See Appendix to Vol. II. of *Simpson's Fluxions*, pp. 339, &c., or pp. 47, 48, 67, 68, of Vol. II. of this work. We thus get for the locus the equation

$$y = x$$

which belongs to a straight line inclined to the horizon at an $\angle$ of

45°, and meeting the top of the cylinder. Hence it follows, that the boundary of the water issuing from the cylinder is the surface of the frustum of a cone, the radii of the upper and lower ends being

$$r \text{ and } r + a$$

where r is the radius of the base of the cylinder, and a its altitude.

717. Let A be the area of any section parallel to the horizon, a that of the orifice, and h the altitude of the vessel; then the time of emptying is (*Vince's Flux.* p. 268,)

$$t = \frac{2A}{a}\sqrt{\frac{h}{g}}.$$

Again, the time of falling through $2h$ is

$$t' = \sqrt{\frac{4h}{g}} = 2\sqrt{\frac{h}{g}}.$$

$$\therefore t : t' :: A : a.$$

718. The area of any section parallel to the horizon is

$$\pi y^2 = p\pi x$$

p being the parameter of the generating parabola, and x the abscissa measured from the vertex. Also if A denote the area of the orifice, and h the whole height of the vessel; then the velocity at the orifice is

$$\sqrt{2gh}$$

when the vessel is quite full; and the quantity discharged in each second is (by the question)

$$A\sqrt{2gh} = nQ$$

Q being the given supply.

Hence

$$Q = A\sqrt{\left(2g \times \frac{h}{n^2}\right)}.$$

and since the water descends with the velocity

$$\frac{A}{\pi p x}\sqrt{2g\frac{h}{n^2}}$$

and ascends with

$$\frac{A}{\pi p x}\sqrt{2g\frac{h}{n^2}}$$

the velocity of the descending surface is

$$v = \frac{A\sqrt{2g}}{np\pi} \times \frac{n\sqrt{x} - \sqrt{h}}{x} \dots\dots\dots\dots (1)$$

Hence the time is

$$t = \int\frac{dx}{v} = \frac{np\pi}{A\sqrt{2g}}\cdot\int\frac{xdx}{n\sqrt{x} - \sqrt{h}}$$

$$= \frac{p\pi}{A\sqrt{2g}} \times \left\{\frac{2}{3}x^{\frac{3}{2}} + \frac{\sqrt{h}}{n}x + \frac{2h}{n^2}\sqrt{x} + \right.$$

$$\left.\frac{2h^{\frac{3}{2}}}{n^3}\,l.(n\sqrt{x} - \sqrt{h}) + C\right\}.$$

Let $t = 0$, when $x = h$; then

$$C = -h^{\frac{3}{2}} \times \left\{\frac{2}{3} + \frac{1}{n} + \frac{2}{n^2} + \frac{2}{n^3}l(\overline{n-1}\sqrt{h}).\right.$$

$$\therefore t = \frac{p\pi}{A\sqrt{2g}} \times \left\{\frac{2}{3}\cdot(x^{\frac{3}{2}} - h^{\frac{3}{2}}) + \frac{\sqrt{h}}{n}\cdot(x-h) + \frac{2h}{n^2}(\sqrt{x} - \sqrt{h})\right.$$

$$\left. + \frac{2h^{\frac{3}{2}}}{n^3}\,l.\,\frac{n\sqrt{x} - \sqrt{h}}{(n-1)\sqrt{h}}\right\} \dots\dots\dots\dots (2)$$

Now when the descending surface becomes stationary, equation (1) gives us

$$x = \frac{h}{n^2}$$

and the time of arriving at this position, equation (2) informs us is infinite, unless when $n = 1$, and then this time is zero. Hence we conclude that unless the influx is equal to the efflux at the very beginning of the discharge, the surface of the fluid will never be quiescent.

Although this conclusion is deduced after the methods expounded in all the Elementary Treatises (see *Prony's Architecture Hydraulique,* Tom. I. p. 342, and *Bossut,* Vol. I.; also *Leybourne's*

Mathematical Repository,) yet it must be confessed it is far from satisfactory. If the velocity of the descending surface be thus rightly estimated, it may easily be shewn that the expression for the time of reaching the state of quiescence will always involve log. (0), or be ∞, whatever may be the form of the vessel; in other words, no such state of quiescence is possible. This seeming contrary to experience, we intend prosecuting the subject somewhat farther at our leisure, and shall probably insert the results in some Scientific Journal.

719. Let r be the radius of the sphere, x the altitude of the fluid at any epoch, and A the area of the orifice; then the velocity at the orifice being

$$\sqrt{2gx}$$

that of the descending surface is

$$\frac{A}{\pi y^2}\sqrt{2gx} = \frac{A\sqrt{2g}}{\pi} \cdot \frac{\sqrt{x}}{2rx - x^2}$$

$$= \frac{A\sqrt{2g}}{\pi} \times \frac{1}{2r\sqrt{x} - x^{\frac{3}{2}}} \ldots\ldots (1)$$

$\therefore$ by the question,

$$2r\sqrt{x} - x\sqrt{x} = maximum,$$

which, by the rule, gives

$$x = \frac{2}{3}r;$$

whence the position of the surface when its velocity is a *minimum* is obtained, and this *minimum* velocity is

$$\frac{2A\sqrt{3g}}{4r^{\frac{3}{2}}\pi}.$$

Although this velocity is a minimum, yet it is not the least velocity of all, that at the orifice being 0.

The reader must observe, that at the orifice the descending surface is = orifice; for otherwise it might appear from equation (1) that this is ∞ instead of 0.

720. Let A be the area of the orifice of the cylinder, h its altitude, and r the radius of its base; then if the water descend through $h - x$ during the time t, the moving force at the end of that time which causes pressure and an increase of velocity at the orifice is

$$\pi r^2 (h - x)$$

and the upward accelerating force is $\therefore$

$$g \frac{\pi r^2 (h-x)}{P + \pi r x^2} = g \frac{h-x}{h+x}.$$

Hence the force which actually accelerates each particle of the fluid downwards through the orifice is

$$F = g + g\frac{h-x}{h+x} = 2g \cdot \frac{h}{h+x}.$$

Hence the velocity at the orifice is due to the altitude

$$2 \cdot \frac{hx}{h+x}$$

and is $\therefore$ $2\sqrt{g} \sqrt{\frac{xh}{h+x}}$.

Hence the velocity at the descending surface is

$$\frac{2A}{\pi r^2} \sqrt{gh} \times \sqrt{\frac{x}{h+x}}$$

and the time is $\therefore$

$$t = \frac{\pi r^2}{2A\sqrt{gh}} \times \int dx \sqrt{\frac{h+x}{x}} \dots\dots\dots\dots (a)$$

$$= \frac{\pi r^2}{2A\sqrt{gh}} \times \{\sqrt{hx+x^2} + \frac{h}{2}\, l.\, \frac{\sqrt{hx+x^2}+x}{\sqrt{hx+x^2}-x} + C\}$$

and which being taken between $x = 0$ and $x = h$ gives

$$T = \frac{\pi r^2}{2A\sqrt{gh}} \times \{h\sqrt{2} + h\, l.\, (\sqrt{2}+1)\}$$

$$= \frac{\pi r^2 \sqrt{h}}{2A\sqrt{g}} \cdot \{\sqrt{2} + l.\, (1+\sqrt{2})\} \dots\dots (1)$$

for the whole time of emptying the cylinder.

Again, the force which accelerates the descent of P is

$$g \cdot \frac{h-x}{h+x} = \frac{d^2s}{dt^2}$$

where ds is the element of space described by P in the time dt. Hence (see eq. a)

$$\frac{d^2s}{dx^2} = \frac{\pi^2r^4}{4A^2h} \cdot \frac{h-x}{x}$$

and supposing dx constant, we get

$$\frac{ds}{dx} = \frac{\pi^2r^4}{4A^2h} \times (hlx - x + C)$$

But since $\frac{ds}{dx} = \frac{ds}{dt} \times \frac{dt}{dx}$

$= \frac{\text{vel. of P}}{\text{vel. of descending surface}} \left(= \frac{0}{\text{Finite q}^{\text{ty}}.} = 0, \text{ when} \right.$

$x = h) \therefore$

$$C = h - h \,.\, l \,.\, (h)$$

$$\therefore \frac{ds}{dx} = \frac{\pi^2r^4}{4A^2h} (h - h \,.\, l\,(h) - x + h \,.\, lx)$$

and integrating again, we get

$$s = \frac{\pi^2r^4}{4Ah} . \{h - (1 - l.h)x \times \frac{x^2}{2} + hxl.x - hx\}$$

$$= \frac{\pi^2r^4}{4A^2h} \cdot \left\{ hxlx - xl.h - \frac{x^2}{2} \right\}$$

which, when $x = h$, gives

$$s = \frac{\pi^2r^4}{4A^2} \left\{ (h - 1)\, l.\, h - \frac{h}{2} \right\}$$

for the space described by P during the discharge of the water.

721. Let θ be the given inclination of the tube to the horizon, α the altitude of the orifice of the vessel, β the observed distance at which the water strikes the plane from the vertical line passing through the orifice; then since the water describes a parabola, whose equation is (see p. 253,)

$$y = x \tan. \theta - \frac{gx^2}{2v^2 \cos.^2 \theta}$$

where y and x are the vertical and horizontal co-ordinates originating at the orifice, and v the velocity of the issuing fluid.

Hence

$$-\alpha = \beta \tan.\theta - \frac{g\beta^2}{2v^2 \cos.^2\theta}$$

But if h be the required altitude of the fluid above the orifice, we have

$$v^2 = 2gh$$

whence by substitution, &c.

$$h = \frac{\beta^2}{4\cos.^2\theta} \times \frac{1}{\alpha + \beta \tan.\theta}.$$

722. Let p be the parameter of the generating parabola, β the radius of the base of the paraboloid, y that of the required orifice, α the given altitude of the water above the vertex, and S the given space descended through in the given time T. Then at any variable altitude $x - \frac{y^2}{p}$ above the orifice, the velocity at the orifice is

$$\sqrt{2g\,.\left(x - \frac{y^2}{p}\right)}$$

and that of the descending surface is

$$\frac{\pi y^2}{\pi p x} \times \sqrt{2g\left(x - \frac{y^2}{p}\right)}$$

Hence

$$t = \frac{p}{y^2\sqrt{2g}} \times \int \frac{x\,dx}{\sqrt{x - \frac{y^2}{p}}}$$

$$= \frac{2p}{3y^2\sqrt{2g}} \times \left(x + \frac{2y^2}{p}\right)\sqrt{x - \frac{y^2}{p}}.$$

Hence

$$\mathrm{T} = \frac{2p}{3y^2\sqrt{2g}} \times \left\{\left(\alpha + \frac{2y^2}{p}\right)\sqrt{\alpha - \frac{y^2}{p}} - \left(\alpha - \mathrm{S} + \frac{2y^2}{p}\right) \times \sqrt{\left(\alpha - \mathrm{S} - \frac{y^2}{p}\right)}\right\}$$

which being reduced and resolved, will give the required value of y.

723. Let β be the required altitude of the hole, a the altitude of the cone; then since the equation to the parabola described is (see p. 253,)

$$y = x \tan. \theta - \frac{gx^2}{2v^2 \cos.^2 \theta}$$

by the question, we have

$$-\beta = \{\beta \tan. 30^\circ + \frac{5}{4}(a-\beta)\} \tan. 30^\circ - \frac{g}{2v^2 \cos.^2 30^\circ}$$

$$\times (\beta \tan. 30 + \frac{5}{4}\overline{a-\beta})^2.$$

But $v^2 = 2g \,.\, (a-\beta)$

Hence by the solution of a quadratic equation, β and $\therefore$ the required position may be found.

724. Let x be any variable altitude of the water, A the area of the orifice, α the length of the axis, and β the extreme ordinate; then since the area of the descending surface is

$$\pi y^2 = \pi a^{2x}$$

the velocity of this surface is

$$\frac{A}{\pi\sqrt{2g}} \times \frac{\sqrt{x}}{a^{2x}}.$$

Hence

$$t = \frac{\pi}{A\sqrt{2g}} \times \int \frac{a^{2x}dx}{\sqrt{x}},$$

an integral which can be taken only between certain limits, as between $x = 0$, and $x = \infty$. See *Whewell's Dynamics*, p. 15.

725. Let $2a$ be the altitude of the water when it is in equilibro, $2b$ the distance between the axes of the two legs; then the centre of gravity of the water in the legs in the state of equi-

librium is the middle point of the line joining the bisections of the water in the legs. Make this point the origin of the vertical and horizontal co-ordinates y, x; then supposing the water to have descended down one of the legs, it will rise as much in the other, and the line which joins the bisections of the axes of the fluid in the legs, being divided in the inverse proportion of the altitudes of the water, will give the centre of gravity corresponding to this position. Hence if m denote the distance to which the centre of gravity of the water has been depressed in one leg and elevated in the other, it is easily shewn that

$$y : x :: m : b$$

we also easily get

$$\sqrt{m^2 + b^2} \times m = \sqrt{x^2 + y^2}$$

$$\therefore \left(b^2 \cdot \frac{y^2}{x^2} + b^2\right) \frac{b^2 y^2}{x^2} = x^2 + y^2$$

$$\therefore b^2 y = x^3$$

or the required curve is the common parabola, whose principal parameter is b^2.

726. Let $y = px^n$ be assumed as the equation to the required parabola. Also let a be the common altitude of the paraboloid and cylinder.

Now (by 696) we have

$$t = \frac{1}{A\sqrt{2g}} \times \int \frac{X dx}{\sqrt{x}}$$

$$\text{But } \frac{X}{\sqrt{x}} = \frac{\pi y^2}{\sqrt{x}} = \pi p^2 x^{2n - \frac{1}{2}}$$

$$\therefore t = \frac{\pi p^2}{A\sqrt{2g}} \times \frac{1}{2n + \frac{1}{2}} \times x^{2n + \frac{1}{2}}$$

$$= \frac{2\pi p^2}{(4n+1)A\sqrt{2g}} \times a^{\frac{4n+1}{2}}$$

for the whole solid.

Again, for the cylinder

$$X = \pi p^2 a^{2n}$$

$$\text{and } t' = \frac{2\pi p^2 a^{2n}}{A\sqrt{2g}} \times \sqrt{a}.$$

$$\therefore t : t' :: 1 : 4n + 1 :: 1 : 9,$$

by the question.

$$\text{Hence } n = \frac{9}{4}$$

and the nature of the parabola is given by its equation

$$y = px^{\frac{9}{4}}.$$

727. If x denote the distance of any element of the orifice ydx (parallel to the horizon) from the surface of the fluid, and Q the quantity of fluid issuing through the whole orifice in a second, we have

$$dQ = 2ydx \times \sqrt{2gx}$$

$$\therefore Q = 2\int ydx\sqrt{2gx}$$

integrated between $x = 0$ and $x = x$, the altitude of the fluid.

Let now the given area of the orifice be A, and that of the descending surface, which is also given, be M; then the velocity of the descending surface is

$$\frac{A}{M} \times Q \propto x^n \text{ by the question}$$

$$\therefore \int ydx\sqrt{2gx} = \frac{M}{2A} \times C \times x^n$$

$$\text{and } y = \frac{nMC}{2A\sqrt{2g}} \times x^{n+\frac{1}{2}}$$

or the required curve is a parabola.

728. The time of emptying any depth x of the fluid is (696)

$$t = \frac{1}{A\sqrt{2g}} \times \int\frac{Xdx}{\sqrt{x}}$$

But here

$$X = \pi y^2 = \pi \frac{b^2}{a^2} . (2ax - x^2)$$

a and b being the semiaxes of the generating ellipse.

$$\therefore t = \frac{\pi b^2}{a^2 A \sqrt{2g}} \times \int (2a \sqrt{x dx} - x^{\frac{3}{2}} dx)$$

$$= \frac{2\pi b^2}{a^2 A \sqrt{2g}} . \left(\frac{2a}{3} x^{\frac{3}{2}} - \frac{x^{\frac{5}{2}}}{5} \right).$$

Make $x = a$, and then $x = \frac{a}{2}$, and the difference of the results gives the time required, *viz.*,

$$\frac{\pi b^2 \sqrt{a}}{A \sqrt{2g}} \times \frac{28\sqrt{2} - 17}{60}.$$

729. By 696

$$t = \frac{1}{A \sqrt{2g}} \times \int \frac{X dx}{\sqrt{x}}.$$

But since, as it may be easily shewn, the section of an oblate spheroid made by a plane $\perp$ to its major axis at the distance x from the vertex, is an ellipse, whose semiaxes are

$$\frac{b}{a} \sqrt{2ax - x^2}, \text{ and } \sqrt{2ax - x^2}$$

a and b being the semiaxes of the spheroid, we have

$$X = \pi \times \frac{b}{a} . (2ax - x^2)$$

Hence

$$t = \frac{2\pi b}{aA \sqrt{2g}} \times \left(\frac{2a}{3} x^{\frac{3}{2}} - \frac{x^{\frac{5}{2}}}{5} \right)$$

$$= \frac{2\pi b}{a A \sqrt{2g}} . (2a)^{\frac{5}{2}} \times \left(\frac{1}{3} - \frac{1}{5} \right)$$

$$= \frac{16 \pi b a^{\frac{3}{2}}}{15 A \sqrt{g}}$$

for the whole spheroid.

730. The velocity at the orifice is

$$\sqrt{2gx}$$

and that of the descending surface is $\therefore$

$$\frac{\mathrm{A}}{\pi y^2} \cdot \sqrt{2gx} = \frac{\mathrm{C}}{y}$$

by the question

$$\therefore\ y^2 = \frac{2g\mathrm{A}^2}{\mathrm{C}^2}\,x$$

or the solid is a paraboloid.

HYDRODYNAMICS IN GENERAL.

731. LET a be the angular velocity round the axis; then when the water and cylinder are relatively at rest, the surface of the water will be that of a certain solid of revolution, whose ordinate, normal, and subnormal, referred to the axis of rotation, will represent the centrifugal force, the re-action of the fluid, and the force of gravity, respectively. Hence if y denote the ordinate, and v the linear velocity of any particle of the fluid in the surface, the centrifugal force is

$$\frac{v^2}{y} = \frac{a^2y^2}{y} = a^2y$$

and the subnormal is

$$\frac{1}{a^2y} \times gy = \frac{g}{a^2}$$

or the generating curve of the volume of the part evacuated by the fluid is the common parabola, whose parameter is

$$\frac{2g}{a^2}.$$

Hence, if r be the radius of the base of the cylinder, the quantity of water thrown out by the revolution of the cylinder is equal in volume to the paraboloid, whose base is πr^2, and altitude $\frac{a^2r^2}{2g}$, *i. e.* to

$$\frac{1}{2} \times \pi r^2 \times \frac{a^2r^2}{2g} = \frac{\pi a^2}{4g} \cdot r^4.$$

and this taken from the given volume of the whole cylinder, will leave the quantity of water required.

732. Let θ be the angle required; then (*Vince's Hydrostatics*, Prop. XXVIII.) the effect $\propto \sin.^2\theta \cos.\theta = \max.$ and by the rule for maxima and minima, we get

$$\sin.\theta = \sqrt{\frac{2}{3}}.$$

733. If p be the parameter of the given paraboloid, and r the radius of its base; then the angular velocity necessary to empty the vessel is (731)

$$\alpha = \sqrt{\frac{2g}{p}}$$

and therefore the time of revolution is

$$360 \sqrt{\frac{p}{2g}}.$$

734. Let α be the required angular velocity; then the volume of fluid evacuated is that of a paraboloid, whose base is πr^2 (r being the radius of the base of the cylinder,) and altitude (see 731,) $\frac{\alpha^2 r^2}{2g}$. Hence, and by the question,

$$\frac{\pi r^2}{2} \times \frac{\alpha^2 r^2}{2g} = \frac{\pi r^2 \times h}{2},$$

h being the height of the cylinder.

$$\therefore \alpha = \frac{\sqrt{2gh}}{r}.$$

735. The descending force is w, and the resistance is $\frac{\pi d^2}{4} s \times v^2$ (*Vince's Hydrostatics*, Prop. XXV.) s being the specific gravity of the fluid, and v the velocity. $\therefore$ the force which accelerates the whole apparatus downwards is

$$F = w - \frac{\pi d^2 s v^2}{4}.$$

$$\therefore vdv = -Fdx = -dx\left(w - \frac{\pi d^2 s w^2}{4}\right)$$

$$\therefore x = \int \frac{-4vdv}{4w - \pi s d^2 v^2}$$

$$= \frac{2}{\pi s d^2} \times \left\{ l.\ (4w - \pi s d^2 v^2) + C \right\}$$

Let $x = a$, when $v + 0$. Then

$$\therefore x - a = \frac{2}{\pi s d^2} \times l.\ \frac{4w - \pi s d^2 v^2}{4w}$$

$$\therefore 4w - \pi s d^2 v^2 = 4we^{\frac{\pi s d^2}{2}(x-a)}$$

$$\therefore v = \frac{2\sqrt{w}}{d\sqrt{\pi s}} \times \left\{1 - e^{\frac{\pi s d^2}{2}(x-a)}\right\}^{\frac{1}{2}} \ldots\ldots (1)$$

Hence

$$dt = \frac{dx\,.\,d \times \sqrt{\pi s}}{2\sqrt{w}\sqrt{\left\{1 - e^{\frac{\pi s d^2}{2}(x-a)}\right\}}}.$$

Let $1 - e^{\frac{\pi s d^2}{2}(x-a)} = e^{2u}$; then

$$dt = \frac{2}{d \times \sqrt{\pi s w}} \times \frac{-e^u du}{1 - e^{2u}}.$$

Again, let $e^u = z$; then

$$dt = \frac{2}{d\sqrt{\pi s w}} \times \frac{-dz}{1 - z^2}$$

$$\text{and } t = \frac{1}{d\sqrt{\pi s w}} \times \left\{ l.\ \frac{1-z}{1+z} + C \right\} \ldots\ldots (2)$$

Let $x = a$; then $z = 0$ and $C = 0$.

Now the greatest velocity that the man can acquire in his descent is that which takes place when the retarding force = the accelerating force. Hence the velocity is

$$v = \sqrt{\frac{4w}{\pi d^2 s}} = \frac{2}{d}\sqrt{\frac{w}{\pi s}}$$

This being substituted in equation (1) gives

$$z = e^u = 1,$$

whence by equation (2) the time of acquiring this velocity is ∞.

In this solution the resistance of the air upon the man's body is not considered; so that it holds good but for the descent of the solid semicircle or cylinder whose weight is w.

736. The resistance $\propto$ area $\times$ density $\times$ (vel.)^2

$$\therefore \mathrm{R} : \mathrm{R}' :: \mathrm{A} \times \mathrm{D} \times \mathrm{V}^2 : 4\mathrm{A} \times 3\mathrm{D} \times 4\mathrm{V}^2$$
$$:: 1 : 48.$$

737. Let r be the radius of the sphere, y that of the smaller end of the required segment; then if R denote the resistance on the circle πr^2, that on the hemisphere is (*Vince's Fluxions*, p. 277,)

$$\frac{\mathrm{R}}{2},$$

and the resistance on the end πy^2 is

$$\frac{\pi y^2}{\pi r^2} \times \mathrm{R} = \frac{y^2}{r^2}\,\mathrm{R}.$$

$\therefore$ the resistance on the segment cut off by πy^2 is (*Vince*, p. 277,)

$$\left(1 - \frac{y^2}{2r^2}\right) \times \frac{y^2}{r^2}\,\mathrm{R}$$

and that upon the convex surface of the frustum is $\therefore$

$$\frac{\mathrm{R}}{2} - \left(1 - \frac{y^2}{2r^2}\right) \times \frac{y^2}{r^2}.$$

Hence the whole resistance upon the frustum is

$$\frac{y^2}{r^2}\,\mathrm{R} - \left(1 - \frac{y^2}{2y^2}\right) \frac{y^2}{r^2}\,\mathrm{R} + \frac{\mathrm{R}}{2}$$

$$\text{or } \frac{\mathrm{R}}{2} \times \left(\frac{y^4}{r^4} + 1\right).$$

Hence by the question

$$\frac{\mathrm{R}}{2} \times \left(\frac{y^4}{r^4} + 1\right) = \frac{3}{4}\,\mathrm{R}$$

which gives

$$y = r \times \frac{1}{2^{\frac{1}{4}}}$$

and this is sufficient to determine the required segment.

738. The sections of a cylinder made by planes parallel to the base are circles. Hence, if R denote the resistance upon the diameter ($2r$) of one of these sections, and a the axis of the cylinder, the resistance upon the semi-surface of the cylinder is (*Vince's Flux.* p. 277).

$$\frac{2}{3} \mathrm{R} \times a$$

Also the resistance upon the end of the cylinder when it moves parallel to the axis is

$$a \mathrm{R} \times \frac{\pi r^2}{2ar} = \frac{\pi r \mathrm{R}}{2}$$

$$\therefore \text{ the resistances are as } \frac{2}{3} a$$

and $\frac{\pi r}{2}$, or as $4a$ and $3\pi r$.

739. Let M be the magnitude of the body, and S and S′ the specific gravities of the fluid and the body, respectively; then the descending force is (*Vince's Hyd.* prop. XVIII)

$$\mathrm{F} = \mathrm{M} \times \mathrm{S} - \mathrm{M} \times \mathrm{S}' = \mathrm{M} \times (\mathrm{S} - \mathrm{S}')$$

which being constant gives by the question,

$$a^2 = \mathrm{F}s = \mathrm{M} \times (\mathrm{S} - \mathrm{S}')s$$

$$\therefore \mathrm{S} = \mathrm{S}' + \frac{a^2}{\mathrm{M}s}.$$

740. For the general construction see the note 192, Vol. II, of *Newton's Prin.* edit. of PP. *Le Seur* and *Jaquier*. From this it appears that the resistance (r) upon the curve KA : resist(R) upon the base KC :: area of KQHEC : area of the rectangle KI.

Now let CI $= m$, CK $= \alpha$, CA $= \beta$, KP $= x$, PH $= y'$, PB$=y$, KB $= s$. Then since

$$LM = \frac{LD^2}{LB}$$

we readily deduce

$$y' = \frac{mdx}{ds}.$$

Hence

$$R : r :: m \times \alpha : m \int \frac{mdx^2}{ds}$$

$$:: \alpha : m \int \frac{dx}{\sqrt{(1 + \frac{dy^2}{dx^2})}}$$

In the problem (*Vince's Flux*. prop. 130)

$$\frac{dy}{dx} \quad \frac{a}{\sqrt{2ax + x^2}}$$

$$\therefore \int \frac{dx}{\sqrt{1 + \frac{dy^2}{dx^2}}} = \int \frac{2ax + x^2}{a + x}\, dx$$

$$= \int (a + x)\, dx - \int \frac{a^2 dx}{a + x}$$

$$= \frac{(a + x)^2}{2} - a^2\, l.\ \frac{a + x}{a} - \frac{a^2}{2}$$

$$= \frac{2ax + x^2}{2} - a^2\, l.\ \frac{a + x}{a}$$

Let $x = \alpha$; then we get

$$R : r :: \alpha : m \times \left\{ \frac{2a\alpha + \alpha^2}{2} - a^2\, l.\ \frac{a + \alpha}{a} \right\}$$

741. Let S, S′ and s denote the specific gravities of the wood, iron, and sea-water respectively. Also, let $2r$ be the diameter of either ball; then since the volume of each ball is

$$\frac{4\pi r^3}{3}$$

the weight of the whole apparatus is

$$\frac{4\pi r^3}{3} S + \frac{4\pi r^3}{3} S' = \frac{4\pi r^3}{3} . (S + S')$$

and the force which accelerates the descent of the balls is (*Vince's Hyd.*)

$$\frac{4\pi r^3}{3} . (S + S' - 2s).$$

Again, the resistance of the fluid upon each globe is (it is $= \frac{1}{2}$ that upon the great $\odot = \frac{1}{2}$ wt. of a cylinder of the fluid of whose base is πr^2, and altitude $\frac{v^2}{2g}$)

$$\frac{1}{2} \times \frac{v^2}{2g} \times \pi r^2 \times s$$

v being the velocity with which they are moving at the time. Hence, the balls descend with the force

$$F = \frac{4\pi r^3}{3} . (S + S' - 2s) - \frac{\pi r s^2}{2g} v^2$$

$$\text{and } vdv = \frac{4\pi r^3}{3} . (S + S' - 2s)\ dx - \frac{\pi r^2 s}{2g} v^2 dx$$

x being the space described from rest.

$$\therefore x = \frac{3g}{r^2\pi} \int \frac{2v\,dv}{8gr. (S + S' - 2s) - 3sv^2}$$

$$= \frac{g}{r^2 s\pi} \times \{C - l. (8gr (S + S' - 2s) - 3sv^2)\}$$

$$= \frac{g}{r^2 s\pi} \times l. \frac{8gr.(S + S' - 2s)}{8gr.(S + S' - 2s) - 3sv^2}$$

$$= \frac{1}{A} \times l. \frac{B}{B - v^2}$$

by supposition.

Hence,

$$v = \sqrt{B} \times \sqrt{(1 - e^{-Ax})}$$

$$\therefore t = \frac{1}{\sqrt{B}} \times \int \frac{dx}{\sqrt{(1 - e^{-Ax})}} .$$

Let $1 - e^{-Ax} = u^2$; then

$$t = \frac{1}{A\sqrt{B}} \times \int \frac{2du}{1 - u^2}$$

$$= \frac{1}{A\sqrt{B}} . l. \frac{1 + u}{1 - u},$$

there being no correction, since when $x = 0$, $u = 0$, and $t = 0$.

Hence, the time of the descent of the balls is

$$t = \frac{1}{A\sqrt{B}} \cdot l. \frac{1 + \sqrt{(1 - e^{-Ax})}}{1 - \sqrt{(1 - e^{-Ax})}}$$

$$= \frac{2}{A\sqrt{B}} \times l. \left\{1 + \sqrt{(1 - e^{-Ax})}\right\} + \frac{x}{\sqrt{B}}.$$

Again, when the iron is disengaged from the wood, the wooden ball will ascend with the force

$$\frac{4\pi}{3} r^3 . (S - s)$$

and, if v' be its velocity at the end of the time t', having then described the space x' from the bottom, the resistance of the water is

$$\frac{1}{2} \times \frac{v'^2}{2g} \times \pi r^2 \times s.$$

Hence, the actual force by which the ball is accelerated upwards is

$$F = \frac{4\pi}{3} r^3 . (S - s) - \frac{\pi r^2 s}{4g} \times v'^2 = \frac{v'dv'}{dx'}$$

$$\therefore x' = \frac{12g}{\pi r^2} \times \int \frac{v'dv'}{16gr . (S - s) - 3sv'^2}$$

$$= \frac{2g}{\pi s r^2} \times \{C - l. (16gr. \overline{S - s} - 3sv'^2)\}$$

$$= \frac{2g}{\pi s r^2} \times l. \frac{16gr. \overline{S - s}}{16gr. \overline{S - s}. - 3sv'^2}$$

$$= \frac{1}{A'} . l. \frac{B'}{B' - v'^2}$$

by supposition.

Hence, as before,

$$v' = \sqrt{B'} \times \sqrt{(1 - e^{-A'x'})}$$

and $t' = \dfrac{2}{A'\sqrt{B'}} \times l. (1 + \sqrt{\overline{1 - e^{-A'x'}}}) + \dfrac{x'}{\sqrt{B'}}$

and for the whole ascent x, we have

$$t' = \frac{2}{A'\sqrt{B'}} \times l.\ (1 + \sqrt{1 - e^{-A'x}}) + \frac{x}{\sqrt{B'}}$$

Now, since by the question the whole time elapsed during the descent and ascent is given, let it $=$ T; then

$$T = t + t' = x \times \left(\frac{1}{\sqrt{B}} + \frac{1}{\sqrt{B'}}\right) + \frac{2}{A\sqrt{B}} \times$$

$$l.\ \{1 + \sqrt{(1 - e^{-Ax})}\} + \frac{2}{A'\sqrt{B'}} \times l.\ \{1 + \sqrt{(1 - e^{-A'x})}\}$$

$$= \sqrt{\frac{3s}{8gr}} \times \left\{\frac{1}{(S + S' - 2s)^{\frac{1}{2}}} + \frac{1}{(2 \times S - s)^{\frac{1}{2}}}\right\} \times x +$$

$$\frac{2\sqrt{3s}}{A\sqrt{8gr}} \times \left\{\frac{1}{\sqrt{(S + S' - 2s)}} \times l.\ (1 + \sqrt{1 - e^{-Ax}}) + \right.$$

$$\left.\sqrt{\frac{2}{S - s}}.\ l.\ (1 + \sqrt{(1 - e^{\frac{-Ax}{s}})})\right\}$$

whence x may be found by approximation.

742. In Newton's construction (Schol. Prop. XXXIV, Vol. II.) since the $\angle$ CQO $= 2 \angle$ CSO $= \angle$ S at the vertex of the cone; therefore in the limit of the altitude OD, or when CQO $= 90°$, CSQ must be $45°$; $\therefore$ OCS $= 45°$, and its supplement $= 135°$.

Now in the solid of least resistance, whatever the nature of that solid may be, the resistance upon the nascent end of it, corresponding to FG, will evidently be the same as upon the nascent truncated cone generated by the tangent at that end. Hence, when this nascent cone is of minimum resistance, the nascent solid will be also of least resistence; and the surface of the truncated solid of least resistance makes therefore with its smaller end an angle of $135°$.

743. Let AC $= \alpha$, CD $= \beta$, and let the co-ordinates at the required point be x, y. Also, let R be the resistance upon the base $\pi\beta^2$; then the resistances upon πy^2, and $\pi AQ^2 = \frac{\pi}{4} y^2$, are

$$\frac{y^2}{\beta^2} \times \text{R and} \frac{y^2}{4\beta^2}\text{R}$$

respectively.

Hence the resistance upon the surface of the frustum of the cone generated by PQA is (*Vince's Fluxions*, prop. 133)

$$\text{R}.\left(\frac{y^2}{\beta^2} - \frac{y^2}{4\beta^2}\right)\frac{y^2}{4x^2 + y^2} = \frac{3\text{R}x}{4a} \times \frac{p}{4x + p} = \frac{3p\text{R}}{4a} \times \frac{x}{p + 4x}.$$

Moreover, if R′ be the resistance upon the whole paraboloid, it will easily be shewn from the above cited prop. of *Vince* that the resistance upon the paraboloidal frustum is

$$\text{R}' - \frac{y^2}{\beta^2}\text{R} \times \frac{2}{y^2} \times \int \frac{ydy}{1 + \frac{dx^2}{dy^2}}$$

$$= \text{R}' - \frac{2\text{R}}{\beta^2}. \int \frac{ydy}{1 + \frac{dx^2}{dy^2}}$$

Hence, by the question, we have

$$\frac{3p\text{R}}{4a}\frac{x}{p + 4x} + \frac{y^2}{4\beta^2}\text{R} + \text{R}'$$

$$- \frac{2\text{R}}{\beta^2}. \int \frac{ydy}{1 + \frac{dx^2}{dy^2}} = minimum$$

∴ by the rule, we get

$$\frac{3p\text{R}}{4a} \times \frac{pdx}{(p + 4x)^2} + \frac{ydy}{2\beta^2}\text{R} -$$

$$\frac{2\text{R}}{\beta^2}. \frac{ydy}{1 + \frac{dx^2}{dy^2}} = 0.$$

But since $y^2 = px$, we have by substitution,

$$\frac{3p\text{R}}{4a}. \frac{pdx}{(p + 4x)^2} + \frac{dx}{4a}\text{R} -$$

$$\frac{\text{R}}{a}. \frac{pdx}{p + 4x} = 0$$

which reduces to

$$(p + 4x)^2 - 4p(p + 4x) = -3p^2$$

and this being resolved, gives

$$x = \frac{p}{2},$$

which will determine the required position of P.

744. If R and R′ denote the resistances upon the circle and hexagon, A and A′ their areas, and θ, θ' the inclinations of the directions to their planes; then (*Vince* prop. 28)

$$R : R' :: \sin.^2 \theta \times \cos. \theta \times A : \sin.^2 \theta' \times \cos. \theta' \times A'$$

But by the question, $\sin. \theta = \sin. 30 = \frac{1}{2}$ and $\sin. \theta' = \sin. 60° = \frac{\sqrt{3}}{2}$. Also, $A = \pi r^2$ (r being the radius of the circle) and

$$A' = 6 \times \frac{r}{2} \times \frac{\sqrt{3}}{2} \times r = \frac{3\sqrt{3}}{2} r^2.$$

$$\therefore R : R' :: \frac{1}{4} \times \frac{\sqrt{3}}{2} \times \pi r^2 : \frac{3}{4} \times \frac{1}{2} \times \frac{3\sqrt{3}}{2} r^2$$

$$:: 2\pi : 9.$$

745. Let a be the length of either of the equal sides of the trapezium, θ their inclination to the other sides. Also, let m be the length of that parallel side which is resisted by the fluid; and suppose A, the resistance on a when moving parallel to m, A′ the resistance when moving perpendicularly to m, and M the resistance upon m. Moreover, let R be the resistance when moving $\perp$ ly to the fluid. Then

$$A : R :: \sin.^3 \theta : 1$$

$$R : M :: a : m$$

$$\therefore A : M :: a \sin.^3 \theta : m.$$

Again,

$$A' : M :: a \times \cos.^3 \theta : m$$

$$\therefore 2A' + M : M :: 2a \cos.^3 \theta + m : m$$

But $M : A :: m : a \sin.^3 \theta$

$$\therefore 2A' + M : A :: 2a \cos.^3 \theta + m : a \sin.^3 \theta$$

the analogy required.

746. The reason is because of the oscillatory motion of the waves. They rise and fall in the manner of a pendulum. (See *Newton*, Vol. II., prop. 44).

747. The faces are inclined to the diagonal of a cube at the angle

$$\theta = \sin.^{-1} \frac{1}{\sqrt{3}}$$

$\therefore$ If R denote the resistance upon the face when its direction is $\perp$ to its surface, the resistance when the direction is parallel to the diagonal is

$$R \times \sin.^3 \theta = \frac{R}{3\sqrt{3}}.$$

But there are three faces thus resisted; therefore the whole resistance is

$$\frac{R}{\sqrt{3}} \text{ Q. E. D.}$$

PNEUMATICS.

748. Let b be the capacity of the barrel, r that of the receiver; then since the defect from the standard altitude has become

$$h - a$$

and density $\propto$ defect (*Vince*, Prop. LXVII.)
we have

$$h : h - a :: \text{density at first} : \text{ditto after } n \text{ turns}$$
$$:: (2b + r)^n : (b + r)^n$$
$$\therefore 2b + r : b + r :: h^{\frac{1}{n}} : (h - a)^{\frac{1}{n}}$$
$$\text{and } b : r :: h^{\frac{1}{n}} - (h - d)^{\frac{1}{n}} : 2\,(h - a)^{\frac{1}{n}} - h^{\frac{1}{n}}$$

749. Let x denote the space through which the elasticity of the air in the tube depresses the mercury; L the length of the tube, h the standard altitude, and a the depth of the air in the tube before inversion; then (*Vince's Hyd.* Prop. LXI.) we have

$$h : x :: x + d - h : a$$
$$\therefore x^2 + (l - h) \times x = ah$$
$$\text{and } x = \frac{-(l - h) \pm \sqrt{(l - h)^2 + 4ah)}}{2}$$

one value of which indicates the space through which the air depresses the mercury, and the other the altitude of mercury it would support if placed above it, by its pressure against the top of the tube.

In the problem $h = 30$
$L = 40$, and $a = 20$; $\therefore$
$x = 7\frac{1}{2}$ inches,
Hence
$h - x = 22\frac{1}{2}$ inches,
the altitude required.

750. Let s be the specific gravity of the air at the Earth's surface, A the area of the given plane, supposed to bear no proportion to the great circle of the earth, and r the radius of the earth; then ns is the weight of a cubic foot of the given uniform medium at the earth's surface, and by the law of gravitation, its weight at the distance

$$\mathrm{R} + x$$

from the earth's centre is

$$\frac{\mathrm{R}^2}{(\mathrm{R}+x)^2} \times ns$$

Hence the increment of the pressure or weight of the medium upon the plane is

$$\frac{\mathrm{R}^2 ns}{(\mathrm{R}+x)^2} \times \mathrm{A}dx$$

and that pressure or weight is

$$\mathrm{AR}^2 ns \times \left(\frac{1}{\mathrm{R}} - \frac{1}{\mathrm{R}+x}\right).$$

Let $x = \infty$. Then $\dfrac{1}{\mathrm{R}+x} = \dfrac{1}{\infty} = 0.$

and the pressure required is measured by

$$ns\mathrm{AR}.$$

Hence it appears *that the pressure of the infinitely high column of any substance whatever is equal to the weight of a column of the same base, and whose altitude is equal to the radius of the earth.*

751. *Generally, let the compressing force* $\propto$ (*density*)$^p \propto \mathrm{D}^p$, *and the force of gravity* $\mathrm{F} \propto \dfrac{1}{(dist.)\,q}$, $\propto \dfrac{1}{\rho^q}$; *required the variation of* D.

Let δ be the density at the surface of the earth, and R the radius of the earth; then since

$$d\mathrm{C} \propto d\mathrm{Q} \times \mathrm{F} \propto -\mathrm{D}d\varrho \times \frac{\mathrm{R}^q}{\rho^q} \propto \text{also } \mathrm{D}^{p-1}d\mathrm{D}$$

$$\therefore \frac{-d\rho}{\rho^q} = \mathrm{M} \times \mathrm{D}^{p-2}d\mathrm{D}$$

M being as yet undetermined.

Hence

$$\frac{1}{q-1} \cdot \rho^{1-q} = \frac{M}{p-1} \cdot D^{p-1} + C$$

$$\therefore \frac{p-1}{q-1} \cdot (\rho^{1-q} - R^{1-q}) = M \times (D^{p-1} - \delta^{p-1}).$$

$$\text{Now } M = \frac{p-1}{q-1} \cdot \frac{\rho^{1-q} - R^{1-q}}{D^{p-1} - \delta^{p-1}}$$

and its value is that with which this fraction vanishes when for ρ and D we put R and δ. Hence by the rule for estimating such fractions

$$M = \frac{-1}{R^q \times \delta^{p-2}}$$

and we get

$$D = \left\{\frac{p-1}{q-1} R^q \delta^{p-2} \times (R^{1-q} - \rho^{1-q}) + \delta^{p-1}\right\}^{\frac{1}{p-1}}.$$

752. Let $y = \delta \times a^x$ be the equation to the *Density-Curve*, (see *Vince's Hyd.* p. 82), y representing the density at the distance x from the surface of the earth, and δ the density at the surface. Then if h be the height of the homogeneous atmosphere, as determined by the barometer, we have (see *Vince*),

$$h = \frac{-ydx}{dy} = \frac{-1}{la}$$

which gives

$$a = e^{-\frac{1}{h}}$$

$$\therefore y = \delta \times e^{\frac{-x}{h}} \quad \ldots\ldots\ldots\ldots\ldots \quad (1).$$

Let D be the density of mercury, determined at the time this formula is applied, and b the height at which it then stands in the tube of the barometer; then

$$h : b :: \frac{1}{\delta} : \frac{1}{D}$$

and the equation (1) becomes

$$y = \delta \times e^{-\frac{\delta}{bD}x} \quad \ldots\ldots\ldots\ldots\ldots \quad (2).$$

Hence the distance from the surface of the earth of the centre of gravity of the cylinder whose altitude is x, is

$$u = \frac{\int xydx}{\int ydx} = \frac{\int xe^{\frac{-\delta}{bD}x}dx}{\int e^{\frac{-\delta}{bD}x}dx}$$

$$= \frac{-xe^{\frac{-\delta}{bD}x} - e^{\frac{-\delta}{bD}x} + C}{-e^{\frac{-\delta}{bD}x} + C'}$$

$$= \frac{e^{\frac{\delta}{bD}x} - x - 1}{e^{\frac{\delta}{bD}x} - 1}.$$

To verify this result, we should have $u = 0$, when $x = 0$. But we then have

$$u = \frac{0}{0}.$$

If, however, we find u by the rule for Vanishing Fractions, we have

$$\frac{dP}{dQ} = \frac{d \,.\, e^{\frac{-\delta}{bD}x} - dx}{d \,.\, e^{\frac{-\delta}{bD}x}} = \frac{0}{1} = 0$$

when $x = 0$.

753. Let W be the weight of the balloon and all its appendages, δ the specific gravity or density of the atmosphere at the surface of the earth when the barometer stands at b feet, and $\frac{\delta}{n}$ that of the gas. Also let y denote the density of that stratum whereat the balloon will cease to ascend, and let c^3 be the capacity of the balloon in cubic feet. Then since the weight of the air displaced by the gas is

$$c^3 \times y$$

and that of the balloon, its appendages and gas, is

$$W + c^3 \times \frac{\delta}{n}$$

$\therefore$ by the question (*Vince*. Prop. XIV.)

$$c^3 \times y = W + c^3 \times \frac{\delta}{n}$$

which gives

$$y = \frac{W}{c^3} + \frac{\delta}{n} \dots\dots\dots\dots\dots\dots (1)$$

Hence, and by the last question,

$$\frac{W}{c^3} + \frac{\delta}{n} = \delta \times e^{\frac{-\delta}{bD}x}$$

which gives

$$x = \frac{bD}{\delta} \times l.\frac{n\delta c^3}{nW + \delta c^3} \dots\dots\dots\dots (2)$$

the height required in feet.

Ex. 1. Let the gas be hydrogen, or $n = 13$; $b = 30$ inches $= \frac{5}{2}$ feet; $D = 14019$ (water being supposed to have the density of 1000); and $\delta = \frac{6}{5}$. Then

$$x = 42057 \times l.\frac{78c^3}{65\,W + 6c^3}.$$

Let it further be supposed that $W = 20$ stone $= 4480$ ounces, and $c^3 = 200000$ cubic feet. Then

$$x = 42057 \times l.\frac{156000}{14912}$$

$$= 42057 \times l.\frac{4875}{466}$$

$$= 42057 \times 2.34768 \text{ feet nearly.}$$

See *Barlow's Tables.*

$$\therefore x = 98736 \text{ feet}$$
$$= 32912 \text{ yards}$$
$$= 18 \text{ miles } 1232 \text{ yards.}$$

This prodigious altitude is owing to the disproportion of the balloon to the weight of the load and materials. The surface of such a machine, even under the form of a sphere, is about 24000 square feet, and the weight of that surface would consequently be considerably greater than the weight here attributed to it.

Ex. 2. *Given* W = 20 *stone, and the other elements as before, to determine the magnitude of the balloon necessary just to lift that weight from the ground.*

Here $x = 0$. Consequently

$$\frac{78c^3}{65\times 4480+6c^3} = 1$$

$$\therefore c^3 = \frac{65\times 4480}{72} = 4044 \text{ cubic feet.}$$

Hence if r be the radius of a spherical balloon of this capacity, we have

$$\frac{4\pi r^3}{3} = 4044$$

and $r^3 = 1000$ nearly.

$\therefore r = 10$ feet nearly.

Even such a vehicle, it appears, would be bulky and expensive. Its exterior would require upwards of 1200 square feet, or 133 square yards of silk.

Professor Playfair, (p. 254, vol. 1, of *Outlines of Natural Philosophy*,) has erroneously treated this subject. The weight of the gas is there wrongly computed.

754. Since the altitudes of fluids in a tube are inversely as their specific gravities, if a be altitude of the water sustained by the air, we have

$$\frac{1}{13568} : 30\tfrac{1}{2} :: \frac{1}{1000} : a$$

$\therefore a = 413.824$ inches.

and the required length is $\therefore$

$$l = \frac{a}{\sin. 30^\circ} = 2a = 827.648 \text{ inches.}$$

755. Generally, we have, by 749,

$$x^2 + (l-h)\,x = ah, \;\therefore h = \frac{x^2+lx}{a+x},$$

where a is the depth of air in the tube before inversion, h the standard altitude, and x the defect from that altitude, caused by the elasticity of the air.

But by the question,

$$x = 33 - \left(\frac{1}{4} + 28\right) = 4\frac{3}{4} \text{ inches}$$

$$l = 33 \text{ inches, and } a = \tfrac{1}{4} \text{ inch.}$$

Therefore

$$h = 35\frac{17}{20} \text{ inches.}$$

756. Let E denote the elasticity of the fluid when occupying a given space $\pi r^2 h$, r being the radius of the base of the tube; then its elasticity, when occupying the space

$\pi r^2 x$

is $E \times \frac{h}{x}$.

Hence if W denote the weight of the piston, the moving force upon it is

$$W - E \cdot \frac{h}{x}$$

and the force which accelerates its descent is

$$F = 1 - \frac{E}{W} \times \frac{h}{x}.$$

Hence $(vdv = -Fdx)$

$$v^2 = 2\left\{\frac{Eh}{W} \times lx - x\right\} + C$$

$$= 2\left\{\frac{Eh}{W} \times l.\frac{x}{h} + h - x\right\}$$

Let $x = h - S$

Then

$$v^2 = 2 \cdot \left\{\frac{Eh}{W} \times l.\frac{h-S}{h} + S\right\}$$

which gives the velocity required.

If the fluid be atmospheric air, its elasticity in a natural state, or when the barometer stands at 30 inches, is about 15lb. upon the square inch, and we have

$$E = \pi r^2 \times 15 \text{ lbs.};$$

and if in addition to the piston there is pressing upon it the whole atmospherical column of air, for W we must put $W + \pi r^2 \times 15$, and we then get

$$v^2 = 2 . \left\{ \frac{15\pi r^2 h}{15\pi r^2 + W} \ l. \frac{h-S}{h} + S \right\}.$$

If the time of acquiring this velocity be required, we have

$$dt = \frac{dx}{v} = \frac{1}{\sqrt{2}} \frac{dx}{\left(h - \frac{Eh}{W} \ l.h + \frac{Eh}{W} \ lx - x\right)^{\frac{1}{2}}}$$

757. In the condenser, the density of the air in the receiver after t turns is (*Vince*, p. 112,)

$$\delta = D \times \frac{(r+tb)}{r} \ldots\ldots\ldots\ldots\ldots (1)$$

D being the density at first, and r and b the capacities of the receiver and barrel.

Again, in the air-pump, after t turns the density, is (*Vince*, 107,)

$$\delta' = D' \times \frac{(b+r)^t}{(2b+r)^t} \ldots\ldots\ldots\ldots\ldots (2)$$

Let $b = \frac{r}{10}$, $t = 3$; then we have

$$\delta = D \times \frac{10+3}{10} = D + \frac{3D}{10}$$

or the density has increased $\frac{3}{10}$ths.

$$\text{Also } \delta' = D' \times \frac{11^3}{12^3}$$

$$\doteq D' - D' \times \frac{353}{1684}$$

or the density has decreased $\frac{353}{1684}$ths.

758. Let R be the radius of the body of the pump, r that of the *suction-tube*. Also let H be the height of a column of water whose weight = weight of a column, having the same base, of the atmosphere, s the length above the water of the suction-pipe, p the play of the piston, u the height of a column of water equivalent to the pressure of the air below the piston, and x the elevation due to the first stroke. Then

$$x + u = \text{H} \ldots\ldots\ldots\ldots\ldots (1)$$

Also

$$u = \frac{\pi r^2 \times s}{\pi \text{R}^2 p + \pi r^2 . (s - x)} \times \text{H}$$

$$= \frac{r^2 s \text{H}}{\text{R}^2 p + r^2 (s - x)}.$$

Hence

$$x^2 - (\text{H} + \frac{\text{R}^2}{r^2} p + s) x = - \frac{\text{R}^2}{r^2} p \, \text{H}$$

$$\therefore x = \frac{1}{2}\left\{\text{H} + \frac{\text{R}^2}{r^2} p + s \pm \sqrt{(\text{H} + \frac{\text{R}^2}{r^2} p + s)^2 - 4 \frac{\text{R}^2}{r^2} p \, \text{H}}\right\}$$

$$= \frac{1}{2}\left\{\text{A} \pm \sqrt{\text{A}^2 - \text{B}}\right\}$$

by supposition.

Hence

$$u = \frac{1}{2}\left\{2\text{H} - \text{A} \mp \sqrt{\text{A}^2 - \text{B}^2}\right\}$$

Again, let x_2 be the elevation of the water in the tube at the end of the second, and u_2 the elastic force of the air within the pump at that period; then as before

$$x_2 + u_2 = \text{H}$$

$$\text{and } u_2 = u \times \frac{r^2 . (s - x)}{\text{R}^2 p + r^2 (s - x_2)}$$

Hence

$$x_2 = \frac{1}{2}\left\{\text{A}^2 \pm \sqrt{\text{A}^2 - \text{B} - 4x\,(\text{H} + s - x)}\right\}$$

$$u_2 = \frac{1}{2}\left\{2\text{H} - \text{A} \mp \sqrt{\text{A}^2 - \text{B} - 4x\,(\text{H} + s - x)}\right\}$$

and so on to x_n and u_n.

Since x and u must be both less than H, it follows that the lower sign of the root is alone applicable.

759. If D be the density of the air at first, and δ that after m turns of the air-pump, we have (*Vince*, p. 107,)

$$\delta = \mathrm{D} \times \frac{(b+r)^m}{(2b+r)^m}$$

$$= \mathrm{D} \times \left(\frac{n+1}{n+2}\right)^m \ldots\ldots\ldots\ldots\ldots (1)$$

Again, let x be the required depth of air in the tube at first, and y the standard altitude; then (*Vince*, p. 92,)

$$y : y - a :: l - a : x$$

$$\therefore xy = (l - a)(y - a) \ldots\ldots\ldots\ldots (2)$$

l being the length of the tube.

Also if y' denote the standard altitude when under the receiver, we have, by the same rule

$$y' : y' - b :: l - b : x$$

But $y' = y \dfrac{\delta}{\mathrm{D}}$. Therefore

$$xy = \frac{\mathrm{D}}{\delta}(l - b)\left(y \frac{\delta}{\mathrm{D}} - b\right) \ldots\ldots\ldots (3)$$

Hence

$$(l - a)(y - a) = \frac{\mathrm{D}}{\delta}(l - b)\left(y \frac{\delta}{\mathrm{D}} - b\right)$$

which gives

$$y = \frac{b\mathrm{D}.(l - b) + a\delta(l - a)}{\delta(a - b)}$$

$$= \frac{b.(n + 2)^m (l - b) + a(n + 1)^m (l - a)}{(n + 1)^m (a - b)}.$$

Hence

$$x = \frac{(l - a)(y - a)}{y}$$

$$= (l-a)(a-b)(n+1)^m \times \frac{b(l-b)(n+2)^m + a(l+b)(n+1)^m}{b(l-b)(n+2)^m + a(l-a)(n+1)^m}$$

760. Let D, d, be the densities at first of the two airs, and δ that of the atmosphere when its elasticity is E. Also let D′, d', denote the densities at any other time t.

Now since the fluids are elastic, they will always fill the vessels; hence the velocity with which the denser air would rush into a vacuum at the end of this time is

$$V = \sqrt{2gh \times \frac{\delta}{D'}}$$

h being the altitude of a cylinder of water equivalent to the weight of the atmosphere.

But the moving forces with which this would take place, and with which the same fluid would rush into the vessel containing the rarer air, are respectively

$$\frac{E}{\delta} \times D', \text{ and } \frac{E}{\delta} \times (D' - d')$$

and these moving forces are as the squares of the velocities

$$\therefore \frac{E}{\delta} D' : \frac{E}{\delta} . (D' - d') :: V^2 : v^2$$

$$\therefore v = \surd 2gh\delta \, \frac{D' - d'}{D'^2}.$$

Again, if C, c, denote the capacities of the vessels, we have

$$C \times D + c \times d = CD' + c \times d'$$

$$\therefore v^2 = 2gh\delta \times \frac{D' - \frac{C}{c} . (D - D') - d}{D'}$$

$$= 2gh \times \frac{\overline{C + c} . D' - \overline{cd + CD}}{cD'^2}$$

which gives the velocity for any degree of density of the denser medium. If D′ be eliminated, a similar formula will be obtained, giving the velocity corresponding to any density of the rarer medium.

761. Generally, supposing the basin and tube cylindrical, let R, r, be their radii in inches, and when the mercury in the tube shall have descended x inches, let that in the basin ascend through X inches; then

$$\pi r^2 \times x = (\pi R^2 - \pi r^2) X$$

$$\therefore X = \frac{r^2 x}{R^2 - r^2}$$

Hence the real depression of the mercury is

$$X + x = \frac{R^2 x}{R^2 - r^2}.$$

In the problem $R = 3r$

$$\therefore \frac{X + x}{x} = \frac{9r^2}{8r^2} = \frac{9}{8}.$$

762. The spaces decrease in Harmonical Progression, (*Vince*, Prop. LXX). Hence if x be the space after the first descent, we have (by the question and the nature of the progression)

$$12 : 6 :: 12 - x : x - 6$$

$$\therefore 18x = 2 \times 72$$

$$\therefore x = 8$$

$\therefore \frac{1}{12} - \frac{1}{8} = -\frac{1}{24}$ is the common difference of the reprocals of the spaces, so that the n^{th} space is

$$1 \div \frac{1}{12} + \frac{n}{24} = \frac{24}{n+2}.$$

763. Let x, x' be the depressions after immersion, the standard altitude being then 30 inches. Then (749) and by the question

$$\left.\begin{array}{l} x^2 + (a - 30)x = 30b \\ x'^2 + (a' - 30)x' = 30b' \end{array}\right\}$$

whence

$$x = \frac{-(a - 30) + \sqrt{(a - 30)^2 + 120b}}{2} = A$$

$$x' = \frac{-(a' - 30) + \sqrt{(a' - 30)^2 + 120b'}}{2} = A'$$

a known.

Again, let h be the new standard altitude, then we have

$$(x+1)^2+(a-h)(x+1)=hb$$

$$\therefore\ h=\frac{(x+1)^2+a.(x+1)}{b}=\frac{A+1}{b}\times(A+a+1)$$

is known. Hence if y denote the corresponding depression in the second tube, we have (749)

$$y=\frac{-(a'-h)+\sqrt{(a'-h)^2+4hb'}}{2}.$$

Q. E. I.

764. The number of degrees in Fahrenheit's thermometer between the freezing and boiling points of water is

$$212-32=180.$$

Hence if the diminution of bulk be $\frac{1}{n}$th of the whole for each of these degrees, $2r$ the diameter of the tube, and B the magnitude of the bulb; then the several diminutions will be

$$\frac{B+\frac{212}{180}a\pi r^2}{n},\ \frac{B+\frac{211\pi r^2 a}{180}}{n},\ \&c.$$

$$\frac{B+\frac{32}{180}\pi r^2 a}{n},$$ and the whole diminution is

$$\frac{180}{n}B+\frac{\pi r^2 a}{n\times 180}\times(32+212)\frac{180}{2}=\pi r^2 a$$

$$\therefore\ B=\frac{\pi r^2 a}{180}\times(n-122)\text{ cubic inches.}$$

765. Generally, let R be the radius of the sphere, r the radius of the tube; then the surface of the segment of the sphere is (*Vince's Fluxions*, p. 110,)

$$S=4\pi R^2-2\pi R\times\sqrt{2Rr-r^2}.$$

Moreover the centre of gravity of this surface is distant from the centre of the base of the segment by (*Vince's Fluxions*, p. 118,)

$$G = \frac{1}{2} \cdot (2R - \sqrt{2Rr - r^2}) + \sqrt{2Rr - r^2}$$

$$= R + \frac{1}{2}\sqrt{2Rr - r^2}.$$

Hence the pressure upon the surface of the ball is

$$P = S \times \overline{G + 10} \times \text{density of the fluid}$$

$$= \text{weight of a volume } S \times (G + 10) \text{ of the mercury.}$$

Now, when the mercury just fills the segment of the sphere, let the weight of a cubic inch of it be w; then since the volume of the segment of the sphere is (*Vince's Fluxions*, p. 95)

$$M = \pi (2R - \sqrt{2Rr - r^2})^2 \times \left(R - \frac{2R - \sqrt{2Rr - r^2}}{3}\right)$$

the weight of a cubic inch when the mercury has risen 10 inches in the tube is

$$\frac{M}{M + 10\pi r^2} w$$

and the required pressure is finally expressed by

$$P = \frac{Mw}{M + 10\pi r^2} \times S \times (G + 10).$$

766. Let δ be the density of the air at first, h the height of a homogenous atmosphere; D the density of the water; then the water would rush into the vessel, completely empty, with a velocity

$$V = \sqrt{2ga}$$

and with a moving force measured by

$$M = \delta h + Da.$$

and when the water shall have risen through the space x, the moving force will be

$$M' = \delta h + Da - \frac{4\pi r^3}{3} \times \frac{1}{\frac{4\pi r^3}{3} - \pi x^2\left(r - \frac{x}{3}\right)} \times \delta h$$

$$= Da - \frac{x^2(3r - x)}{4r^3 - x^2(3r - x)}\, \delta h.$$

But $M \propto$ square of the velocity. Hence the velocity at this stage of the filling, *viz.*

$$v = V \times \sqrt{\frac{M'}{M}} = \left\{2ga.\frac{Da - \frac{x^2.(3r - x)}{4r^3 - x^2(3r - x)}\,\delta h}{\delta h + Da}\right\}^{\frac{1}{2}}$$

is known.

Let x be 0. Then at first the water rushes into the globe with the velocity

$$v = a\sqrt{\frac{2gD}{\delta h + Da}}.$$

Again, when $v = 0$, we have

$$x^2.(3r - x) = Da\,(4r^3 - x^2\,\overline{3r - x})$$

$$\therefore\ x^2(3r - x) = \frac{4Dar^3}{1 + Da}.$$

Hence, the portion of the sphere which the water then occupies, is

$$\frac{\pi}{3}. \times x^2(3r - x) = \frac{4\pi}{3}.\frac{ar^3}{1 + Da}.$$

By the resolution of a cubic equation x also may be found.

767. Let a be the altitude and r the radius of the base of the cylinder, and A the area of the orifice. Also let S be the density at first of the air, E its elasticity at first; s its density, and e its elasticity at the end of the time t. Also, let V and v denote the initial and subsequent velocities of the issuing medium.

Now, supposing Q $\times$ S the mass of air expelled when the piston has descended through $a - x$, we have

$$Q \times S = \pi r^2 \times aS - \pi r^2 x \times s \ldots\ldots (1)$$

and since the expelling force at first and afterwards are

$$E \text{ and } \frac{s}{S} \times E$$

$\therefore$ we have

$$1 : \frac{s}{S} :: SV^2 : sv^2$$

since the moving forces are as the quantities of motion generated in a given time.

Hence, $v = V$ or it is constant.

Again, the quantity discharged in the time dt is

$$V \times A \times s \times dt$$

and it is also measured by

$$-\pi r^2 a \times ds$$

$$\therefore dt = -\frac{\pi r^2 a}{AV} \cdot \frac{ds}{s}$$

$$\therefore t = \frac{\pi r^2 a}{AV} \times l. \frac{S}{s}$$

$$\therefore s = S \times c^{-\frac{AVt}{\pi r^2 a}}$$

which being substituted in (1) gives the quantity of air expelled during t'', *viz.*

$$Qs = \pi r^2 S \times (a - xe^{-\frac{AVt}{\pi r^2 a}}),$$

x being a known function of t'.

Let $x = 0$. Then

$$Qs = \pi r^2 a S$$

the whole content, as it ought.

OPTICS.

768. If q, q', denote the distances of the object and its image from the centre of the reflector whose radius is r, we have (see *Coddington's Optics*, p. 8)

$$\frac{1}{q'} = \frac{1}{q} + \frac{2}{r}.$$

Also, when the object, O, is very small, compared with the radius of the reflector, it and its image O′ may be considered circular arcs, and in the limit, we have

$$\begin{aligned} \mathrm{O} : \mathrm{O}' &:: q : q' \\ &:: q : \frac{rq}{r+2q} \\ &:: r + 2q : r. \end{aligned}$$

But, if θ, θ' be the angles subtended by O, O′ at the vertex, we get

$$\begin{aligned} \mathrm{O} : \mathrm{O}' &:: (r+q)\theta : (r-q')\theta' \\ &:: (r+q)\,\theta : \left(r - \frac{rq}{r+2q}\right)\theta' \\ &:: (r+2q)\,\theta : r\theta' \end{aligned}$$

$$\therefore r + 2q : r :: (r+2q)\,\theta : r\theta'$$

$$\therefore \theta = \theta'.$$

769. If Δ, Δ'' denote the distances of the object O and image O′ from the lens; then (*Coddington's Optics*, p. 120)

$$\mathrm{O} : \mathrm{O}' :: \Delta : \Delta''$$

But in the lens (*Coddington*, p. 61.)

$$\begin{aligned} \frac{1}{\Delta''} &= (m-1)\left(\frac{1}{r} + \frac{1}{r'}\right) + \frac{1}{\Delta} \\ &= \frac{1}{\mathrm{F}} + \frac{1}{\Delta}. \end{aligned}$$

F being the distance of the principal focus

$$\therefore o : o' :: \Delta : \frac{F\Delta}{\Delta + F} :: \Delta + F : F.$$

Now, by the question, $o : o' :: 1 : 2$, and since the image is to be erect, the object and image are on the same side of the lens, and the distance of the principal focus on the other side is negative

$$\therefore 1 : 2 :: F - \Delta : F$$

$$\therefore \Delta = \frac{F}{2} = \frac{m-1}{2} \times \frac{r+r'}{rr'}.$$

When $r = r'$

$$\Delta = (m-1)\, r.$$

770. The principal focal length, measured from the centre is (*Coddington*, p. 66)

$$F = -\frac{mr}{2(m-1)}.$$

Therefore, by the question,

$$-r = -\frac{mr}{2(m-1)}$$

and $m = \dfrac{\text{sin. incidence}}{\text{sin. refraction}} = 2.$

771. Let a be the distance between the object and the eye, and x the distance of the lens from the object; then (*Coddington*, p. 120) the

$$\text{Visual angle } \theta \propto \frac{1}{(F - \Delta) \times (\Delta'' + k)}$$

But $\dfrac{1}{\Delta''} = \dfrac{1}{\Delta} - \dfrac{1}{F}$ (*Coddington*, p. 61.)

$$\therefore \theta \propto \frac{1}{(F - \Delta)\, k + \Delta \times F}$$

and by the question

$$(F - \Delta)\, k + \Delta \times F = \text{minimum}$$

But $\Delta = x$, and $k = a - x$

$$\therefore x^2 - ax + a\text{F} = \text{min.}$$
$$\therefore 2xdr - adr = 0$$

and $x = \frac{a}{2}$

which gives the position required.

772. Since the angle of deviation increases with the angle (φ) of incidence, (see *Wood's Optics*, p. 46,) the rod will be most bent, when this latter angle is a maximum. Now, the angle of refraction being 90°, is a maximum, and then

$$\sin. \varphi = m \times \sin. 90 = m$$

$\therefore$ generally the maximum deviation is

$$90^\circ - \sin.^{-1} m.$$

In the problem $m = \frac{3}{4}$, and the deviation is

$$90^\circ - \sin.^{-1} \frac{3}{4} = 90^\circ - 48^\circ\ 36'$$
$$= 41^\circ - 24'.$$

773. Let r be the radius of the base, and h the height of the cylinder. Then, if φ, φ' be the angles of incidence and refraction, we have

$$\sin. \varphi = m \sin. \varphi' = \frac{4}{3} \sin. \varphi'$$

But $\sin. \varphi = \dfrac{2r}{\sqrt{(4r^2 + h^2)}}$.

$$\therefore \sin. \varphi' = \frac{3}{2} \cdot \frac{r}{\sqrt{(4r^2 + h^2)}} \ldots\ldots (1)$$

Therefore by the question,

$$r + x \times \tan. \varphi' : 2r :: x : h$$

x being the required depth.

Hence,

$$x = \frac{hr}{2r - h \tan. \varphi'} = \frac{h\sqrt{(7r^2 + 4h^2)}}{2\sqrt{(7r^2 + 4h^2)} - 3h}$$

774. The density of the sun's rays, or the brightness of his image when viewed with a reflector or refractor $\propto$

$$\frac{\text{area of aperture} \times \text{power}}{(\text{Focal Length})^2} : \text{(see } \textit{Wood}\text{, p. 120.)}$$

Hence, supposing the power the same in both the reflector and lens we have, by the question

$$\frac{\pi r^2}{F^2} = \frac{\pi r'^2}{F'^2}$$

$2r$, $2r'$ being the linear apertures, and F, F′ the focal lengths.

But $F' = 2F$ (*Wood*, pp. 90, 95.)

$$\therefore r = \frac{r'}{2}.$$

775. Since the object O is very small, it may be considered an arc of a circle concentric with a vertical section of the cylinder. Hence the image O′ will also be a circular arc concentric with the object, and if q, q' denote their radii, we have

$$O : O' :: q : q'.$$

But (*Wood*, Prop. 43, Cor. 1.)

$$q - f : f :: q : q'$$

f being the focal distance

$$\therefore O : O' :: q - f : f$$

which gives the magnitude of the image, when q and f are known.

776. In the sphere the angle of incidence at the second surface is always equal to the angle of refraction at the first surface. Hence it will readily appear, upon drawing the figure, that the angle of incidence at the first surface is equal to the angle of refraction at the second, and therefore the ray intercepted between the surfaces will be the base of two opposite isosceles Δ, whose vertices are at the centre and intersection of the first and last directions of the ray. Now, when the distance of the centre from this radiant chord is given, the chord itself is given, and the

angle it subtends—which call β; and then denoting by ϕ, ϕ' the angles of incidence and refraction, we have

$$\phi' = \frac{\pi}{2} - \frac{\beta}{2}.$$

$$\sin. \phi = m \sin. \phi' = m \cos. \frac{\beta}{2}.$$

Hence, the angle of deviation, or

$$2(\phi - \phi') = 2 \sin.^{-1}\left(m \cos. \frac{\beta}{2}\right) - \pi + \beta$$

which is constant. Q. E. D.

777. If F, F' be the focal lengths of the simple lenses, and ϕ that of the compound lens; then (*Coddington*, p. 68)

$$\frac{1}{\phi} = \frac{1}{F} + \frac{1}{F'}$$

But by the question F = F', and (*Coddington*, p. 62)

$$\frac{1}{F} = -(m-1)\left(\frac{1}{r} + \frac{1}{r'}\right) = -\frac{2(m-1)}{r}$$

when $r = r'$

$$\therefore \frac{1}{\phi} = -\frac{4(m-1)}{r}.$$

778. The apparent image is the caustic formed by the intersections of the refracted rays proceeding from the eye as a radiant point to the linear surface of the water, which is in the same plane with the given straight line.

Let x, y be the co-ordinates of this caustic measured along the $\perp$ upon the surface from the eye, and originating in the intersection of the $\perp$ and surface; also, let θ, θ' be the inclinations of any incident and refracted ray to the $\perp$, and Δ the distance of the eye from the surface; then (*Coddington*, p. 81)

$$x \tan. \theta' - \Delta \tan. \theta + y = 0$$

$$x \frac{d\theta'}{\cos.^2 \theta'} - \Delta \frac{d\theta}{\cos.^2 \theta} = 0$$

and $\sin. \theta = m \sin. \theta'$.

Hence

$$\frac{d\theta}{d\theta'} = \frac{x}{\Delta} \cdot \frac{\cos.^2 \theta}{\cos.^2 \theta'} = \frac{m \cos. \theta'}{\cos. \theta}$$

$$\therefore \frac{\cos. \theta}{\cos. \theta'} \left(\frac{m\,\Delta}{x}\right)^{\frac{1}{3}}.$$

$$\therefore \tan. \theta' = \left(\frac{m\,\Delta}{x}\right)^{\frac{1}{3}} \times \frac{1}{m} \times \tan. \theta$$

But $\tan. \theta' = \dfrac{\sin. \theta'}{\cos. \theta'} = \dfrac{\sin. \theta}{\sqrt{1 - \dfrac{\sin.^2 \theta}{m^2}}}$

$$= \frac{\sin. \theta}{\sqrt{m^2 - \sin.^2 \theta}}$$

$$\therefore \left(\frac{m\,\Delta}{x}\right)^{\frac{1}{3}} \times \frac{1}{m} \times \sqrt{(m^2 - \sin.^2 \theta)} = \sqrt{1 - \sin.^2 \theta}.$$

Hence we find.

$$\tan. \theta = \frac{m^{\frac{2}{3}}}{\Delta^{\frac{1}{3}}} \times \sqrt{\frac{x^{\frac{2}{3}} - m^{\frac{2}{3}} \Delta^{\frac{2}{3}}}{m^2 - 1}}$$

and $\tan. \theta' = \dfrac{1}{x^{\frac{1}{3}}} \times \sqrt{\dfrac{x^{\frac{2}{3}} - m^{\frac{2}{3}} \Delta^{\frac{2}{3}}}{m^2 - 1}}$,

which being substituted in the first equation, give

$$y = (\Delta^{\frac{2}{3}} m^{\frac{2}{3}} - x^{\frac{2}{3}}) \sqrt{\frac{x^{\frac{2}{3}} - m^{\frac{2}{3}} \Delta^{\frac{2}{3}}}{m^2 - 1}}$$

the equation to the caustic, which being traced by the known rules for describing curves in general, the problem will be completely resolved.

779. Let $2t$ be the thickness of the lens, $2b$ its breadth, and r the focal length; then, since the focal length is equal to the radius (*Wood*, Art. 171), by the nature of the circle we have

$$b^2 = 2r\,t - t^2$$

But t being very small, $t^2 = 0$ nearly

$$\therefore b^2 = r \times 2t. \qquad \text{Q. E. D.}$$

780. If q be the distance of the focus of incident, and q'

that of the emergent rays from the centre of the sphere; then (*Coddington*, p. 66)

$$\frac{1}{q''} = \frac{1}{q} - 2.\frac{m-1}{mr}$$

∴ the focal length is

$$q'' = \frac{mrq}{mr - 2(m-1)q} = nr$$

by the question. Hence

$$m = \frac{2qn}{q(2n+1) - nr}$$

the ratio required.

781. If Δ' be the real and apparent ⊥ depths of the object; then (*Coddington*, p. 45)

$$\Delta' = \Delta \times m$$

m being the given ratio of refraction. Hence, having found the real situation of the object, let a ⊥ be let fall from the situation fired from upon the line joining the object and image, and call this ⊥ p. Also, let the distance of this ⊥ from the surface of the water be d, and β denote the inclination of the required direction to the surface of the water; then

$$p : d + \Delta :: 1 : \tan.\,\theta$$

$$\therefore \tan.\,\theta = \frac{d + m\,\Delta'}{p}$$

which gives the ∠ required.

782. If a circle be described passing through the two extremities of the flag-staff and touching the horizon, the ∠ subtended by the flag-staff at the horizon will be the greatest possible; because all others must fall without the circumference of this circle.

783. Let a be the given distance of the object from the reflector, b that of the eye, and x, y, the co-ordinates of the

required locus measured from the extremity of the rectilinear locus of the object; then, from the conditions of the problem, and the nature of reflection, it is easily seen that

$$x = x' + x'' = \sqrt{a^2 - \frac{a^2}{b^2}\,y^2} + \sqrt{b^2 - y^2}$$

$$= \frac{a}{b} \cdot \sqrt{b^2 - y^2}$$

or the locus is an ellipse whose semi-axes are b and a.

784. Let a, b, c, be the length of the three objects, each subtending at the eye the same $\angle\,\theta$. Also, let x be the first side of the angle subtended by a, and φ its opposite $\angle$; then we have

$$a : x :: \sin.\,\theta : \sin.\,\varphi$$

$$a + b : x :: \sin.\,2\theta : \sin.\,(\varphi - \theta)$$

$$a + b + c : x :: \sin.\,3\theta : \sin.\,(\varphi - 2\theta)$$

which give

$$\frac{\sin.\,2\theta}{\sin.\,\theta} = \frac{a+b}{a} \times \frac{\sin.\,(\varphi - \theta)}{\sin.\,\varphi}$$

and $$\frac{\sin.\,3\theta}{\sin.\,\theta} = \frac{a+b+c}{a} \times \frac{\sin.\,(\varphi - 2\theta)}{\sin.\,\varphi}.$$

Hence, by expanding the sines of the multiple arcs, we get

$$\cot.\,\varphi = \frac{b-a}{b+a} \cdot \cot.\,\theta$$

and $$\cot.\,\varphi = \frac{2\cos.^2\theta\,(b + c - a) - (b + c)}{2(a + b + c)\sin.\,\theta.\cos.\,\theta},$$

$$\therefore 2(a + b + c) \cdot \frac{b-a}{b+a} \cdot \cos.^2\theta = 2\cos.^2\theta \times (b+c-a) - b+c$$

which gives

$$\cos.\,\theta = \frac{1}{2} \times \sqrt{\frac{(b + c)(a + b)}{ac}}$$

$$\therefore \sin.\,\theta = \frac{1}{2}\sqrt{\frac{4ac - (b + c)(a + b)}{ac}}$$

$$\therefore \cot.\,\theta = \sqrt{\frac{(b + c)(a + b)}{4ac - (b+c)(a + b)}}$$

$$\therefore \cot.\varphi = \frac{b-a}{\sqrt{a+b}} \times \sqrt{\frac{b+c}{4ac-(b+c).(a+b)}}.$$

Hence

$$\sin.\varphi = \frac{1}{2}.\sqrt{\frac{(a+b)(4ac - \overline{b+c}.\overline{a+b})}{a(ca-b^2)}}$$

$$\therefore x = a \times \frac{\sin.\varphi}{\sin.\theta}$$

$$= \sqrt{\frac{ac.(a+b)}{ac-b^2}}$$

which determines the position of the eye.

785. Describe a circle passing through the two extremities of the given object, and touching the circle (see p. 7, vol. I.); the point of contact gives the maximum or minimum, according as the circles touch with their convexities or concavities.

786. The focal length is (*Coddington*, p. 68)

$$F = \frac{r}{m-1}.$$

But $m = 1.5$.

$\therefore F = 2r$

787. The image of the ring will be equal to the object, and equally distant from the surface of the reflector. Let α, β, be the visual angles of the diameter $2r$ of the object and image, a the distance of the centre of the object from the reflector, and b the distance of the eye from that centre; then

$$r = b \tan.\frac{\alpha}{2}$$

and we easily get

$$\cos.\beta = \frac{b^2 - r^2 + 4a^2}{\sqrt{\{b^4+(8a^2+2r^2)b^2+(4a^2-r^2)^2\}}}.$$

Hence α and β may be found by the table, and the apparent magnitudes of the object and image may be compared.

788. Since the image is always at the same distance from the reflector as the object, the reflector will always bisect that part of the revolving line which is intercepted by the given fixed lines. Hence, if r, r' denote the distances of the axis of revolution from the two given lines, measured along the revolving line, when in such a position as to be equally inclined to both of them; 2β be the inclination of the given lines, and θ the angle described from r, and ρ the radius vector of the required locus, it may easily be shewn that

$$\rho = \frac{\cos.\beta}{2} \times \frac{r\cos.(\beta+\theta)+r'\cos.(\beta-\theta)}{\cos.(\beta+\theta).\cos.(\beta-\theta)}$$

$$\text{or} = \frac{\cos.\beta}{2}.\{r\sec.(\beta-\theta)+r'\sec.(\beta+\theta)\}$$

which is the polar equation to the required locus.

The student may amuse himself with deducing other equations to the curve, and by tracing it through all its ramifications.

789. Since, by hypothesis, the object is very distant compared with its magnitude, it may be considered a circular arc concentric with the reflector. Hence also the image is a circular arc similar and concentric with the object, (*Wood*, p. 113,) and if O, O' be the absolute magnitudes of the object and image, we have

$$O : O' :: q : q'$$

q and q' being the radii of the object and image.

Again, let the given distance between the eye and image be d, and α, α' the known $\angle$ at the eye subtended by the object and image, and x the distance of the object from the eye, and r the radius of the reflector; then from the great distance of the object from the eye it may be considered a circular arc, having the eye for its centre,

$$\therefore O = x \times \alpha$$

$$O' = d \times \alpha'$$

$$\therefore x \times \alpha : d \times \alpha' :: q : q'$$

$$\text{But } q : q' :: QF : FE :: f - q : f$$

$$\therefore\ q' = \frac{fq}{f-q}.$$

Moreover (*Coddington*, p. 55,)

$$m = \frac{q}{r-q} \cdot \frac{r-q'}{q'}$$

$$\therefore\ q' = \frac{qr}{mr - \overline{m-1}.q}$$

$$\therefore\ \frac{f}{f-q} = \frac{r}{mr - \overline{m-1}.q}$$

$$\therefore\ q = \frac{fr\,(1-m)}{r-(m-1)f}.$$

Hence

$$x \times \alpha : d \times \alpha' :: f-q : f$$
$$:: rm - (m-1)f : r - \overline{m-1}\,f$$

But (*Coddington*, p. 56,)

$$f = -\frac{m}{m-1}\,r$$

$$\therefore\ x \times \alpha : d \times \alpha' :: rm + mr : r + mr$$
$$:: 2m : m+1$$

$$\therefore\ x = \frac{2m}{m+1} \times \frac{\alpha'}{\alpha}\,d.$$

Hence

$$q' = q + d - x = \frac{fq}{f-q}$$

$$\therefore\ d - x = \frac{q^2}{f-q}.$$

$$\text{But } q = \frac{fr\,(1-m)}{r - \overline{m-1}\,f} = \frac{mr}{m+1}$$

$$\therefore\ d - x = \frac{r}{2} \times \frac{m-1}{m+1}$$

$$\therefore\ r = 2 \cdot \frac{m+1}{m-1} \cdot (d-x)$$

which determines the magnitude of the reflector.

Hence

$$q = \frac{2m}{m-1} \cdot (d-x)$$

and $x - q$ is therefore known, which, being the distance of the centre of the reflector from the eye, gives the position of the reflector.

790. Let a be the altitude of the cylinder, r the radius of its base, also let m be the ratio of refraction; then the depth of the image of the centre of the base is

$$\frac{a}{m}$$

and if ϕ denote the inclination of the line joining the eye and edge of the cylinder to the vertical, we have

$$r = \frac{a}{m} \tan. \theta$$

$$\therefore \tan. \theta = \frac{mr}{a}$$

which gives the direction of vision. Again, let h be the height of the eye above the fluid, and x the distance of the person from the vessel; then

$$x = h \tan. \theta = \frac{mhr}{a}.$$

791. The rays of light proceeding from the wick form a frustum of a cone, whose axis is parallel to the wall, and smaller end the top of the cover. This cone of light being of indefinite length is cut by the plane of the wall, which, being parallel to the axis, the section will be the common hyperbola.

If the top of the cover be not horizontal, the luminous figure on the wall will be an ellipse.

792. Changing the notation to that of the problem, and making r positive and r' negative, we have (*Coddington*, p. 65,)

$$\frac{1}{f} = \left(\frac{1}{n} - 1\right)\left\{\frac{1}{r'} + \frac{1}{r}\left(1 - \frac{\frac{1}{n} - 1}{\frac{1}{n}} \cdot \frac{t}{r}\right)^{-1}\right\}$$

$$= \frac{1-n}{n} \cdot \left\{\frac{1}{r} + \frac{1}{r'} + \overline{1-n} \cdot \frac{t}{r^2}\right\} \text{ nearly,}$$

the terms involving t^2, t^3, &c. being neglected because of their comparative smallness.

793. Generally, (*Wood*, p. 113,)

$$PR : pr :: QF : FE$$

$\therefore$ by the question

$$7 : 1 :: QF : FE$$

and $6 : 1 :: QE \, (= 5 \text{ feet}) : FE$

$$\therefore FE = \frac{5}{6} \text{ feet.}$$

794. He must use a double concave lens, whose focal length is (*Wood*, p. 133,)

$$F = \frac{QE \times Eq}{Qq} = \frac{14 \times 3}{11} = 3 \text{ feet } 9\frac{9}{11} \text{ inches.}$$

795. Let ϕ be the $\angle$ of incidence of the given ray, r the radius of the reflector, and x the distance of the reflected ray from the centre; then

$$d : r :: \sin. \phi : \sin. (60 - \phi)$$

and $x : r :: \sin. \phi : \sin. (60 + \phi)$.

From the first proportion we get

$$\tan. \phi = \frac{d\sqrt{3}}{2r + d}$$

and this being substituted in the second, gives

$$x = \frac{rd}{r+d}.$$

Hence, since the geometrical focus bisects the radius, the distance of the reflected ray from it is

$$\frac{r}{2} \mp \frac{rd}{r+d}$$

according as the focus of incident rays is nearer to the reflector than its centre is, or not.

796. Generally (*Coddington*, p. 66,)

$$f = \frac{mr}{2(m-1)}.$$

But here $m = \frac{4}{3}$, and $r = \frac{1}{10}$

$$\therefore f = \frac{2}{3} \times \frac{3}{10} = \frac{1}{5} \text{ of an inch}$$

797. Generally, (*Coddington*, p. 61,)

$$\frac{1}{\Delta''} = \frac{1}{\Delta} - \frac{1}{F}$$

But by the question

$$\Delta = 4, \text{ and } \Delta'' = 9$$

$$\therefore F = \frac{36}{5} = 7\frac{1}{5} \text{ inches.}$$

798. The density of the rays in the sun's image $\propto$ as the area of the aperture directly, and inversely as the focal length of the lens. If, therefore, d, d' be the densities for the sphere of water and the plano-convex lens of glass, we have

$$d : d' :: \frac{1}{f} : \frac{2}{f'}$$

But, (*Coddington*, pp. 66 and 63,)

$$\frac{1}{f} = 2 . \frac{m-1}{mr} \text{ and } \frac{1}{f'} = \frac{m'-1}{r'}$$

and by the question

$$m = \frac{4}{3}, m' = \frac{3}{2} \text{ and } r' = 2r$$

$$\therefore d : d' :: \frac{1}{2r} : \frac{1}{r} :: 1 : 2.$$

799. The glass must be a double concave lens, whose focal length is (*Wood*, p. 138,)

$$\text{FE} = \frac{\text{QE} \times \text{E}q}{\text{Q}q} = \frac{12 \times 3}{9} = 4 \text{ feet.}$$

800. Generally, *required the ratio of the sine of incidence to that of refraction, when, the cylinder being the* p*th part full, the eye (placed so as to see the farther extremity of the vessel when empty,) shall just see the centre.*

Let r be the radius of the cylinder's base, ϕ, ϕ' the angles of incidence and refraction from the eye into the medium, and x the distance of the centre of the base from the $\perp$ at the point of incidence; then

$$\sin.\phi : \sin.\phi' :: r + x : x$$

$$\text{and } r + x : 2r :: \frac{1}{p} : 1$$

$$\therefore x = \frac{r}{p}.(2 - p)$$

$$\therefore \sin\phi : \sin.\phi' :: 2 : 2 - p.$$

In the problem $p = \dfrac{4}{3}$.

$$\therefore \sin.\phi : \sin.\phi' :: 3 : 1.$$

801. When the object is between the principal focus and the centre of the lens, the image is erect. (*Wood*, p. 114.)

Also since the object and image are concentric and similar arcs, we have, by the question

$$1 : 3 :: \text{QE} : \text{E}q :: \text{QF} : \text{FE} \; (\textit{Wood}, \text{p. } 99,)$$
$$:: \text{FE} - \text{QE} : \text{FE}$$

$$\therefore \text{FE} = \frac{3}{2}\text{QE} = \frac{3}{2} \times 4 = 6 \text{ inches.}$$

802. Let x be the distance of the centre of the base from the vertex of the segment of the sphere whose surface is the nth part of the whole surface; then (*Vince's Flux.* p. 110,) this partial surface is

$$2\pi r x = \frac{1}{n} \cdot 4\pi r^2$$

by the question.

$$\therefore x = \frac{2r}{n}.$$

Again, since the extreme rays proceeding from the sphere to the eye will touch the sphere, we easily prove that the required distance from the vertex of the sphere to the eye is

$$d = \frac{rx}{r-x} = \frac{2nr}{n-2}.$$

803. The image recedes from the reflector as fast as the reflector from the fixed object, and in the same direction. Therefore the image recedes twice as fast from the object as the reflector.

804. The rays proceeding from the circumference of the circle to the eye form an oblique cone, which being produced, will be cut by a vertical plane, and the section thus made will be a circle when it is the subcontrary section of the cone. (See p. 18, and Fig. 15.) Hence the eye must be in such a position that the subcontrary section of the cone may be vertical.

Let a be the distance of the given locus of the eye from the centre of the given horizontal circle, whose radius call r, and x the required altitude of the eye. Also let θ, θ' be the $\angle$ subtended at the circumference, by the rays proceeding from the extremities of the diameter which meets the given locus. Then if the section be vertical we must have

$$\theta' = 90 + \theta$$

$$\text{and } x = (2r + a) \tan.\theta = a \tan.(180 - 90 - \theta)$$
$$= a \cdot \cot.\theta$$

$$\therefore \tan.^2\theta = \frac{a}{2r+a}, \text{ and } \tan.\theta = \sqrt{\frac{a}{2r+a}}$$

Also $x = \sqrt{a(2r+a)}$
which determines the required position.

805. Let r, r' be the radii of the spheres, and $r + r' + d$ the distance between their centres; then if any point be taken in the line joining the centres, and from it tangents be drawn to both spheres, the surfaces S, S′ visible at that point will be the portions cut off by planes through the extremities of the tangents ⊥ to d. Let x, x' be the abscissæ measured from the vertices of the spheres along d, of these portions; then (*Vince's Flux.* p. 110,)

$$S = 2\pi r x,\ S' = 2\pi r' x'$$

$$\therefore 2\pi r x + 2\pi r' x' = \text{max.}$$

$$\therefore r dx + r' dx' = 0.$$

Again, it is easily shewn that

$$d = \frac{rx}{r-x} + \frac{r'x'}{r'-x'} \quad \ldots\ldots\ldots\ldots (1)$$

and this gives

$$\frac{dx}{(r-x^2)} + \frac{dx'}{(r'-r')^2} = 0$$

$$\therefore \frac{r-x}{r'-x'} = \sqrt{\frac{r'}{r}} \quad \ldots\ldots\ldots\ldots (2)$$

Hence, and from equation (1), we get

$$x = \frac{(r'+d)\,r - r'^2\sqrt{\frac{r'}{r}}}{r+r'+d}$$

$$\text{and } x' = \frac{(r+d)\,r' - r^2\sqrt{\frac{r}{r'}}}{r+r'+d}$$

and the distance from the required to the surface whose radius is r, is

$$\frac{x}{r-x} = \frac{(r'+d)\,r - r'^2\sqrt{\frac{r'}{r}}}{r^2 + r'^2\sqrt{\frac{r'}{r}}}.$$

806. Let a be the height of the person's eye, b the length of the mirror, whose top is supposed to be on a level with the eye, and x the variable distance of the person from the glass; then since the image is equal and similar to the object, and at the same distance from the reflector's surface, if P be the length visible, we have (by similar $\triangle$)

$$P : b :: 2x : x$$
$$\therefore P = 2b.$$

807. Let m be the given ratio of the sines of incidence and refraction, r the radius of the sphere, m' the given ratio of the rays included by the sphere; also let p, p' be the unknown distances of these rays from the centre of the sphere; then it may easily be shewn that

$$\frac{p}{p'} = m$$

and $$\frac{2\sqrt{r^2 - p^2}}{2\sqrt{r^2 - p'^2}} = m'$$

which give

$$p = mr\sqrt{\frac{1 - m'^2}{m^2 - m'^2}}$$

the distance of the incident ray from the centre.

808. Let a be the $\perp$ distance of A from CP, (see Fig. to the Enunciation,) or of B from CQ; also let b be the distance of C from these perpendiculars, and suppose x, x' the distances of P and Q from the $\perp$ from A and B, and θ, θ' the $\angle$ of incidence at A and B; then we easily get these equations

$$x = a \tan. \theta$$
$$x' = a \tan \theta'$$
$$\frac{b - x}{b - x'} = \frac{\cos. (C - \theta)}{\cos. \theta}$$
$$\frac{b - x'}{b + x'} = \frac{\cos. (C - \theta')}{\cos. \theta'}$$

Hence are readily derived

$$\frac{b-x}{b+x'} = \cos. C - x \,.\, \frac{\sin. C}{a}$$

$$\text{and } \frac{b-x'}{b+x} = \cos. C - x' \,.\, \frac{\sin. C}{a}$$

and by further elimination and reduction, we finally obtain

$$x^2 - \frac{a \sin. C - b(1-\cos. C)}{1-\cos. C}\, x = \frac{ab \sin.^2 C + b^2 \sin. C - 2ab}{\sin. C \,.\, (1-\cos. C)}$$

which being resolved gives the positions of P.

The problem may be resolved more easily, although not so strictly, by considering the symmetry of the loci of P and Q with respect to the points A, C, B. It is evident, although not very easily demonstrable, that $\theta = \theta'$, and $x = x'$.

809. The caustic is the common cycloid, whose base is equal to, and coincident with, the base of the given semi-cycloid. (*Wood*, Prop. 100.)

Again, let dx denote the constant element of the base, and ds the corresponding variable element of the arc of the caustic; then the densities of the rays of light upon these elements will be inversely as their magnitudes; that is, since the light will be uniform upon the base,

$$\text{density} \propto \frac{1}{ds}.$$

But if y be the ordinate of the reflector parallel to the rays, by the nature of the curve, we have

$$dy : dx :: \text{vers. } \theta : \sin. \theta$$

$$\text{and } dy + ds = 0. \quad (\textit{Codd.}\ \text{p. } 22.)$$

$$\therefore \text{Density} \propto \frac{\sin. \theta}{\text{vers. } \theta}.$$

It does not therefore $\propto$ as in the enunciation.

810. The whole number of images is (*Codd.* p. 34,)

$$\frac{360^\circ}{60} = 6.$$

811. The focal length of a double convex lens is, (*Codd.* p. 62,)

$$\frac{r r'}{m(r+r')}$$

$\therefore$ by the question

$$2r = \frac{r}{2m}$$

and $m = \frac{1}{4}$ the ratio required.

812. The distance of the focus of refracted rays from the surface of a spherical refractor is (*Codd.* p. 56,)

$$\Delta' = \frac{mr}{m-1}$$

But for glass, $m = \frac{3}{2}$.

$$\therefore \Delta' = 3r,$$

so that the parallel rays will just reach in their convergence the plane reflector, and be reflected back again, converging to the spherical surface.

813. Let α denote the inclination of the two mirrors, and θ that of the first incident and second reflected ray intersecting at the luminous point; then it may be easily shewn that

$$\theta = \pi - 2\alpha.$$

Again, let a be the distance of the given point at which the ray is always reflected from the intersection of the two mirrors, and suppose the radiant point moving in the arc whose radius is x, to be now in the middle point of that arc; then it easily appears that

$$x : a :: \sin.\left(\frac{\pi - 2\alpha}{2} + \frac{\alpha}{2}\right) : \sin.\frac{\pi - 2\alpha}{2}$$

$$:: \sin.\left(\frac{\pi}{2} - \frac{\alpha}{2}\right) : \sin.\left(\frac{\pi}{2} - \alpha\right)$$

$$:: \cos.\frac{\alpha}{2} : \cos.\alpha$$

$$\therefore x = a \times \frac{\cos.\frac{\alpha}{2}}{\cos.\alpha}$$

the radius required.

814. Let q be the distance of any section of the cylinder $\perp$ to the coincident axis from the centre of the reflector, and q' that of the centre of the image of the circumference of this section, also let R, R′ denote the radii of the section and image; then

$$R : R' :: q : q'$$

$$:: R' \propto \frac{q'}{q}$$

But in the convex reflector (*Coddington*, p. 8)

$$\frac{1}{q'} = \frac{1}{q} - \frac{2}{r}$$

$$\therefore R' \propto \frac{r + 2q'}{r} \propto \text{ also } \frac{r}{r - 2q}$$

Let $q = a$ (the distance of the end of the cylinder from the reflector), then $q' = \dfrac{ra}{r - 2a}$; also when $q = \infty$, $q' = \dfrac{-r}{2}$.

Consequently the vertex of the image is in the principal focus, and the distance of any section of the image from this vertex being put $= d$, we have $d = q' + \dfrac{r}{2}$;

hence $R' \propto \dfrac{r + 2 \times \left(d - \frac{r}{2}\right)}{r} \propto d$, which is the property of the cone.

The base of the cone has a radius $R' = R . \frac{r}{r - 2a}$,

and its altitude is $\frac{r}{2} + \frac{ra}{r - 2a}$.

815. If δ, δ' denote the distances of the foci of incidence and reflection of any pencil of rays from the point of incidence $\delta_{,}$ $\delta'_{,}$ their next values, and D, D′ values of them for a given point of the reflector; then λ being the length of the caustic between these two points of incidence we easily prove, (*Wood, Prop.* 97)

$$d\lambda = \delta + \delta' - (\delta_{,} + \delta'_{,}) = d(\delta) + d(\delta')$$

$$\therefore \lambda = \delta + \delta' + C = \delta + \delta' - (D + D').$$

In the problem $\delta - D =$ ordinate (y) of the circle, and $D' = 0$ for the beginning of the caustic,

$$\therefore \lambda = y + \delta'$$

Also, (*Wood*, p. 244) $\delta' = \frac{1}{2} y$

$$\therefore \lambda = \frac{3}{2} y.$$

816. Since the image is always at the same distance from the reflector which produces it as the object, and on the opposite side of it, that part of the distance between the eye and image, which is included between the image and reflector is equal to the distance between the object and point of incidence. The remaining parts of the paths of the object and image to the eye coincide. $\therefore$ the two paths are equal.

817. The $\angle$ between the incident and emergent rays is (*Wood Prop.*, 93)

$$2(n + 1)\theta' - 2\theta$$

θ and θ' being the $\angle$ of incidence and refraction. But when the rays emerge parallel to the direction of incidence

$$2(n + 1)\theta' - 2\theta = 0$$

$$\therefore \theta : \theta' :: n + 1 : 1. \qquad \text{Q. E. D.}$$

818. The longitudinal aberration is (*Coddington*, p. 14)

$$\frac{r \text{ vers. } \theta}{2}$$

r being the radius of the reflector, and $\theta = \frac{1}{2}$ the aperture.

Hence, if α denote the lateral aberration, we have

$$\alpha : \frac{r \text{ vers. } \theta}{2} :: r \sin. \theta : \frac{r}{2}$$

$$\therefore \alpha = r \sin. \theta \times \text{vers. } \theta.$$

But the diameter (d) of the least circle of aberration is half the lateral aberration nearly, and the diameter (d') of the aperture is $2r \sin. \theta$. Consequently we have

$$d : d' :: \frac{r \sin. \theta \times \text{vers. } \theta}{2} : 2r \sin. \theta$$

$$\frac{1}{d^2} : \frac{1}{d'^2} :: 4\pi r^2 : \pi \left(\frac{r \text{ vers. } \theta}{2}\right)^2$$

the required proportion between the densities.

819. Sin. $\mathrm{I} = m$ sin. R, and sin. $i = m$ sin. r, and the angle of deviation is (*Codd.* p. 48)

$$\mathrm{I} + i - \alpha$$

α being the angle of the prism; also

$$\mathrm{R} + r = \alpha.$$

But the deviation is to be a minimum

$$\therefore d\mathrm{I} + di = 0.$$

$$\text{Also } d\mathrm{R} + dr = 0$$

$$\text{and } \frac{d\mathrm{I}}{d\mathrm{R}} = m \frac{\cos. \mathrm{R}}{\cos. \mathrm{I}}, \text{ and } \frac{di}{dr} = m \frac{\cos. r}{\cos. i}$$

$$\therefore \frac{\cos. \mathrm{R}}{\cos. \mathrm{I}} = \frac{\cos. r}{\cos. i}$$

$$\therefore \frac{1 - m^2 \cos.^2 \mathrm{I}}{\cos.^2 \mathrm{I}} = \frac{1 - m^2 \cos.^2 i}{\cos.^2 i}$$

which gives $\cos. \mathrm{I} = \cos. i$

and $\therefore \mathrm{I} = i$.

Similarly $\mathrm{R} = r$.

820. If Q be the distance of the focus of refraction at the first surface, we have (*Codd.* p. 56)

$$\frac{1}{Q} = \frac{n}{(n+1)a} - \frac{n}{(n+1)r} = \frac{r-nd}{(n+1)rd}$$

Also

$$\frac{1}{q'} = \frac{n+1}{Q+t} - \frac{n}{r}$$

$$\therefore \frac{1}{q'} = -\frac{n}{r} + \frac{(n+1)}{Q} \times \left(1 + \frac{t}{Q}\right)^{-1}$$

$$= -\frac{n}{r} + \frac{(n+1)}{Q} - \frac{n+1}{Q^2} . t \text{ nearly}$$

$$= -\frac{n}{r} + \frac{r-nd}{rd} + \frac{(nd-r)^2}{(n+1)r^2d^2} t$$

$$= \frac{r-2nd}{rd} + \frac{(nd-r)^2}{(n+1)r^2d^2} t.$$

Again, (*Codd.* p. 61,)

$$\frac{1}{q} = -\frac{2n}{r} + \frac{1}{d}$$

$$\therefore \frac{1}{q'} = \frac{1}{q} + \frac{(nd-r)^2}{(n+1)r^2d^2} t.$$

821. This problem may be simplified by enouncing it, *Given the position of a horizontal straight line, and of a vertical plane mirror, required the inclination of another mirror to the former, that the angle at a given point of the line subtended by the image of the line in the image of the former mirror may be zero.*

Let θ be the angle required, and α the given inclination of the straight line to the surface of the given reflector; then if φ denote the inclination of the second image of the line to the given mirror, we easily get

$$\varphi = \theta + \theta - \alpha = 2\theta - \alpha.$$

But by the question $\varphi = 0$.

$$\therefore \theta = \frac{\alpha}{2}.$$

822. If p be the number of reflections, A the $\angle$ of incidence, and B that of refraction; then the $\angle$ between the incident and emergent ray is (*Wood*, Prop. XCIII.)

$$2\theta \backsim 2(p+1)\theta'.$$

Also for the rays to emerge parallel, we must have (*Wood*, Prop. XCIV.)

$$\tan.\theta : \tan.\theta' :: p+1 : 1$$

$$\text{and } \sin.\theta = m\sin.\theta' = \frac{4}{3}\sin.\theta'.$$

$$\therefore p+1 = \frac{\tan.\theta}{\tan.\theta'} = \frac{\sin.\theta}{\sin.\theta'} \cdot \frac{\cos.\theta'}{\cos.\theta}$$

$$= \frac{4}{3} \cdot \frac{\cos.\theta'}{\cos.\theta}$$

Hence, and by the question,

$$\theta - \frac{4}{3}\theta' \frac{\cos.\theta'}{\cos.\theta} = \text{max. or min.}$$

and putting its differential $= 0$, we get

$$\frac{d\theta}{d\theta'} = \frac{4\cos.\theta(\cos.\theta' - \theta'\sin.\theta')}{3\cos.^2\theta - \theta'.\cos.\theta'\sin.\theta}$$

But since

$$\sin.\theta = \frac{4}{3}\sin.\theta'$$

we have

$$\frac{d\theta}{d\theta'} = \frac{4}{3}\frac{\cos.\theta'}{\cos.\theta}.$$

Hence, by substitution, we get, after the proper reductions,

$$\frac{3\cos.^2\theta}{\sin.\theta} = \frac{\cos.^2\theta'}{\sin.\theta'}$$

$$\therefore \frac{3\cos.^2\theta}{\cos.^2\theta'} = \frac{\sin.\theta}{\sin.\theta'} = \frac{4}{3}$$

$$\therefore \frac{\cos.\theta}{\cos.\theta'} = \pm\frac{2}{3}.$$

Hence

$$p+1 = \frac{4}{3} \times \frac{\cos.\theta}{\cos.\theta'} = \pm 2$$

and $p = \pm 2 - 1 = 1$.

823. Let r be the radius of the hemisphere, x and y the co-ordinates, measured from the edge of the bowl, of the nearest point which comes into view when it is filled with water; then it readily appears that the depth (d) of the image of this point : r :: x : r.

$$\therefore d = x.$$

Hence

$$d : y :: \sin. \theta' : \sin. \theta :: 3 : 4$$

θ and θ' being the angles of incidence and refraction.

$$\therefore y = \frac{4}{3} d = \frac{4}{3} . x$$

$$\text{But } y^2 = 2rx - x^2 = \frac{16}{9} x^2$$

$$\therefore \frac{25}{9} x = 2r$$

$$\therefore x = \frac{18}{25} r$$

$$\text{and } y = \frac{4}{3} x = \frac{24}{25} r$$

the depth required.

824. If Δ and Δ' denote the distances of the foci of incident and refracted rays, we have (*Coddington*, p. 56)

$$\frac{1}{\Delta'} = \frac{m-n}{mr} + \frac{n}{m\Delta}$$

$$\therefore \Delta' - \Delta = \frac{mr\Delta}{(m-n)\Delta + nr} - \Delta = \text{max.}$$

and putting its differential $= 0$, we get

$$\frac{mr}{(m-n)\Delta + nr} - \frac{(m-n)\, mr\, \Delta}{(\overline{m-n} . \Delta + nr)^2} - 1 = 0$$

which gives

$$\Delta = \frac{r\sqrt{n}}{\sqrt{m} + \sqrt{n}}$$

and this substituted in the expression for the maximum gives

$$\Delta' - \Delta = \frac{\sqrt{m} - \sqrt{n}}{\sqrt{m} + \sqrt{n}} \times r.$$

825. Since the object is small it may be considered a circular arc concentric with the sphere, and its image after reflection will therefore be a circular arc, also concentric with the sphere, and similar to the object, (*Wood*, Prop. XV.)

Again, if Δ denote the distance of the image from the refractor, in order that its image, caused by refraction, may be *distinct*, we must have (*Wood*, Prop. XXXIX.)

$$\Delta - r : r :: \text{sin. I} : \text{sin. R} :: 4 : 3$$

$$\therefore \Delta = \frac{7}{3} r$$

Hence the distance of the image by reflection from the reflecting surface is

$$2r - \frac{7}{3} r = - \frac{r}{3}$$

and it is $\therefore$ exterior to the sphere; and the distance of the object from the same surface is $\therefore$ (*Wood*, p. 26,)

$$\frac{f^2}{\frac{r}{2} + \frac{r}{3}} = \frac{3r}{10}.$$

Hence the distances of the object O and image O′ by reflection from the centre are

$$r - \frac{3r}{10} \text{ and } r + \frac{r}{3}$$

$$\therefore \text{O} : \text{O}' :: \frac{7}{10} : \frac{4}{3} :: 21 : 40.$$

$$\text{and O}' = \frac{40}{21} \times \text{O} \quad\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Again, since the distance of the image by reflection from the reflecting surface is

$$\Delta = \frac{7}{3} r$$

the distance of its image caused by refraction from that surface is got from the expression

$$\frac{1}{\Delta'} = -\frac{m-1}{r} + \frac{m}{\Delta}$$

$$= -\frac{1}{3r} + \frac{5}{7r} = \frac{5}{21r}$$

which gives

$$\Delta' = \frac{21}{5} r.$$

Hence the distance of this image, which is concentric and similar to the former, from the centre is

$$\Delta' + r = \frac{26}{5} r$$

and if O″ be its magnitude, we have

$$O'' = O' \times \frac{26}{5} \times r \times \frac{3}{4r} = \frac{40}{21} \times \frac{13}{10} \times 3 \times O$$

$$= \frac{52}{7} O.$$

Now let a be the distance of the eye from the centre; then its distance from O″ is $\frac{26}{5} r - a$

and the visual angle is

$$\theta = \frac{5 \times 52 \times O}{7 \times (26r - a)}.$$

Also, since the distance of the principal focus of the sphere from the centre is (*Coddington*, p. 66,)

$$f = -\frac{mr}{2(m-1)} = -2r.$$

$\therefore$ the visual angle of the object when at the principal focus is

$$\theta' = \frac{O}{a + 2r}$$

$\therefore \ \theta : \theta' :: a + 2r : 250 \times 7 \cdot (26r \sim a)$.

826. From the centre of the reflector let fall a $\perp$ upon the rectilinear object produced; then taking this as an axis, the image will be a portion of a conic section, whose major-axis coincides with that $\perp$, and whose focus is the centre of the reflector. Let θ be the traced angle, and ρ the radius vector of this conic section, c the distance of the foot of the object from the centre, and r the radius of the reflector; then (*Coddington* p. 36,)

$$\frac{1}{\rho} = \frac{2}{r} + \frac{1}{c} \cos. \theta$$

$$= \frac{1}{a\,(1-e^2)} + \frac{e}{a\,(1-e^2)} \cos. \theta$$

where

$$e = \frac{r}{2c}, \ a = \frac{2c^2 r}{4c^2 - r^2}.$$

Hence minor axis is

$$b = a\sqrt{1 - e^2} = \frac{cr}{\sqrt{(4c^2 - r^2)}}$$

and the equation between its rectangular co-ordinates is known, *viz*.

$$y^2 = \frac{b^2}{a^2} \cdot (a^2 - x^2).$$

Again, let α be the inclination of the object to the given axis, d its length, and n the part of it produced. Also let ϕ denote the inclination of the tangent at any point of the image to its major axis; then

$$\tan. \phi = \frac{dy}{dx} = \frac{b}{a} \cdot \frac{-x}{\sqrt{a^2 - x^2}}$$

$$= \frac{b}{a} \frac{\sqrt{(b^2 - y^2)}}{y}$$

let $y = n$; then the inclination of the image of the given object to the major axis is had from

$$\tan.\ \phi = \frac{b}{a}\ \frac{\sqrt{(b^2 - n^2)}}{n}.$$

and therefore the inclination to the given axis is known, *viz.*,

$$\phi \sim \left(\frac{\pi}{2} - \alpha\right).$$

The portion of the image due to the given object is the arc of the conic section intercepted by the straight lines drawn from the extremities of the given object at right-angles to it.

ASTRONOMY.

827. SINCE the day is given, the sun's place in the ecliptic is known, and therefore, by the tables, its declination; which suppose D. Also let L denote the given latitude of the spectator. Hence the meridian altitude of the sun is

$$\alpha = 90^\circ - L \pm D$$

+ or — being used according as the given time is in summer or winter. This is the $\angle$ subtended by the tower; consequently its altitude is

$$h = a \times \tan. \alpha = a \times \cot. (L \mp D).$$

828. Once for all let us investigate a general formula for the *aberratic curve.*

Let ρ and p denote the radius vector and $\perp$ upon the tangent of the given orbit; ρ' and p' the corresponding ones to the aberratic curve. Also let θ be the $\angle$ described by ρ; θ' the $\angle$ described by ρ'. Then since the angular velocity of p = the angular velocity of ρ' (*Woodhouse's Ast.* 2d edit., p. 304,) we have (see 443)

$$d\theta' : d\theta :: \frac{dp}{p} : \frac{d\rho}{\rho}$$

Again, $\rho' \propto$ velocity of the earth

$$\propto \frac{1}{p} = \frac{c}{p}$$

$$\therefore \frac{dp}{p} = \frac{d\rho'}{\rho'}$$

$$\therefore d\theta' : d\theta :: -\frac{d\rho'}{\rho'} : \frac{d\rho}{\rho}$$

But in all curves

$$d\theta = \frac{d\rho}{\rho} \cdot \frac{p}{\sqrt{\rho^2 - p^2}}$$

$$\therefore \frac{d\rho}{\rho} \cdot \frac{p}{\sqrt{(\varrho^2 - p^2)}} : \frac{d\rho'}{\varrho'} \cdot \frac{p'}{\sqrt{(\rho'^2 - p'^2)}} :: \frac{d\rho}{\varrho} : \frac{-d\rho'}{\rho'}$$

$$\therefore \frac{p}{\sqrt{\rho^2 - p^2}} = - \frac{p'}{\sqrt{\varrho'^2 - p'^2}}$$

which gives

$$\left.\begin{aligned} p' &= p \cdot \frac{\varrho'}{\rho} = \frac{c}{\varrho} \\ \text{Also } \varrho' &= \frac{c}{p} \end{aligned}\right\}.$$

These two equations will give the equation to the aberratic curve.

Ex. (1.) In the question

$$p^2 = \frac{L}{4} \cdot \varrho$$

L being the principal parameter of the parabolic orbit;

$$\therefore \varrho' = \frac{c}{p} = \frac{2c}{\sqrt{L\rho}} = 2\sqrt{\frac{cp'}{L}}$$

$$\therefore p' = \frac{L}{4c} \rho'^2$$

But if a be the distance of the centre of polar co-ordinates from the centre of a circle, its equation is

$$p = \frac{r^2 - a^2 + \varrho^2}{2r}$$

and consequently when that pole is in the circumference

$$p = \frac{\varrho^2}{2r}$$

Hence it appears that the aberrative curve is a circle whose radius is

$$\frac{2c}{L}.$$

Ex. 2. Let the orbit be an ellipse whose equation is

$$p^2 = \frac{b^2\rho}{2a - \varrho}.$$

Then

$$\rho' = \frac{c}{p} = \frac{c}{b}\sqrt{\frac{2a - \varrho}{\varrho}}$$

$$= \frac{c}{b} \sqrt{\left(\frac{2a}{c} p' - 1\right)}$$

$$\therefore p' = \frac{\rho'^2 + \frac{c^2}{b^2}}{\frac{2ac}{b^2}}$$

so that the aberratic curve is a circle whose radius is

$$\frac{ac}{b^2}$$

and the distance of whose centre from the centre of co-ordinates is

$$\frac{c}{b} \sqrt{(a^2 - b^2)}.$$

Ex. 3. Let the orbit be the logarithmic spiral whose equation is

$$p = m\rho.$$

Then

$$\rho' = \frac{c}{p} = \frac{cm}{\rho} = \frac{\frac{cm}{c}}{p'} = mp'.$$

$$\therefore p' = \frac{\rho'}{m}.$$

Therefore the aberratic curve is also a logarithmic spiral.

829. Let α, α' be the altitudes when the sun is due east and at six o'clock, L the latitude of the place, and D the declination of the sun; then from the two right-angled Δ whose sides are

$$90^\circ - \mathrm{L},\ 90^\circ - \mathrm{D},\ 90 - \alpha$$

$$\text{and } 90^\circ - \mathrm{L},\ 90^\circ - \mathrm{D},\ 90 - \alpha'$$

we easily get

$$r \sin. \mathrm{D} = \sin. \mathrm{L} \sin. \alpha$$

$$r \sin. \alpha' = \sin. \mathrm{L}. \sin. \mathrm{D}$$

and eliminating sin. D,

$$\sin.^2 \mathrm{L} = \frac{r^2 \sin \alpha'}{\sin. \alpha}$$

which gives the latitude of the place.

830. Let a, b denote the semi-axis of the planet's orbit; then its area is πab

Hence if a circle be taken whose radius is

$$r = \sqrt{(ab)}$$

its area will be equal to that of the orbit; and if, with an uniform motion, a body be supposed to go through the whole circumference in the periodic time of the planet, the $\angle$ so described by the radius in an hour, will measure the *mean* horary motion (α) of the planet in its orbit.

Again if $d\theta$ denote the true $\angle$ described by the radius vector ρ in the same hour, by *Kepler's* law of the equable description of areas, we have

$$\text{Area described by } \rho = \frac{\rho^2 d\theta}{2}$$

$$\text{and area described by } r = \frac{r^2 \alpha}{2}$$

$$\therefore \rho^2 d\theta = r^2\alpha = ab\alpha$$

$$\therefore \rho d\theta = \frac{ab\alpha}{\rho}$$

which is the true horary motion of the planet in its orbit.

Let the planet's orbit Np (Fig. 106.) and ecliptic Nm intersect in the node N, then make

$$\text{P}p = \frac{ab\alpha}{\rho}$$

and describe the arcs PM, pm, Pr $\perp$ NM and pm. Now since Pp is very small compared with the whole extent of the orbit, Ppr may be considered rectilinear, and we have

$$\text{P}r : \text{P}p :: \sin. \text{NPM} : 1$$

$$\text{M}m : \text{P}r :: 1 : \cos. \text{PM}$$

$$\therefore \text{M}m : \text{P}p :: \sin. \text{NPM}. \cos. \text{PM} : \cos.^2\text{PM}$$

$$:: \cos. \text{N} : \cos.^2\text{PM}$$

$$\therefore \text{M}m = \frac{ab\alpha}{\rho} \times \frac{\cos. \text{N}}{\cos.^2\text{PM}}$$

which gives the horary motion required.

In like manner the horary motion in latitude is found to be

$$pr = \text{P}p \times \tan. \text{PM}. \cot. \text{NP}$$

Now $\quad = \frac{ab\alpha}{\rho}$. tan. PM. cot. NP
to th
ho

831. Let L be the given co-latitude, D the sun's co-declination, and x his co-altitude at the time required. Also let the hour angle from noon be measured by the $\angle\ \theta$; then L, x and D make a spherical $\triangle$, which gives

$$\cos.\ \theta = \frac{\cos.\ x - \cos.\ L.\ \cos.\ D}{\sin.\ L.\ \sin.\ D.} \ldots\ldots\ 1$$

$$\therefore\ dx = d\theta.\ \frac{\sin.\ \theta}{\sin.\ x}.\ \sin.\ L.\ \sin.\ D \ldots\ldots\ (2)$$

But from the same $\triangle$ we have

$$\sin.\ \theta : \sin.\ (\text{azimuth}) :: \sin.\ x.\ \sin.\ D$$

$$\therefore\ dx = d\theta \times \sin.\ L.\ \sin.\ \text{azimuth}.$$

Hence the variation of the sun's altitude is greatest when his azimuth is 90°, or when he is crossing the *prime vertical.*

Again, in this case

$$\cos.\ \text{azimuth} = \frac{\cos.\ D - \cos.\ x.\ \cos.\ L}{\sin.\ x.\ \sin.\ L} = 0$$

gives $\cos.\ x = \frac{\cos.\ D}{\cos.\ L}$

$$\text{And } \therefore \cos.\ \theta = \frac{\cos.\ x - \cos.\ D.\ \cos.\ L}{\sin.\ D.\ \sin.\ L}$$

$$= \frac{\cos.\ D \times (1 - \cos.^2 L)}{\sin.\ D.\ \sin.\ L.\ \cos.\ L.}$$

$$= \frac{\cos.\ D}{\sin.\ D}.\ \frac{\sin.\ L}{\cos.\ L}$$

$$= \tan.\ L. \times \cot.\ D$$

which gives θ or the time from noon.

832. Let the moon's horizontal parallax be P; her periodic time t; the apparent and real semidiameters of the sun α and r', the radius of the earth r; the length of the sidereal year t', and the distances of the sun and moon from the earth ρ and ρ'; then we have (see 453)

orbit;

$$t = \frac{2\pi\rho^{\frac{3}{2}}}{\sqrt{\mu}}, t' = \frac{2\pi\rho'^{\frac{3}{2}}}{\sqrt{\mu'}}$$

μ and μ' being the quantities of m... respectively.

But

$$r = \rho \sin. P, r' = \rho' \sin. \alpha$$

$$\therefore \mu : \mu' :: \frac{\rho^3}{t^2} : \frac{\rho'^3}{t'^2}$$

$$:: \frac{r^3}{t^2 \sin.^3 P} : \frac{r'^3}{t'^2 \sin.^3 \alpha}$$

But if d, d' be the densities of the earth and sun,

$$\mu : \mu' :: r^3 d : r'^3 d'$$

$$\therefore d : d' :: t'^2 \sin.^3 \alpha : t^2 \sin.^3 P.$$

833. In consequence of the daily revolution of the earth about its axis, the sun appears to describe, with an uniform motion, a circle whose plane is parallel to that of the equator. Let S, S′, S″, S (Fig. 107,) be the intersection of the sun's path with the surface of the earth, P P′ being the earth's axis, Q 12Q′ the equator, and PSP′ the meridian of the place. Then, at the hours of 12, 1, 2, if the sun be supposed to be in positions corresponding to S, S′, S″, and the earth to be hollow, the shadow of the axis PC will be thrown upon the equator in the directions CQ, C1, C2, which are the intersections of the circles of latitude passing through S, S′, S″ with the equator; and since the motion of the sun is uniform, those circles will be at equal intervals, and will consequently give

$$Q1 = 12 = \&c.$$

Hence, if at Z, the place of the spectator, there be placed a circular plate qq' with its plane parallel to the equator, and with a gnomon $cp \perp$ to it; the shadow of this gnomon will uniformly describe the circumference of the plate, and will therefore indicate the time of the day according as the graduation may be effected. If hours merely are to be noted, at q write 12, take $q1 = 15° = 12 = 23$, &c. Then when the shadow is at q, 1, 2, &c., the hour will be 12, 1, 2, &c., respectively.

This is called the Equatorial Dial.

Now, instead of the shadows being cast upon the plane parallel to the equator, which is the simplest case, let them fall upon the horizon, as in Fig. 107. In this case H1, 12, 23, &c., are not equal. Their magnitudes must be calculated by the resolution of the right-angled $\triangle$ P1H, P2H, &c.

Let L be the known latitude of the place; then

$$PH = L$$

$$\text{And } \angle\ 1PH = 15^\circ$$

Hence, by the rules of spherical trigonometry

$$\tan. H1 = \sin. L. \tan. 15^\circ$$
$$\tan. H2 = \sin. L. \tan. 2 \times 15^\circ$$
$$\tan. H3 = \sin. L. \tan. 3 \times 15^\circ$$
$$\&c. = \&c.$$

which give, by the tables,

$$H1, H2, H3, \&c.$$

and therefore the hour $\angle$ HC1, 1C2, 2C3, &c.

Upon the horizontal plate hh' let hh' be the meridian line, cp parallel to the earth's axis, and from the centre c draw $c1$, $c2$, $c3$, &c., making the $\angle$ $hc1$, $1c2$, &c. = HC1, 1C2, &c. The shadows of the gnomon cp will indicate the hours of the day.

To construct a Vertical North or South Dial.

The hour circles P′1, P′2, &c., (Fig. 108,) will intersect the prime vertical ZN, making the several right-angled $\triangle$ PN1, PN2, &c., and, as before, we have

$$\tan. N1 = \sin. PN. \tan. 15^\circ = \cos. L. \tan. 15^\circ$$
$$\tan. N2 = \cos. L. \tan. 2 \times 15^\circ.$$
$$\&c. = \&c.$$

whence the $\angle$ NC1, 1C2, &c., or the $\angle$ $nc1$, $1c2$, &c., may be found, and the dial constructed.

If P be the north pole, the figure represents the position of the gnomon for a dial facing the south. The construction for the Vertical North Dial is nearly the same.

To construct a Vertical East or West Dial.

Let cp (Fig. 109,) parallel to the earth's axis be the gnomon, and 6, 11 the dial plate parallel to the meridian of the place, also

let cp be on the eastern side of the plate; then the shadows upon the plate will evidently be all parallel to the gnonom and to one another. Moreover, at six o'clock the sun, being due east, will cast the shadow perpendicularly upon the dial, and the $\angle\, p 6 11$ therefore is a right $\angle$. Hence we have

$$67 = p6 \times \tan. 15 = a \times \tan. 15^\circ$$
$$68 = a \times \tan. 2 \times 15^\circ$$
$$\&c. = \&c.$$

which give the positions of the hour lines.

Similarly may be constructed a Vertical West Dial.

In the problem we are required to find the $\angle$ between *the hour lines of* 12 *and* 3 *in the horizontal dial.*

We have already found that

$$\tan. H3 = \sin. L \times \tan. 45^\circ.$$
$$= \sin. L$$

which gives the $\angle$ required by the tables.

834. By the preceding problem we have

$$6\ 11 = a \tan. 5 \times 15^\circ$$
$$\text{and } 6\ 10 = a \tan. 4 \times 15^\circ.$$
$$\therefore 3 = 11\ 10 = a \times (\tan. 5 \times 15^\circ - \tan. 4 \times 15^\circ$$

which gives

$$a = \frac{3}{2} \cdot \frac{2\sqrt{3} - 1}{\sqrt{3} - 2}.$$

835. Let C (11) (Fig 108) be the position of the substyle, then by the question in the right-angled $\triangle$ NP′ (11), NP′ the co-latitude and $\angle$ NP′ (11) = 15° are known, and we may therefore find the $\angle$ P′N (11), which is the inclination of the dial to the meridian. By trigonometry we have

$$\cos. P'N = \cot. 15^\circ \times \cot. P'N(11)$$
$$\text{or } \sin. L = \cot. 15^\circ \times \cot. I.$$
$$\therefore \tan. I = \frac{\cot. 15^\circ}{\sin. L},$$

which gives the required positi .

836. The sun being due east and due west at 6 o'clock A.M. and P.M., will not shine upon a Vertical South Dial before the one time, nor after the other, in any latitude, or on any day. Consequently the greatest number of hours that can be indicated by such a dial is twelve; and this evidently happens at both equinoxes, or when the sun rises and sets at 6 o'clock; and from this time during the whole summer, since the sun never reaches the east and west points, except when under the horizon, the number of hours indicated by the Vertical South Dial will be the same as that of daylight.

837. By 833, we have

$$\text{tan. } H4 = \text{sin. } L \times \text{tan. } 4 \times 15^\circ$$

$$\text{and tan. } H3 = \text{sin. } L. \text{ tan. } 3 \times 15^\circ$$

$$\text{tan. } H2 = \text{sin. } L. \text{ tan. } 2 \times 15^\circ$$

$$\therefore \text{tan. } (3, 4) = \frac{\text{tan. } (H4) - \text{tan. } (H3)}{1 + \text{tan. } H4 \times \text{tan. } H3}$$

$$= \frac{\text{sin. } L \times (\text{tan. } 60^\circ - \text{tan. } 45^\circ)}{1 + \text{sin. }^2L \text{ tan. } 60^\circ. \text{ tan. } 45^\circ}$$

$$= \frac{\text{sin. } L \times (2\sqrt{3} - 1)}{2 + \text{sin. }^2L \times \sqrt{3}}.$$

$$\text{Also tan. } (2, 3) = \frac{\text{sin. } L (\text{tan. } 45 - \text{tan. } 30)}{1 + \text{sin. }^2L. \text{ tan. } 45. \text{ tan. } 30}$$

$$= \frac{\text{sin. } L (\sqrt{3} - 2)}{2(\sqrt{3} + \text{sin. }^2L)}$$

$\therefore$ by the question

$$m : n :: \frac{2\sqrt{3} - 1}{2 + \sqrt{3} \text{ sin. }^2L} : \frac{\sqrt{3} - 2}{2(\sqrt{3} + \text{sin. }^2L)}$$

which gives, when reduced,

$$\text{sin. } L = \left(\frac{m\sqrt{3} - n}{n\sqrt{3} - m}\right)^{\frac{1}{2}}$$

838. Let the $\angle$ 2PH $= \theta$ (Fig. 10) and generally suppose the difference between this $\angle$ and the $\angle$ 10 PH $= \alpha$ a given quantity. Then by 833 we get

$$\tan. H2 = \sin. L \times \tan. 30^\circ = \frac{\sin. L}{\sqrt{3}} = \tan. (H\,10)$$

which gives

$$H2 = H\,(10) = m$$

by supposition.

Again, let $P2 = x$, $P\,10 = y$.

then

$$\sin. x : \sin. m :: \sin. H : \sin. \theta$$
$$\sin. m : \sin. y :: \sin. (\theta - \alpha) : \sin. H$$

$$\therefore \frac{\sin. x}{\sin. y} = \frac{\sin. (\theta - \alpha)}{\sin. \theta} \ldots\ldots\ldots\ldots (1)$$

Also

$$\cos. \theta = \frac{\cos. m - \cos. x \cos. L}{\sin. x \sin. L.} \ldots\ldots\ldots (2)$$

$$\cos. (\theta - \alpha) = \frac{\cos. m - \cos. y \cos. L}{\sin. y. \sin. L} \ldots\ldots\ldots (3)$$

$$\text{And } \cos. H = \frac{\cos. x - \cos. m. \cos. L}{\sin. m. \sin. L} = -\frac{\cos. y - \cos. m. \cos. L}{\sin. m. \sin. L}$$

Hence

$$\cos. x + \cos. y = 2 \cos. m. \cos. L \ldots\ldots (4)$$

Whence by the arithmetic of sines

$$x, y \text{ and } \theta$$

may be found; and these being known, it is easy to find

$$90^\circ - \angle PH2$$

which will give the dip required.

839. By 833, we have

$$\tan. (H3) = \sin. L \times \tan. 3 \times 15^\circ$$
$$= \sin. L$$

Again,

$$\tan. (H6) = \sin. L \times \tan. 6 \times 15$$
$$\tan. H4 = \sin. L \times \tan. 4 \times 15$$

$$\therefore \tan. (4, 6) = \frac{\tan. H6 - \tan. H4}{1 + \tan. H6 \times \tan. H4}$$

$$= \frac{\sin. L \times (\infty - \sqrt{8})}{1 + \sqrt{8}. \times \infty \times \sin.^2 L}$$

$$= \frac{1}{\sin. L \times \sqrt{8}}$$

Hence, by the question

$$\sin. L = \frac{1}{\sin. L \times \sqrt{8}}$$

$$\therefore \sin. L = \frac{1}{8^{\frac{1}{4}}}.$$

840. Let α be the angular distance of the shadow of the gnomon from the meridian at a second past 4 o'clock, and β that at 4 o'clock; then, by 833, we have

$$\tan. \alpha = \sin. L. \tan. \left(4 \times 15^\circ + \frac{1}{240}\right)$$

$$\tan. \beta = \sin. L. \tan. (4 \times 15^\circ)$$

$$\therefore \tan. (\alpha - \beta) = \frac{\tan. \alpha - \tan. \beta}{1 + \tan. \alpha. \tan. \beta}$$

$$= \sin. L \times \frac{\tan. \left(60 \frac{1}{240}^\circ\right) - \tan. 60^\circ}{1 + \sin.^2 L \times \tan. 60 \frac{1}{240} \times \tan. 60^\circ}$$

But by the question the angle due to a second at noon is

$$\frac{\alpha - \beta}{2}$$

$$\therefore \tan. \frac{\alpha - \beta}{2} = \sin. L \times \tan. \frac{1}{240}^\circ$$

$$\text{And } \tan. (\alpha - \beta) = \frac{2 \tan. \frac{\alpha - \beta}{2}}{1 - \tan.^2 \frac{\alpha - \beta}{2}}$$

$$\therefore \frac{2 \sin. L \times \tan. \frac{1}{240}^\circ}{1 - \sin.^2 L \times \tan.^2 \frac{1}{240}^\circ} = \sin. L \times$$

$$\frac{\tan.\left(60\frac{1}{240}\right) - \tan.\ 60^\circ}{1 + \sin.^2 L \times \tan.\ 60\frac{1}{240} \times \tan.\ 60}$$

which gives

$$\sin.^2 L = \frac{T \times (2t + 1) - 1}{(T + t)\ (2T + t^2)}$$

T and t being put for tan. 60° and tan. $\frac{1}{240}$°.

841. Let ZHZ′ (Fig. 110,) be the meridian of the place, PP′ the earth's axis, QQ′ the equator, SS′ the path of the sun, and HH′ the horizon, whose intersection with SS′ is s. Then sPH measures the time of sunrise, from midnight. Let PH the latitude be denoted by L, Ps the co-declination of the sun by D, and by the rules of trigonometry, we have

$$\cos.\ (sPH) = \frac{\tan.\ L}{\tan.\ D}$$

which gives sPH or the time required.

842. By the preceding problem

$$\tan.\ D = \frac{\tan.\ L}{\cos.\ sPH}$$

$$= \frac{\tan.\ L}{\cos.\ 45^\circ} = \sqrt{2} \times \tan.\ L$$

which gives the co-declination of the star, and therefore the declination.

843. We will premise the solution of this problem, with *a general investigation of the nature of the curve traced by the shadow of a point, elevated above the horizon, upon the horizontal plane.*

Let A (Fig. 111, a, b) be the given point, AC $\perp$ horizon, CB the shadow at noon, and CP for any azimuth φ. From P

let fall PM $\perp$ CB, and make CM $= x$, PM $= y$, CA $= h$, ZP $=$ co-lat. of the place $=$ L, PS $=$ co-declination of the sun $=$ D, and ZS $=$ co-altitude of the sun $= a$; then by trig.

$$\cos.\,\phi = \frac{\cos.\,D - \cos.\,a \times \cos.\,L}{\sin.\,a \times \sin.\,L}$$

But

$$\sin.\,a = \frac{CP}{AP} = \frac{\sqrt{(x^2 + y^2)}}{\sqrt{(h^2 + x^2 + y^2)}}$$

$$\cos.\,a = \frac{CA}{AP} = \frac{h}{\sqrt{(h^2 + x^2 + y^2)}}$$

and $\cos.\,\phi = \dfrac{x}{\sqrt{(x^2 + y^2)}}$

Hence we get

$$\cos.\,D\sqrt{(h^2 + x^2 + y^2)} = h \cos.\,L + x \sin.\,L \;\ldots\ldots\; (q)$$

which gives

$y^2 \cos.^2 D + x^2 (\cos.^2 D - \sin.^2 L) - x \times h \sin.\,2L + h^2 (\cos.^2 D - \cos.^2 L) = 0 \ldots\ldots$ (1) the equation to a conic section, which is *a parabola, an ellipse, or hyperbola*, according as $\cos.^2 D - \cos.^2 L$ is zero positive or negative.

The transverse axis $2a$ is found by making $y = 0$; for then

$$x^2 - \frac{h \sin.\,2\,L}{\cos.^2\,D - \sin.^2\,L}\,x = -\,h^2 . \frac{\cos.^2 D - \cos.^2 L}{\cos.^2 D - \sin.^2 L}$$

$$\text{and } x = \frac{h}{2 \cos.^2 D - \sin.^2 L)} \times (\sin.\,2D \pm \sin.\,2L)$$

and $2a$ = sum of these values of x.

$$= \frac{h \sin.\,2D}{\cos.^2 D - \sin.^2 L} \;\ldots\ldots\ldots\ldots\ldots\; (2)$$

Also the other axis is twice the maximum value of y, which being found by putting the differential of the value of $y^2 \cos.^2 D$ equal to 0, thence getting x and substituting in equat. (1), gives

$$2b = \frac{2h \sin.\,D}{\sqrt{(\cos.^2 D - \sin.^2 L)}} \;\ldots\ldots\ldots\; (3)$$

The curve traced on the horizon being thus determined, let us proceed to deduce from it the curve traced on any plane bpC, which

is $\perp$ to the meridian ACB, and inclined to the horizon at the $\angle$ bCB $= \angle\, p$CP $= \beta$.

Let pm (which is $\perp$ Cb) $- = y'$

$cm = x'$.

And

$$x = x' \times \frac{\sin.(\beta + M)}{\sin. M}$$

$$= x' \times \{\sin.\beta. \times \cot. M + \cos.\beta\}$$

$$= \frac{x'}{h} \times \{x \sin.\beta + h \cos.\beta\}$$

$$\therefore\ x = \frac{x' h \cos.\beta}{h - x' \sin.\beta} \ldots\ldots\ldots\ldots\ldots (4)$$

Also

$$\text{AM} : \text{A}m :: y : y'$$

$$\text{A}m : h :: \cos.\beta : \sin.(\beta + M)$$

$$\therefore\ \text{AM} : h :: \frac{y}{y'} : \frac{\sin.(\beta+M)}{\cos.\beta}$$

$$:: \frac{y}{y'} : \cos. M \tan.\beta + \sin. M$$

$$:: \frac{y}{y'} : \frac{x}{\text{AM}} \tan.\beta + \frac{h}{\text{AM}}$$

$$\therefore\ \frac{hy}{y'} = x \tan.\beta + h$$

$$= \frac{x'h \cos.\beta}{h - x' \sin.\beta} \times \tan.\beta + h$$

$$\therefore\ y = \frac{hy'}{h - x' \sin.\beta} \ldots\ldots\ldots\ldots\ldots (5)$$

These values of x and y being substituted in equation (1) give

$y'^2 \cos.^2 D + x'^2 \{\cos.^2 L - \sin.^2\beta - \sin.^2 D + 2 \sin.\beta \times \sin. L\,(1 + \sin.\beta \sin. L)\} - hx' \{\cos.\beta \sin. 2L + 2 \sin.\beta \times (\cos.^2 D - \cos.^2 L)\} + h^2 (\cos.^2 D - \cos.^2 L) = 0, \ldots\ldots (6)$

the equation to a conic section, which is a *parabola*, an *ellipse*, or an *hyperbola*, according as the co-efficient of x'^2 is zero, positive or negative.

The semiaxes of this curve may be found in the same way as those of the horizontal trace were determined.

In the problem it is required to find β when the *trace* whose equation is (6) is a parabola.

In this case

$$\cos.^2 L - \sin.^2 \beta - \sin.^2 D + 2 \sin. \beta . \sin. L(1 + \sin. \beta . \sin. L) = 0$$

which gives

$$\sin. \beta = \frac{\sin. L \pm \sqrt{(1 - \frac{\sin.^2 2L}{2} - \sin.^2 D . \cos. 2L)}}{\cos. 2L}$$

844. Let D = the declination of the sun (it = 23° 28′ nearly,) when in either tropic, 90° − H the hour angle from midnight at which the sun rises at one place, and H + h, that at which the sun rises at the other place, and let the required latitudes of these places be called L, L′. Then from the two right-angled spherical Δ whose legs are

D, H; D, H + h

and whose $\measuredangle$ opposite to D are 90° − L, 90° − L′, we easily get

$$\tan. D = \sin. H \times \tan. (90° - L)$$

$$\tan. D = \sin. (H + h) \times \tan. (90° - L')$$

$$\therefore \tan. L' = \cot. D \times \sin. (H + h)$$

$$= \cot. D \times \{\cos. h \sin. H + \sin. h. \cos. H\}$$

$$= \cot. D \{\cos. h . \tan. D . \tan. L + \sin. h \times \sqrt{(1 - \tan.^2 D \tan.^2 L)}\}$$

which gives the latitude of one place, when that of the other is known.

845. Let A, A′ be the right ascensions of the stars, D, D′ their co-declinations, and x the required distance, measured on a great circle of the sphere. Then we have a spherical Δ whose sides are D, D′ and x, and the angle between D and D′ is A ∽ A′. Consequently

$$\cos. (A \backsim A') = \frac{\cos. x - \cos. D . \cos. D'}{\sin. D . \sin. D'}$$

and $\cos. x = \sin. D . \sin. D' \times \cos. (A \backsim A') + \cos. D \times \cos. D'$ which gives x.

The rectilinear distance may also be obtained, being

$$2 \sin. \frac{x}{2}.$$

In fact we easily deduce from the above

$$\sin.^2 \frac{x}{2} = \sin.^2 \frac{D + D'}{2} - \sin. D . \sin. D' \times \cos.^2 \frac{A - A}{2}$$

$$= \sin.^2 \frac{D + D'}{2} - \sin.^2 M$$

$$= \sin. \left(\frac{D + D'}{2} + M\right) . \sin. \left(\frac{D + D'}{2} - M\right)$$

$\sin.^2 M$ being put for the second term.

846. To avoid unnecessary prolixity in this subject, let us once for all give a general view of what is termed by some writers the doctrine of the sphere.

In the spherical $\triangle$ *whose sides are the codeclination* D, *the colatitude of the place* L, *the zenith distance* Z, *and two of whose* $\angle$ *are the hour* $\angle$ *from noon* H, *and azimuth* a; if any three of these quantities be given, the other two may be found by the rules and formulæ of trigonometry.

Case 1. *Given* D, L, Z, *to find* H, a.

Let $S = \frac{D + L + Z}{2}$. Then by *Woodhouse's Trigonometry*, we get

$$\sin.^2 \frac{H}{2} = \frac{\sin. (S - D) \sin. (S - L)}{\sin. D . \sin. L} \;\ldots\ldots\; (1)$$

$$\text{and } \sin.^2 \frac{a}{2} = \frac{\sin. (S - L) \sin. (S - Z)}{\sin. L \times \sin. Z} \;\ldots\ldots\; (2)$$

Case 2. *Given* D, L, H, *to find* Z, a.

Here we have

$$\sin.^2 \frac{Z}{2} = \sin. \left(\frac{D + L}{2} + M\right) . \sin. \left(\frac{D + L}{2} - M\right) \ldots (3)$$

M being such that

$$\sin.^2 M = \sin. D \ . \ \sin. L \ . \ \cos.^2 \frac{H}{2}.$$

and thence

$$\sin. \alpha = \frac{\sin. D}{\sin. Z} \times \sin. H \ \ldots\ldots\ldots\ldots\ldots\ldots (4)$$

Case 3. *Given* D, L, α, *to find* Z, H.

Since

$$\cos. \alpha = \frac{\cos. D - \cos. L \ . \ \cos. Z}{\sin. L \ . \ \sin. Z}$$

we thence deduce

$$\cos. Z = \frac{\cos. D . \cos. L \pm \sin. L \cos. \alpha \sqrt{\{\sin.^2 D - \sin.^2 L(1+\cos. \alpha)\}}}{1 - \sin.^2 L \sin.^2 \alpha} \ldots (5)$$

and to find H we then have

$$\sin. H = \frac{\sin. Z}{\sin. D} \times \sin. \alpha \ \ldots\ldots\ldots\ldots\ldots\ldots (6)$$

Case 4. *Given* L, Z, H, *to find* D, α.

This case in like manner gives

$$\cos. D = \frac{\cos. Z \cos. L \pm \sin. L \sin. H \sqrt{\{\sin.^2 Z - \sin.^2 L(1+\cos. H)\}}}{1 - \sin.^2 L \sin.^2 \alpha} \ldots (7)$$

Also

$$\sin. \alpha = \frac{\sin. H}{\sin. Z} \ . \ \sin. D \ \ldots\ldots\ldots\ldots\ldots\ldots (8)$$

Case 5. *Given* L, Z, α, *to find* D, H.

Here

$$\sin.^2 \frac{D}{2} = \sin. \left(\frac{L+Z}{2} + N'\right) \sin. \left(\frac{L+Z}{2} - N'\right) \ldots (9)$$

wherein

$$\sin.^2 N' = \sin. L \ . \ \sin. Z \cos.^2 \frac{\alpha}{2};$$

Also

$$\sin. H = \frac{\sin. Z}{\sin. D} \sin. \alpha \ \ldots\ldots\ldots\ldots\ldots\ldots (10)$$

Case 6. *Given* D, Z, H, *to find* L, α.
Here

$$\sin.\alpha = \sin.D \times \frac{\sin.H}{\sin.Z} \quad \ldots\ldots\ldots\ldots\ldots \quad (11)$$

and similarly to case (3) we get

$$\cos.L = \frac{\cos.D.\cos.Z \pm \sin.Z\cos.H \times \sqrt{\{\sin.^2D - \sin.^2Z(1+\cos.H)\}}}{1 - \sin.^2 Z \sin.^2 H} \quad (12)$$

Case 7. *Given* D, Z, α, *to find* L. H.
Here we have

$$\sin.H = \frac{\sin.\alpha}{\sin.D} \times \sin.Z \quad \ldots\ldots\ldots\ldots\ldots \quad (13)$$

and as in Case (3)

$$\cos.L = \frac{\cos.D.\cos.Z \pm \sin.Z\cos.\alpha\sqrt{\{\sin.^2D - \sin.^2Z(1+\cos.\alpha)\}}}{1 - \sin.^2 Z \sin.^2 \alpha} \quad (14)$$

Case 8. *Given* D, H, α, *to find* L, Z.
Equation (13) will give Z ;
and then L is obtained from (14).

Case 9. *Given* L, H, α, *to find* D, Z.
By *Napier's Analogies*, we have

$$\left.\begin{aligned} \tan.\frac{D+Z}{2} &= \frac{\cos.\frac{\alpha - H}{2}}{\cos.\frac{\alpha - H}{2}}\tan.\frac{L}{2} \\ \tan.\frac{D-Z}{2} &= \frac{\sin.\frac{\alpha - H}{2}}{\sin.\frac{\alpha + H}{2}}\tan.\frac{L}{2} \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (15)$$

Case 10. *Given* Z, H, α, *to find* D, L.
Here

D is given from (13)

and then we get L from (12).
The problem is resolved by Case 5.

847. In *Tycho Brahe's System of the World*, the sun is supposed to move round the earth at rest, and the planets to move round the sun. Now, generally speaking, a planet will appear retrograde, in any system, when moving in that part of its *actual path* which is convex to the earth. (*Woodhouse*, p. 558.) Hence if that path be any where convex to the earth, the planet will have retrogradation.

Let P be the periodic time of the Sun round the earth, p that of Venus round the sun, and R, r the distances of the sun from the earth, and of Venus from the Sun: then at the inferior conjunction, the *actual orbit* of Venus will be *convex* or *concave* to the Earth, according as (see 467,)

$$Rp^2 - rP^2$$

is negative or positive.

But if $R = 1$, we know from the tables that $r = .723832$. Also $P = 365.25$ and $p = 224.7$. These quantities being substituted in the above formula, give a negative result. Consequently, Venus will appear retrograde in the Tychonic system.

848. The difference of the meridian altitudes of the two stars is = the difference of their declinations. Call this given difference d, and let d' denote the difference of the altitudes at one o'clock, and α the given difference of the azimuths at that time, and $90 - L$ the required latitude. Also let z, z' denote the zenith distances. Then from the spherical $\triangle$ whose sides are z, z' and d, we have

$$\cos.\alpha = \frac{\cos.d - \cos.z\cos.z'}{\sin.z\,.\,\sin.z'}$$

But $z' - z = d'$.

$$\therefore \cos.z' = \cos.(z + d')$$

and we get by substitution and the arithmetic of sines

$$\cos.2z - \tan.d'\,.\,\sin.2z = A,$$

wherein

$$A = \frac{2\cos.d - \cos.d'\,(1 + \cos.\alpha)}{\cos.d'\,(1 - \cos.\alpha)}$$

Hence

$$1 - \sin.^2 2z = A^2 + 2A\tan.d'\sin.2z + \tan.^2 d'\sin.^2 2z.$$

$$\therefore \sin.^2 2z + \frac{2A \tan. d'}{1+\tan.^2 d'} \sin. 2z = \frac{1-A^2}{1+\tan.^2 d'}$$

which being resolved gives z, and therefore $z' = z + d'$.

Hence is known the $\angle$ between (z, d); for

$$\sin. (z, d) : \sin. \alpha :: \sin. z' : \sin. d.$$

But $\sin. L : \sin. (z, d) :: \sin. z : \sin. 15^\circ$

$$\therefore \sin. L : \sin. \alpha :: \sin. z . \sin. z' : \sin. d . \sin. 15^\circ$$

$$\therefore \sin. L = \frac{\sin. \alpha \times \sin. z . \sin. z'}{\sin. d \times \sin. 15^\circ}$$

which gives

$$90^\circ - L.$$

849. Let the sun's apparent diameter be D, his co-declination be x, and the latitude of the place be $90 - L$. Also let θ be the hour angle corresponding to the time of rising; then (Fig. 110,)

$$\theta\ (= \sigma\sigma') : ss' :: 1 : \sin. x$$

$$ss' : D\ (= ns) :: 1 : \sin. (nsP)$$

$$\therefore \theta = \frac{D}{\sin. x \times \sin. nsP}.$$

$$\text{But } \cos. (nsP) = \frac{\cos. L}{\sin. x}$$

$$\therefore \theta = \frac{D}{\sqrt{(\sin.^2 x - \cos.^2 L)}}$$

and θ is therefore a minimum when $x = 90^\circ$.

850. Let L, L′ be the latitudes of the places, D the co-declination of the sun, and H, H′ the hour angles measured from noon at the time of setting; then by means of the spherical right-angled Δ, two of whose sides are

L, D; and L′, D;

we get, (by *Napier's* Rules,)

$$\tan. L = \cos. H \times \tan. D,$$

$$\text{and } \tan. L' = \cos. H' \times \tan. D.$$

ut $H \sim H' =$ difference of longitude $= h$ a given quantity.

$$\therefore \frac{\tan. L}{\tan. D} = \cos. (H' + h) = \cos. H' \cos. h - \sin. H' \sin. h$$

$$= \cos. h \times \frac{\tan. L'}{\tan. D} - \sin. h \sqrt{\left(1 - \frac{\tan.^2 L'}{\tan.^2 D}\right)}$$

$$\therefore \sqrt{(\tan.^2 D - \tan.^2 L')} = \frac{\cos. h}{\sin. h} \times \tan. L' - \frac{\tan. L}{\sin. h}$$

$$\therefore \tan.^2 D = \tan.^2 L' + \frac{1}{\sin.^2 h} \times (\cos. h \tan. L' - \tan. L)^2$$

$$= \frac{\tan.^2 L' + \tan.^2 L - 2 \cos. h \tan. L . \tan. L'}{\sin.^2 h}$$

which will give the co-declination D.

851. On any two days in the year, which are equally distant from the time of the equinox, it may easily be shewn that the time of daylight on the one day is equal to the duration of darkness on the other; and the same is true for all other days equally remote from that epoch. Hence the truth of the problem is manifest.

852. Since the declination of the sun when in either tropic is 23° 27′ 30″, it is evident that to people living under the equinoctial, he must rise 23° 27′ 30″ from the east towards the north, when he is in the northern tropic.

853. Let O denote the obliquity of the ecliptic, D the sun's declination, and A his right ascension at the required time; then from the right-angled spherical △ whose legs are D and A, we easily derive

$$\tan. D = \tan. O \times \sin. A \text{ .... } (1)$$

$$\therefore \frac{d.D}{\cos.^2 D} = \tan. O \times d.A \cos. A$$

But, by the question,

$$d.D = d.A$$

$$\therefore \cos. A \times \tan. O = 1 + \tan.^2 D = 1 + \tan.^2 O \times \sin.^2 O$$

which gives

$$\cos.^2 A + \cot. O \cos. A = 1 + \cot.^2 O$$

$$\therefore \cos. A = \frac{-\cot. O \pm \sqrt{(4 + 5 \cot.^2 O)}}{2}.$$

and therefore A is known.

Again, if l denote the longitude of the sun, or its distance from Aries, we have by the same $\triangle$

$$\tan. l = \frac{\tan. A}{\cos. O},$$

which gives the sun's place in the ecliptic.

854. Let ρ, ρ', be the distances of the earth and saturn from the sun, and x that of the moon from the earth. Then when saturn shines with a full orb, the exterior angle of elongation being a right angle, his distance from the earth is

$$R = \sqrt{(\varrho'^2 - \varrho^2)}.$$

Also when the moon is in the full, her distance from the sun is

$$r = \sqrt{(\rho^2 - x^2)}.$$

Again, the intensity of the light which bodies receive from a luminary $\propto \frac{1}{(\text{dist.})^2 \text{ from that luminary}}$ (see *Wood's Optics.*)

Hence the real intensities upon the moon and saturn are *measured* by

$$\frac{1}{\rho^2 - x^2} \text{ and } \frac{1}{\rho'^2}.$$

These intensities, in the progress of the light reflected to the earth are diminished in the ratio of

$$\frac{1}{x^2}, \text{ to } \frac{1}{\rho^2 - \varrho'^2}$$

and the *apparent* brightness $\propto$ magnitude of the disk $\times$ intensity. Hence, by the question,

$$\frac{3}{2} \times \frac{\pi s^2}{\varrho'^2(\varrho^2-\varrho'^2)} = \frac{\pi m^2}{x^2.(\rho^2-x^2)}$$

s and m being the radii of the disks of saturn and the moon.

$$\therefore x^4 - \varrho^2 x^2 = \frac{2}{3} \cdot \frac{m^2}{s^2} \rho'^2 (\rho'^2 - \rho^2)$$

whence by the solution of a quadratic x^2 and $\therefore x$ may be found.

855. Let D, D′, be the given co-declinations, A, A′, the right ascensions, α the common azimuth, z the given zenith distance, and z' the one not given, and L the co-latitude of the place. Then from the spherical Δ, whose sides are,

$$z, L, D; \ z' L D'$$

we have

$$\cos. \alpha = \frac{\cos. D - \cos. z \cdot \cos. L}{\sin. z \cdot \sin. L} \ldots\ldots\ldots (1)$$

$$\cos. \alpha = \frac{\cos. D' - \cos. z' \cdot \cos. L}{\sin. z' \cdot \sin. L} \ldots\ldots\ldots (2)$$

Also

$$\cos. (A' - A) = \frac{\cos. (z' - z) - \cos. D \times \cos. D'}{\sin. D \cdot \sin. D'}$$

$$\therefore \cos. (z' - z) = \cos. D \cdot \cos. D' + \sin. D \cdot \sin. D' \cos. (A' - A)$$

and this gives (see *Woodhouse's Trigonometry*,)

$$\sin.^2 \frac{z'-z}{2} = \sin. \left(\frac{D+D'}{2} + M\right) \sin. \left(\frac{D+D'}{2} - M\right) \ldots\ldots (3)$$

wherein

$$\sin.^2 M = \sin. D \cdot \sin. D' \cos.^2 \frac{A'-A}{2}.$$

Hence is found $z' - z$, and therefore z which being substituted in equation (2) will, by means of equating (1) and (2), give the value of L in terms of D, D′, and A′ − A.

856. Let $sn = h$ (Fig. 110,) the horizontal refraction; then $\sigma\, \sigma'$ measures the required acceleration.

But

$$h = ss' \times \sin. ss'n = ss' . \cos. \text{Latitude},$$
$$= \sigma\sigma' . \cos. \text{dec.} \cos. \text{Latitude},$$

whence $\sigma\sigma'$ is known.

857. Let α be the given $\angle$ subtended by the diameter of the sun at the mean distance a, and θ the angular distance from perigee, then at the distance

$$\varrho \left(= \frac{a . (1 - e^2)}{1 + e \cos. \theta}\right)$$

the apparent diameter is

$$\alpha' = \frac{a}{\varrho} \times \alpha = \frac{1 + e \cos. \theta}{1 - e^2} \times \alpha.$$

858. The distance (z) of the pole of the ecliptic from the zenith of the place $\doteq$ the inclination of the plane of the ecliptic to the horizon. Hence, if L denote the co-latitude of the plane, O the obliquity, then it is evident, upon inspecting the Δ whose sides are z, O, and L, that z is least, when the pole of the ecliptic is in the meridian. Hence in this case there are situated in the circumference of the meridian the poles of the equator and ecliptic, which, therefore intersect in the pole of the meridian, (see *Creswell's Spherics*;) that is, *Aries rises in the east or west point of the compass when the ecliptic is least inclined to the horizon.*

659. Let the given sum of the sun's co-altitude (z), and co-declination (D) be

$$m = z + D.$$

Also let H be the hour $\angle$ from noon, and L the co-latitude of the place; then

$$\cos. H = \frac{\cos. z - \cos. D . \cos. L}{\sin. D . \sin. L}$$

$$= \frac{\cos.(m - D) - \cos. D . \cos. L}{\sin. D . \sin. L}$$

which, by the arithmetic of sines will give D, and therefore

$$z\ (= m - D.)$$

860. If the moon's velocity were augmented, her periodic time would be diminished, and the monthly motion of the node of her orbit, which is due to the disturbing force of the sun, would also be diminished. But the difference between the sidereal and synodical periods, is owing to the motion of the nodes. Therefore, &c. &c.

861. Let z, z' be the two given zenith distances of the sun, and α, α' the given azimuths; also let L denote the co-latitude, and D the co-declination, which is supposed to undergo no variation between the times of observation. Then from the Δ whose sides are

$$L, z, D; \ L\, z'\, D;$$

we have

$$\cos. D = \cos. z . \cos. L + \sin. z . \sin. L . \cos. \alpha$$

$$\cos. D = \cos. z' . \cos. L + \sin. z' . \sin. L . \cos. \alpha'$$

$$\therefore \cos. z + \sin. z . \cos. \alpha . \tan. L = \cos. z' + \sin. z' . \cos. \alpha' \times \tan. L$$

$$\therefore \tan. L = \frac{\cos. z - \cos. z'}{\sin. z' . \cos. \alpha' - \sin. z . \cos. \alpha}$$

which gives L, and therefore $90° - L$.

862. Let ϱ, ϱ' be the distances of the sun from the earth and Jupiter; then the apparent magnitudes of the sun's diameter are as

$$\frac{1}{\varrho}, \frac{1}{\varrho'}, \text{ or as } \rho', \rho.$$

863. Let a denote the earth's mean distance, $a - ae$ his least distance; then since the *apparent* diameter $\propto$ inversely as the distance, by the question, we have

$$\frac{1}{a} : \frac{1}{a - ae} :: 2 : 5 :: 1 - e : 1$$

$$\therefore e = \frac{3}{5}.$$

Hence the semi-axis major is

$$b = a\sqrt{a - e^2} = a \cdot \frac{4}{5}$$

and $\frac{b}{a} = \frac{4}{5}$

the proportion required.

864. Let L denote the co-latitude of the place; D, D′ the co-declinations of the stars subtending their common azimuth of 90°; h the given difference of their right ascensions; and H, H + h the hour angles from noon. Then, by Napier's Rules for the resolution of right angled spherical △, we get

$$\tan. L = \cos. H \cdot \tan. D$$
$$\tan. L = \cos. (H + h) \tan. D'$$

Hence, by the arithmetic of sines, H being eliminated, we get

$$\frac{\tan. D}{\tan. D'} \cdot \tan. L = \cos. h \tan. L + \sin. h \cdot \sqrt{(\tan.^2 D - \tan.^2 D')}$$

Moreover, if a be the given distance of the stars when on the prime vertical from the △, whose sides are D, D′ and that distance, we get

$$\cos. h = \frac{\cos. a - \cos. D \cos. D'}{\sin. D . \sin. D'}$$

and this, together with the above equation involving D, D′, will give, when reduced, the declinations required.

865. Let the given sum of the azimuth (α) and hour ∠ from noon H, be

$$m = \alpha + H.$$

Also let D, and z, be the given co-declination and zenith distance

of the sun, and L the co-latitude of the place. Then from the Δ, whose sides are D, z, L, we have

$$\cos. \mathrm{H} = \frac{\cos. z - \cos. \mathrm{D} . \cos. \mathrm{L}}{\sin. \mathrm{D} . \sin. \mathrm{L}}$$

$$\text{and } \cos. \alpha = \frac{\cos. \mathrm{D} - \cos. z . \cos. \mathrm{L}}{\sin. z . \sin. \mathrm{L}}$$

But $\cos. \mathrm{H} = \cos. (m - \alpha) = \cos. m . \cos. \alpha + \sin. m . \sin. \alpha$.

$$\therefore \frac{\cos. z - \cos. \mathrm{D} . \cos. \mathrm{L}}{\sin. \mathrm{D} . \sin. \mathrm{L}} = \cos. m \times \frac{\cos. \mathrm{D} - \cos. z . \cos. \mathrm{L}}{\sin. z . \sin. \mathrm{L}}$$

$$+ \frac{\sin. m \times}{\sin. z \sin. \mathrm{L}} \{ \sin.^2 z . \sin.^2 \mathrm{L} + \cos. \mathrm{D} - \cos. z . \cos. \mathrm{L})^2 \}^{\frac{1}{2}}$$

which, being reduced, will give L.

866. Let h be the difference of the times of setting, H, H + h the hour $\angle$ from noon, D, D′ the co-declinations of the two stars, and L the co-latitude of the plane; then from the Δ whose sides are

$$90^\circ, \mathrm{D}, \mathrm{L};\ 90^\circ, \mathrm{D}', \mathrm{L}$$

we have

$$\cos. \mathrm{H} = \frac{\cos. 90^\circ - \cos. \mathrm{D} . \cos. \mathrm{L}}{\sin. \mathrm{D} . \sin. \mathrm{L}} = -\cot. \mathrm{D} . \cot. \mathrm{L}$$

and $\cos. (\mathrm{H} + h) = -\cot. \mathrm{D}' . \cot. \mathrm{L}$

$\therefore \cot. \mathrm{D}' \cot. \mathrm{L} = \cos. h . \cot. \mathrm{D} . \cot. \mathrm{L} + \sin. h . \sqrt{(1 - \cot.^2 \mathrm{D} . \cot.^2 \mathrm{L})}$; which being reduced and adapted to logarithmic computation will give L, and therefore the required latitude.

867. If l denote the longitude of the sun, l' that of the star, and λ its latitude; then (*Woodhouse*, p. 299) the aberration in latitude is

$$20' . 25 \times \sin. (l' - l) \sin. \lambda$$

which is therefore 0 when

$$l = l',$$

which makes known the longitude of the sun, and therefore the day of the year.

But since the longitudes of the sun and star are equal on this day, their right ascensions are also equal, and they consequently cross the meridian at the same moment, or at twelve o'clock.

The problem may be easily generalized; by stating it, *Required at what time of the day a star, whose longitude and latitude are given, crosses the meridian, on that day of the year when its aberration in latitude is a given quantity* α.

In this case the longitude of the sun may be found from the formula

$$\alpha = 20'.25 \,.\, \text{sin.}\, (l' - l)\, \text{sin.}\, \lambda$$

and thence the right ascension of the sun (A) from

$$\text{tan. A} = \text{cos. O} \times \text{tan.}\, l,$$

O being the obliquity of the ecliptic.

Moreover, the right ascension (A′) of the star is obtained from

$$\text{tan. A}' = \text{cos. O} \times \text{tan.}\, l'$$

and $A \backsim A'$ the time required measured from noon is therefore known.

868. This problem relates to navigation—to oblique sailing.

Since the vessel is steered constantly to the same point of the compass, her path along the surface of the sea cuts the meridians at the same $\angle$, *viz.*, 45°. Hence, if ds denote the constant increment of her route, the corresponding increment of latitude is

$$ds \times \text{cos.}\, 45^\circ = \frac{ds}{\sqrt{2}}$$

which is also constant.

Hence the whole path described from the equator to the pole is

$$90^\circ \times \sqrt{2}.$$

Again, *to find the difference of the longitude*, suppose the degrees of longitude to be of the same length for every parallel of latitude, (as in *Mercator's Projection*,) then the degrees of latitude must increase as the secant of the latitudes.

Hence, if λ denote the latitude, its increment, in order to make that of the longitude the same for all parallels, must be

$$d\,.\,\lambda' = \frac{d\lambda}{\cos.\lambda}$$

$$\therefore \lambda' = \int \frac{d\lambda}{\cos.\lambda} = \log.\tan.\left(45^\circ + \frac{\lambda}{2}\right) \ldots\ldots (1)$$

Hence, if λ' be calculated for any two latitudes of the vessel, *viz.*, λ, μ, the difference of the longitudes will evidently be

$$D = (\lambda' - \mu')\tan.\theta \ldots\ldots\ldots\ldots\ldots (2)$$

θ measuring the distance of the point of the compass from the north or south, or the constant inclination of the route to the meridians.

In short, all problems of this kind are resolved by the two formulæ

$$\left.\begin{aligned} R &= \lambda \sec.\theta \\ \text{and } D &= l.\frac{\tan.\left(45^\circ + \frac{\lambda}{2}\right)}{\tan.\left(45^\circ + \frac{\mu}{2}\right)} \times \tan.\theta \end{aligned}\right\}$$

R being the length of the route.

Ex. 1. In the problem, $\lambda = 90^\circ$, $\mu = 0$, and $\theta = 45$.

$\therefore R = 90^\circ \times \sqrt{2}$, as before.

and $D = l \times \tan.90^\circ = l.\,\infty = \infty$.

So that the ship would never actually reach the pole, although she would go on for ever approaching more nearly to it.

Ex. 2. Generally, let $\lambda = 90^\circ$, $\mu = 0$. Then

$$\left.\begin{aligned} R &= 90^\circ.\sec.\theta, \\ \text{and } D &= l\,.\,\frac{\tan.90^\circ}{\tan.45} \times \tan.\theta \end{aligned}\right\}$$

which shews that in whatever direction, other than direct north or south, a ship might sail, she would never by possibility reach the pole.

If she steer to the north or south, then we have $\theta = 0$ and $\therefore$

$$R = 90^\circ$$

$$\text{and } D = \tan. 0 \times l. \frac{\tan. 90}{\tan. 45}$$
$$= l. (\tan. 90°)^{\tan. 0} = l. (\infty)^0.$$
$$= l. 1 = 0,$$

correct results.

869. Let the inclination of the plane to the horizon be α, and its inclination to the meridian β. Then taking a place whose horizon is parallel to the plane, it is evident the sun will begin to shine upon the plane when it rises at this assumed place.

Let L be the given co-latitude, L′ the assumed one. Then from the Δ, whose sides are the distance of the zeniths of the places, (*viz.* α), and L, L′, we get

$$\cos. \beta = \frac{\cos. L' - \cos. L . \cos. \alpha}{\sin. L . \sin. \alpha}$$

and $\therefore \cos. L' = \cos. L . \cos. \alpha + \sin. \alpha . \sin. L . \cos. \beta$, which gives (when adapted to logarithms) L′.

Hence, if D be the co-declination of the sun, we get, from the right angled Δ, whose $\perp$ and hypothenuse are $90° - L'$, and D,

$$\cot. H = \cot. L' \times \cot. D,$$

where H is the hour angle from the meridian of the assumed place.

But inclination of the meridians of two places = inclination of their horizons. Consequently

$$H \pm \alpha$$

the time required is known.

870. Let L be the co-latitude of the place, D the declination of the sun on one day, D′ that on another, and d the difference of the lengths of the meridian shadows of the tower, whose altitude is x. Then since the meridian altitude of the sun is

$$L \pm D,$$

we have

$$x = l \tan. (L \pm D) = (l + d) \tan. (L \pm D')$$

l being the actual length of one shadow.

Hence,

$$l = \frac{d \times \tan.(L \pm D')}{\tan.(L \pm D) - \tan.(L \pm D')}$$

$$\text{and } \therefore x = \frac{\tan.(L \pm D . \tan.(L \pm D')}{\tan.(L \pm D) - \tan.(L \pm D')}$$

871. Let L be the latitude of the place, D the co-declination, and A the right ascension of the star. Also let D′ be the co-declination of the sun, which is known from the tables for the *given day*. Then from the right-angled $\triangle$, whose $\perp$ and hypothenuse are L and D′, we have

$$\cos. H = \tan. L . \cot. D',$$

H being the hour $\angle$ from midnight, and indicating the hour of sunrise. The time when the star rises is

$$H \pm A.$$

872. A writer, in *Leybourn's Mathematical Repository* gives the following solution of this problem :—

"The greatest apparent distance of a star from the north pole, arising from aberration and nutation together, is when the sun's longitude, and also that of the ☽'s ascending node, are three signs before the ♓, reckoning according to the order of the signs; and that apparent distance will be greatest of all when the star is in the solstitial colure; but it does not appear that the period can be accurately determined when such an event will take place, for any proposed star. We can, however, calculate the time which seems to be required by the question, if it falls within the limits of some proposed revolution of the ☽'s nodes, or if it is restricted to a given year; thus, for instance, let the year be 1809, and suppose the star's right ascension = 108°, and its declination 60° north. Let a great ⊙ be drawn to the E from the star $\perp$ to the meridian through the star, and it will meet the ecliptic in 25° 41′ of Libra, which is the ⊙'s place, when the aberration in declination southward is a maximum, (obliquity of ecliptic being 23° 27′ 44″);

this answers to October 19, at London. On that day the star comes to the meridian at $22\frac{1}{2}'$ before six in the morning. The longitude of the ☽'s node at that time is $6^s\ 23°\ 45'$, and the other is $6^s\ 25°\ 26'$; the star is therefore very nearly at its maximum distance from the north pole, which distance will not sensibly vary during three or four days at that time."

873. Let D, D′ be the co-declinations of the stars, L the latitude of the place, and H, H + h the angular distances from the meridian at the moment of rising, h being the difference of the given right ascensions; then from the right-angled △, whose ⊥ and hypothenuses are

D, L; D′ L,

we get

$$\cos.(H+h) = \tan. L . \cot. D$$
$$\cos. H = \tan. L . \cot. D',$$

and eliminating H, we have

$$\tan. L . \cot. D = \cos. h \tan. L \cot. D' - \sin. h \times \sqrt{\{1 - \tan.^2 L . \cot.^2 D'\}}$$

which gives

$$\cot.^2 L = \frac{\cot.^2 D + \cot.^2 D' - 2\cos. h . \cot. D . \cot. D'}{\sin.^2 h}.$$

874. Let A, D, be the right ascension and co-declination of the star; A′, D′ the right ascension and co-declination of the sun; also let L be the latitude of the place, and H, H′ the hour angles for the star and sun when rising; then from the right-angled △, whose hypothenuses are D, D′, we have

$$\cos. H = \tan. L . \cot. D$$
$$\cos. H' = \tan. L . \cot. D'$$

But $H' - H = A' - A$,

$$\therefore \cos. H' . \cos. H + \sin. H . \sin. H' = \cos. (A' - A)$$
$$\therefore \cos. (A' - A) = \tan.^2 L . \cot. D . \cot. D' + \sqrt{\{(1 - \tan.^2 L . \cot.^2 D) . (1 - \tan.^2 L \cot.^2 D')\}}.$$

Again, if O be the obliquity of the ecliptic, it is easily shown that

$$\cot. D' = \tan. O \sin. A' \ldots\ldots (1)$$

which being substituted in the above equation, gives

$\cos. (A' - A) = \tan.^2 L \cot. D. \tan. O \times \sin. A' + \sqrt{\{(1 - \tan.^2 L \cot.^2 D) (1 - \tan.^2 L \tan.^2 O \times \sin.^2 A')\}}$. which being reduced and resolved, will give

$$A'.$$

Hence, by Equat. (1) is known

$$D'$$

and therefore by the tables *the day of the year is known.*

The time of rising on that day is known from

$$\cos. H' = \tan. L. \cot. D'.$$

875. "*To delineate the phases of the moon,*" (see Woodhouse's Astron., p. 552.)

"*To find the* ∠ *which the line joining the moon's cusps make with the horizon at any time,* we will premise the following,

Lemma. *Required the inclination of the plane passing through the centres of the earth, sun, and moon, to the ecliptic.*

Let S, E, m' (Fig. 112,) be the intersection of these planes, S, M, E being the positions of the sun, moon, and earth. Draw $Mm \perp$ ecliptic, $mm' \perp$ SE, and join Em, EM, $m'M$; then the inclination required is measured by

$$\angle Mm'm.$$

But $Mm = mm'. \tan. Mm'm = Em \tan. MEm$.

$$\therefore \tan. Mm'm = \tan. MEm \times \frac{Em}{mm'}$$

$$= \frac{\tan. MEm}{\sin. SEm}$$

$$= \frac{\tan. (☽\text{'s geocentric latitude})}{\sin. (☽\text{'s elongation in longitude})}$$

which gives the $\angle Mm'm$.

Again, let MM' (Fig. 113,) the line joining the moon's cusps be produced to meet the horizon in H', and let Mm', the plane passing through the earth, sun, and moon, intersect the ecliptic

$M'm'$, which also cuts the horizon in H, in the point m'. Then, by the nature of phases, (see *Woodhouse*,) the $\angle M = 90°$; the $\angle m'$ is given by the *Lemma*, and the $\angle M'HH'$ being measured by the *altitude of the nonagesimal*, is found from *Woodhouse*, p. 741; consequently,

$$\begin{aligned} \angle H' &= 180° - M' - H \\ &= 180° - H - (M + m') \\ &= 90° - (H + m') \end{aligned}$$

is known.

876. Let A be the sun's right ascension, L the latitude of the place, D the sun's declination, and x the ascensional difference. Also let O be the obliquity of the ecliptic then, in the first place, from the right-angled $\triangle$ (D, A) we have

$$\tan. D = \tan. O \times \sin. A$$

which gives D.

Again, from the right-angled $\triangle$ (D, x) we have

$$\begin{aligned} \sin. x &= \tan. D \times \tan. (\angle \text{ opposite } x) \\ &= \tan. D \times \tan. L \\ &= \tan. O \sin. A \times \tan. L \end{aligned}$$

which is a general investigation of the ascensional difference.

877. From the $\triangle$ $(p, 90° - l, 90° - a)$, we have

$$\cos. t = \frac{\sin. a - \sin. l \cos. p}{\sin. p \cos. l}$$

$$\therefore \text{vers. } t = 1 - \cos. t =$$

$$\frac{\sin. p. \cos. l + \sin. l. \cos. p - \sin. a}{\sin. p \cos. l}$$

$$= \frac{\sin. (p + l) - \sin. a}{\sin. p \cos. l}$$

$$= \frac{2 \cos. \frac{p+l+a}{2} \sin. \frac{p+l-a}{2}}{\sin. p. \cos. l}$$

$$= 2 \cos. \frac{p+l+a}{2} \sin. \left(\frac{p+l+a}{2} - a\right) \text{cosec. } p \text{ sec. } l$$

878. The apparent diameter of the sun $\propto$ inversely as the radius vector of the orbit. Hence since

$$\varrho = \frac{a.(1-e)}{1-e\cos.\theta}$$

θ being measured from *apogee*, the apparent diameter $\propto 1-e\cos.\theta$.

Hence the augmentation of the apparent diameter is

$$\frac{1-e\cos.\theta}{a(1-e^2)} - \frac{1}{a(1+e)} = \frac{1-e\cos.\theta-1+e}{a.(1-e^2)}$$

$$= \frac{e.(1-\cos.\theta)}{a(1-e^2)} \propto \text{vers.}\,\theta$$

whatever be the magnitude of the eccentricity.

879. *The visible illumined part of the disk : whole disk :: vers. exterior* $\angle$ *of elongation* : 2. See *Woodhouse, p.* 552, or *Vince*.

But in the case of the moon, the distance of the sun from the earth being incomparably greater than that of the moon, the exterior $\angle$ of elongation may be considered the $\angle$ of elongation itself.

$$\begin{aligned} \text{Hence visible part : whole disk} &:: \text{vers. } 60^\circ : 2 \\ &:: 1-\cos.60^\circ : 2 \\ &:: 1 : 4 \end{aligned}$$

$$\therefore \text{ visible part : dark part } :: 1 : 3$$

880. Let D, A; L, λ be the declination, right ascension, longitude, and latitude of the star; d(D) d(A) the given aberrations in declination and right ascension. Then, if σ, σ' (Fig. 114,) be the two places of the star, P, π the poles of the equator and ecliptic, Pσn, $\pi\sigma m$ referred to the parallels of the equator and ecliptic, σn σm will denote the co-declination and the co-latitude; and we have

$$\left.\begin{aligned} \sigma' n &= \cos.\,\mathrm{D} \times d\mathrm{A} \\ \sigma' m &= \cos.\,\lambda \times d\mathrm{L} \end{aligned}\right\} \quad \ldots\ldots\ldots (1)$$

Again, since the aberration is small, the Δ $\sigma r m'$, $\sigma' r n$ may be considered rectilinear. Hence the $\angle$ $n\sigma' r = \angle$ $m\sigma r$ = angle of position (= P).

And it is easily shown that

$$\sigma' m = \sigma n \times \sin. P \pm \sigma' n \cos. P$$

$$\sigma m = \sigma n \times \cos. P \mp \sigma' n \sin. P.$$

But from (1)

$$dL = \frac{\sigma' m}{\cos. \lambda}; \text{ and } \sigma m = d\lambda, \ \sigma n = dD$$

$$\left.\begin{array}{l} \therefore dL = \dfrac{1}{\cos. \lambda} \times (dD \sin. P \pm dA \cos. \lambda \cos. P) \\ \text{And } d\lambda = dD \times \cos. P \mp dA \cos. D \sin. P \end{array}\right\}$$

See also *Woodhouse's* chap. on Aberrations.

881. The surface is expressed generally by

$$S = \int 2\pi y ds$$

s being the arc of the generating curve.

$$\text{But } ds = \frac{-r dy}{\sqrt{(r^2 - y^2)}}$$

$$\therefore S = 2\pi r \int \frac{y dy}{\sqrt{(r^2 - y^2)}}$$

$$= 2\pi r. \{C + \sqrt{(r^2 - y^2)}\}$$

Let $S = 0$, when $y = r$; then $C = 0$, and

$$S = 2\pi r \sqrt{r^2 - y^2}$$

$$= 2\pi r^2 . \sin. \text{lat.}$$

Let lat. $= 90$ and 30; then the whole surface and the surface between the specified parallels are

$$\therefore 2S = 4\pi r^2 \text{ and } 2S' = 4\pi r^2 \times \frac{1}{2}$$

respectively; so that

$$2S = 2 \times 2S' \qquad \text{Q. E. D.}$$

882. Let D, D′, x be the co-declinations of the stars and planet, a the given difference of the right-ascensions of the stars, and y the difference between the right-ascensions of the planet and that star whose declination is D.

Also let d, d' be the given distance of the stars from the planet then from the Δ (D, d, x) D′, d', x) we get

$$\left.\begin{aligned} \cos. y &= \frac{\cos. d - \cos. D. \cos. x}{\sin. D. \sin. x} \\ \text{And } \cos. (y+a) &= \frac{\cos. d' - \cos. D'. \cos. x}{\sin. D' \sin. x} \end{aligned}\right\}$$

which, by the Arithmetic of Sines, will give us y and x.

883. The shadow will evidently be a parallelogram deviating from a rectangle by half a right-angle. Its breadth will therefore $=$ breadth of the wall $\times \cos. 45^\circ = \frac{1}{\sqrt{2}} \times$ breadth of the wall.

The datum of the sun's altitude is unnecessary. It will give the length of the shadow.

884. This is easy, since

merid. alt. $=$ co-lat. $\pm$ dec.

midnight depress. $=$ co-lat. $\mp$ dec.

$$\therefore \text{co-lat.} = \frac{\text{merid. alt.} + \text{med. dep.}}{2}$$

$$\text{And } \pm \text{dec.} = \frac{\text{merid. alt.} - \text{mid. dep.}}{2}$$

885. Since the star is *known*, his declination is given by the tables. Let D be the co-declination, a its altitude when on the *Prime Vertical*, and L the co-latitude of the place; then from the right-angled $\triangle$ (D, a, L) we have

$$\cos. D = \cos. a \cos. L$$

$$\therefore \cos. L = \frac{\cos. D}{\cos. a}$$

which gives L.

886. Let $s s'$ (Fig. 115,) be two successive positions of the sun, P being the pole and z the zenith. Then if Pz the co-latitude of the place be put $=$ L, the co-dec. $=$ D, the hour $\angle$

$zPs = H$, and azimuth $szP = a$, &c., as in the figure, we have

$$\sigma' n = da.\ \sin zs.$$
$$ss' = dH.\ \sin.\ D.$$

and $\sigma' n = ss' \times \sin.\ s'sn = dH \sin.\ D \cos.\ zsP$

$$\therefore\ da = dH \times \frac{\sin.\ D.\ \cos.\ zsP}{\sin.\ zs}$$

$$= dH \times \frac{\cos.\ L - \cos.\ zs.\ \cos.\ D}{\sin.^2 zs}.$$

Hence, putting $zs = z$, we have, by the question,

$$\frac{\cos.\ L - \cos.\ z.\ \cos.\ D}{\sin.^2 z} = \text{min.}$$

which gives by the rule,

$$\cos.^2 z - \frac{2 \cos.\ L}{\cos.\ D} \cos.\ z = -1$$

And $\therefore\ \cos.\ z = \dfrac{\cos.\ L \pm \sqrt{(\cos.^2 L - \cos.^2 D)}}{\cos.\ D}$.

Hence z is known.

Again,

$$\cos.\ H = \frac{\cos.\ z - \cos.\ D.\ \cos.\ L}{\sin.\ D.\ \sin\ L}$$

which therefore gives the time required.

887. Let L be the co-latitude, z the zenith distance, a the azimuth, and D the co-declination; then from the Δ (z, L, D) we have

$$\cos.\ a = \frac{\cos.\ D - \cos.\ L.\ \cos.\ z}{\sin.\ L.\ \sin.\ z}$$

which gives (*Woodhouse's Trig.*)

$$\sin.^2 \frac{D}{2} = \sin.\ \left(\frac{L+z}{2} + M\right) \sin.\ \left(\frac{L+z}{2} - M\right)$$

wherein

$$\sin.^2 M = \sin.\ L.\ \sin.\ z.\ \cos.\ ^2\frac{a}{2}$$

This gives the declination.

Again, the declination being found, it would be easy to find the

right ascension in terms of the obliquity of the ecliptic. But we are required to express it otherwise.

Since the day and hour of observation are given, let the known right-ascension and hour $\angle$ of the sun, be denoted by A, and H; also let A′, H′ be the right-ascension and meridian distance of the star; then

$$A' - A = H - H'$$
$$\therefore A' = A + H - H'$$

But H′ is known from

$$\sin. H' = \frac{\sin. \alpha}{\sin. D}. \sin. z$$

$\therefore$ A′ is known.

888. Let L be the latitude of the place, D the known co-declination of the star; then from the Δ (L, D, 45°) we have

$$\cos. D = \cos. L. \cos. 45$$
$$\therefore \cos. L = \cos. D \times \sqrt{2}.$$

889. Let L denote the co-latitude, D the co-declination of the sun, and α the required azimuth; then from the right-angled Δ (L, D) we have

$$\tan. \alpha = \frac{\tan. D}{\sin. L}.$$

890. Since the day and hour are given, the right-ascension (A), and hour $\angle$ (H) of the sun are known, also since the latitude and longitude of the star are given, its right-ascension (A′) and declination (90° − D) may be found (see *Vince.*) Hence the angular distance of the star from the meridian (H′) is known from

$$H' - H = A' - A,$$

and the latitude of the place (90° — L) is given. Consequently if 90 — z, α denote the required altitude and azimuth, from the Δ (z, D, L) we have

$$\cos. H' = \frac{\cos. z - \cos. L. \cos. D}{\sin. L. \sin. D}$$

$$\therefore \sin.^2 \frac{z}{2} = \sin. \left(\frac{L + D}{2} + M\right) \sin. \left(\frac{L + D}{2} - M\right)$$

where

$$\sin.^2 M = \sin. L. \sin. D. \cos.^2 \frac{H'}{2};$$

and z being thus known, we have

$$\sin. \alpha = \sin. H' \times \frac{\sin. D}{\sin. z}$$

which gives α. We have therefore found the altitude and azimuth.

Again, let z, P, σ (Fig. 116,) be the places of the zenith, pole, and star; QQ′, CC′ the equator and ecliptic intersecting in the first point Aries R, and cutting the meridian in the points Q′, C′. Then it is evident that $Q'R = A - H$.

Also the $\angle$ R = 23° 28′, and Q′ is a right $\angle$. Consequently

$$\tan. Q' C' = \sin. (A - H) \tan. 23^\circ 28'.$$

$$\sin. \angle Q'C'R = \sin. (A - H) \times \frac{\sin. 23^\circ 28}{\sin. Q'C'}$$

$$\text{and } \sin. C'R = \frac{\sin. Q'C'}{\sin. 23^\circ 28'}$$

which give Q′C′ and $\angle zC'z'$

Hence in the Δ $zz'C'$, we have

$$\left.\begin{aligned} zC' &= 90^\circ - L - Q'C' \\ \angle zC'z' &= 180^\circ - Q'C'R \\ \text{and } \angle C'zz' &= 180^\circ - \alpha \end{aligned}\right\}$$

which will give, when resolved, the $\angle C'z'z$ and $c'z'$.

Consequently,

$$C'z'z \text{ and } Rz' \;(= C'z' - RC')$$

are known, which are the other *quæsita*, of the problem.

891. Since the day of the year is given, the sun's right-ascension (a) is known from the tables. Let also the given co-declinations and right-ascensions of the stars be denoted by

$$D, D'; A, A'.$$

Moreover, let H be the hour $\angle$ or distance of the sun from noon at the instant the stars are on the same azimuth; then the angular distances of the stars from the meridian are evidently

$$h = \text{H} + \text{A} - a, \text{ and } h' = \text{H} + \text{A}' - a.$$

Hence, if L = the co-latitude of the place, and z, z' the zenith distances from the $\triangle$ (z, L, D), (z', L, D′), we get

$$\cos.(\text{H} + \text{A} - a) = \frac{\cos. z - \cos. \text{L}. \cos. \text{D}}{\sin. \text{L}. \sin. \text{D}.}$$

$$\cos.(\text{H} + \text{A}' - a) = \frac{\cos. z' - \cos. \text{L}. \cos. \text{D}'}{\sin. \text{L}. \sin. \text{D}'}$$

Also from the $\triangle$ (D, D′, $z' - z$) we get

$$\cos.(\text{A}' - \text{A}) = \frac{\cos.(z' - z) - \cos. \text{D}. \cos. \text{D}'}{\sin. \text{D}. \sin. \text{D}'}$$

$$\therefore \sin.^2 \frac{z' - z}{2} = \sin. \left(\frac{\text{D} + \text{D}'}{2} + \text{M}\right). \sin. \left(\frac{\text{D} + \text{D}'}{2} - \text{M},\right)$$

M being such that

$$\sin.^2 \text{M} = \sin. \text{D}. \sin. \text{D}'. \cos.^2 \frac{\text{A}' - \text{A}}{2}$$

Hence $z' - z$ is known ($= m$), which therefore gives

$$\cos.(\text{H} + \text{A} - a = \frac{\cos. z - \cos. \text{L}. \cos. \text{D}}{\sin. \text{L}. \sin. \text{D}}$$

$$\text{and } \cos.(\text{H} + \text{A}' - a) = \frac{\cos.(z + m) - \cos. \text{L}. \cos. \text{D}'}{\sin. \text{L}. \sin. \text{D}'}$$

by which two equations, z being eliminated, we shall have H expressed in terms of A, A′, a, L, D, D′; which will therefore be known.

892. When the hour angle from noon = azimuth from the south, the zenith distance is evidently equal to the co-declination (D). Hence if L be the co-latitude, and H the required hour $\angle$, we have

$$\cos. \text{H} = \frac{\cos. \text{D} - \cos. \text{D}. \cos. \text{L}}{\sin. \text{D}. \sin. \text{L}}$$

$$= \cot. \text{D}. \frac{1 - \cos. \text{L}}{\sin. \text{L}}$$

$$= \cot. D \frac{2 \sin.^2 \frac{L}{2}}{2 \sin. \frac{L}{2}. \cos. \frac{L}{2}}$$

$$= \cot. D. \tan. \frac{L}{2}.$$

893. Let L, D, z be the co-latitude of the place, the sun's co-declination and zenith distance; also let a be his azimuth, and H the hour angle from noon; then from the $\triangle$ (L, D, z), we get

$$\sin.^2 \frac{a}{2} = \frac{\sin. (S-L). \sin. (S-z)}{\sin. L. \sin. z}$$

$$\text{where } S = \frac{L+D+z}{2}$$

and thence

$$\sin. H = \sin. a \times \frac{\sin. z}{\sin. D}.$$

Moreover, since the sun's declination is given, the day of the year is known from the tables. Consequently the azimuth and hour $\angle$ are found.

Again, since the co-latitude of the sun is 90°, and the obliquity of the ecliptic, or distance between the poles of the equator and ecliptic, is 23° 28′ from the $\triangle$ (90°, 23° 28′, D) we have

$$\cos. P = \cot. 23°. 28', \times \cot. D,$$

P being the angle of position.

894. Since the day is given, the earth's longitude (L) is known from the tables. Let L′ and λ be the given longitude and latitude of the star, and θ the required inclination; then from the right-angled $\triangle$ (90° + L − L′, λ, θ) we get

$$\cos. \theta = \cos. \lambda. \cos. (90 - \overline{L' - L})$$
$$= \cos. \lambda. \sin. (L' - L)$$

895. At the moment the star and sun are seen rising together, let the time from noon be H, and let L be the co-latitude

of the place, A the right-ascension of the sun, D his co-declination, A′, D′, H′ the right-ascension, co-declination, and angular distance from the meridian of the star. Then from the quadrantal △ (90°, L, D), (90°, L, D′), we have

$$\cos. H = \frac{\cos. 90 - \cos. L. \cos. D}{\sin. L. \sin. D}$$

$$= -\cot. L. \cot. D$$

$$\cos. H' = -\cot. L. \cot. D'.$$

But $H' - H = A' - A$

$$\therefore \cos. (H + A' - A) = -\cot. L \cot. D' \; . \; . \; . \; . \; . \; (1)$$

But since the obliquity of the ecliptic is 23° 28′, we have

$$\cot. D = \sin. A. \tan. 23° \; 28'$$

$$\therefore \cos. H = -\cot. L. \sin. A. \tan. 23° \; 28' \; . \; . \; . \; . \; (2)$$

Now, from equation (1) we get

$$H - A + A'$$

and $\therefore H - A$

is known. Hence, by substituting in equation (2), we get both H and A, which give the hour of the day, and (by the tables) the day of the year.

896. Let θ, θ' be the true anomalies corresponding to the distances ρ, ρ'; then from the equation to a parabola $\left(\varrho = \frac{a}{\cos.^2 \frac{\theta}{2}}\right)$

we get

$$\sqrt{\varrho} : \sqrt{\varrho'} :: \cos. \frac{\theta'}{2} : \cos. \frac{\theta}{2}$$

$$\therefore \sqrt{\varrho} + \sqrt{\varrho'} : \sqrt{\varrho} - \sqrt{\varrho'} :: \cos. \frac{\theta'}{2} + \cos. \frac{\theta}{2} : \cos. \frac{\theta'}{2} - \cos. \frac{\theta}{2}$$

$$:: 2\cos. \frac{\theta + \theta}{2}. \cos. \frac{\theta - \theta'}{2} : 2 \sin. \frac{\theta + \theta'}{2}. \sin. \frac{\theta - \theta'}{2}$$

$$:: \cot. \frac{\theta + \theta'}{2} : \tan. \frac{\theta - \theta'}{2}.$$

897. Let L, L', L'', be the longitudes, λ, λ', λ'' the co-latitudes of the three places z, z', z''. Also let d, d', d'', be the distances between

$$z, z'; \; z', z''; \; z, z''.$$

Then from the Δ

$$(d, \lambda, \lambda'), \; (d', \lambda', \lambda''), \; (d'', \lambda, \lambda'')$$

we get

$$\cos.(\mathrm{L} - \mathrm{L}') = \frac{\cos. d - \cos. \lambda. \cos. \lambda'}{\sin. \lambda. \sin. \lambda'}$$

$$\cos.(\mathrm{L}' - \mathrm{L}'') = \frac{\cos. d' - \cos. \lambda'. \cos. \lambda''}{\sin. \lambda'. \sin. \lambda''}$$

$$\cos.(\mathrm{L} - \mathrm{L}'') = \frac{\cos. d'' - \cos. \lambda. \cos. \lambda''}{\sin. \lambda. \sin. \lambda''}$$

whence

$$\sin.^2\frac{d}{2} = \sin.\left(\frac{\lambda+\lambda'}{2} + \mathrm{M}\right) \sin.\left(\frac{\lambda+\lambda'}{2} - \mathrm{M}\right)$$

$$\sin.^2\frac{d'}{2} = \sin.\left(\frac{\lambda'+\lambda''}{2} + \mathrm{M}'\right). \sin.\left(\frac{\lambda'+\lambda''}{2} - \mathrm{M}'\right)$$

$$\sin.^2\frac{d''}{2} = \sin.\left(\frac{\lambda+\lambda''}{2} + \mathrm{M}''\right). \sin.\left(\frac{\lambda+\lambda''}{2} - \mathrm{M}''\right)$$

M, M', M'' being such that

$$\sin.^2\mathrm{M} = \sin. \lambda. \sin. \lambda'. \cos.^2\frac{\mathrm{L} - \mathrm{L}'}{2}$$

&c. = &c.

which therefore give the three distances d, d' d''.

Hence if D, D', D'' denote the angles respectively opposite to d, d', d'', we have

$$\cos. \mathrm{D} = \frac{\cos. d - \cos. d. \cos. d'}{\sin. d'. \sin. d''}$$

$$\cos. \mathrm{D}' = \frac{\cos. d' - \cos. d. \cos. d''}{\sin. d. \sin. d''}$$

$$\cos. \mathrm{D}'' = \frac{\cos. d'' - \cos. d. \cos. d'}{\sin. d, \sin. d'}$$

which give

D, D', D''

Again, let x be the common distance of the three places from the required place, and L‴ and λ‴ its longitude and co-latitude; then from the △ (λ, λ''', x), $(\lambda', \lambda''', x)$, $(\lambda'', \lambda''', x)$, we get

$$\cos.(L - L''') = \frac{\cos. x - \cos. \lambda. \cos. \lambda'''}{\sin. \lambda. \sin. \lambda'''} \quad . \quad . \quad . \quad . \quad (1)$$

$$\cos.(L' - L''') = \frac{\cos. x - \cos. \lambda'. \cos. \lambda'''}{\sin. \lambda'. \sin. \lambda'''} \quad . \quad . \quad . \quad . \quad (2)$$

$$\cos.(L'' - L''') = \frac{\cos. x - \cos. \lambda'' \cos. \lambda'''}{\sin. \lambda'' \sin. \lambda'''} \quad . \quad . \quad . \quad . \quad (3)$$

$\therefore \sin. \lambda, \sin. \lambda'''. \cos.(L - L''') + \cos. \lambda \cos. \lambda''' =$
$\sin. \lambda'. \sin. \lambda''' \cos.(L' - L''') + \cos. \lambda' \cos. \lambda''' \quad . \quad . \quad . \quad (4)$

And $\sin. \lambda. \sin. \lambda'''. \cos.(L - L''') + \cos. \lambda. \cos. \lambda''' =$
$\sin. \lambda'' \sin. \lambda''' \cos.(L'' - L''') \cos. \lambda'' \cos. \lambda''' \quad . \quad . \quad . \quad . \quad . \quad (5)$

which *two equations* will give, when reduced, λ‴ and L‴; and we shall then have *more* than resolved the problem.

By taking the △ $(d, d' d'')$ we may easily show that

$$\cos. \theta = \cot. x. \frac{1 - \cos. d}{\sin. d} = \cot. x \tan. \frac{d}{2}$$

$$\text{And } \cos.(D' - \theta) = \cot x. \tan. \frac{d'}{2}$$

θ being the inclination of x to d.

Hence, by reduction,

$$\cot.^2 x = \frac{\sin.^2 D''}{\tan.^2 \frac{d}{2} + \tan.^2 \frac{d'}{2} - 2 \tan. \frac{d}{2} \tan. \frac{d'}{2} \cos. D''}$$

which gives x.

Hence θ is known.

Also the ∠ between λ and d' is found from

$$\cos.(\lambda, d') = \frac{\cos. \lambda'' - \cos. \lambda. \cos. d'}{\sin. \lambda. \sin. d'}$$

Hence we have the ∠ between x and λ, and from the △ $(\lambda'''\ x, \lambda)$ we get

$$\cos.(x, \lambda) = \frac{\cos. \lambda''' - \cos. \lambda \cos. x}{\sin. \lambda \sin. x}$$

which gives λ‴.

and λ''' being substituted in the equat. (1), gives

$$L - L''' \text{ and } \therefore L'''.$$

This latter method of finding λ''' and L''' is evidently more commodious in practice than the former.

898. Since the star is given, its co-declination (D), and right-ascension (A) are known by the tables.

Let H be the hour angle from noon at sunrise, A′ the sun's right-ascension, and D′ his co-declination, and L the co-latitude of the place. Then

$$\cos. H = -\cot. D' \cot. L$$

and from the right-angled $\triangle$ $(90^\circ - D', A')$, we have

$$\cot. D' = \tan. 23^\circ 28' \times \sin. A'$$

$$\therefore \cos. H = -\sin. A'. \cot. L. \tan. 23^\circ 28'$$

But since the star is due south, $A' - A = H$,

$$\therefore \frac{\sin. (A+H)}{\cos. H} = -\tan. L. \cot. 23^\circ 28'$$

and reducing, we finally get

$$\tan. H = -\left\{\tan. A + \frac{\tan. L. \cot. 23^\circ 28'}{\cos. A}\right\}$$

which gives H the hour of the day.

Hence $A' = A + H$ the sun's right-ascension is known, which gives the day of the year.

899. Let L be the co-latitude, z the zenith distance, and D the co-declination of the sun at six; then from the right-angled $\triangle$ (z, L, D) we have

$$\therefore \cos. D = \frac{\cos. z}{\cos. L}$$

which gives D, and therefore by the tables, the day of the year.

900. Let D be the sun's co-declination, 2H the hour $\angle$ from noon at rising, and L the co-latitude; then from the $\triangle$ (right-angled by the question)

$$\cot. D = \cos. H. \cot. L$$

and from the $\triangle$ (90°, L, D) we get

$$\cos. 2H = -\cot. L. \cot. D$$

$$2\cos.^2 H - 1 = -\cot. L \cot. D$$

$$= 2\cot.^2 D. \tan.^2 L - 1$$

$$\therefore \tan.^2 L - \cot. D. \tan. L = 2\cot.^2 D$$

$$\therefore \tan. L = \frac{\cot. D \pm 3\cot. D}{2}$$

$$= 2\cot. D \text{ or } -\cot. D$$

which gives the latitude L.

901. The perpendiculars will evidently meet at the centre of the earth, being $\perp$ to its surface. Hence the $\angle$ between them is measured by the arc of the meridian, which is intercepted by the two horizons, or it is the difference of the latitudes of the two places.

902. Let θ be the angle which any great circle, passing through the star, makes with the horizon, the altitude of the star being a, and the distance between the intersections of the vertical and great circle with the horizon being x; then from the right-angled $\triangle$ (x, a) we have

$$\tan. \theta = \frac{\tan. a}{\sin. a} \propto \frac{1}{\sin. x}$$

and θ is therefore least when $x = 90°$, or when

$$\tan. \theta = \tan. a, \text{ and } \theta = a. \qquad \text{Q. E. D.}$$

903. For the sun's place, when the aberration in declination $= 0$, (see *Woodhouse*, p. 269.)

If the unresolved aberration $(\sigma\sigma')$ were constant, the aberration in right-ascension would plainly be greatest when that in declination $= 0$. But $\sigma\sigma' \propto \sin. \sigma T$, and consequently varies, $\therefore$ &c.

904. Let z, z' be the zenith distances of the sun when the coincidences of the extremities of the shadows with the bases

of the rods take place; also D the co-declination of the sun, and L the co-latitude of the place.

In the first place, since the shadows, although in opposite directions, are in the same straight line, between the first and second shadow, the sun must have passed through 180 degrees of azimuth.

In the next place, we have, by the question,

$$6 = 20 \text{ cot. } z, \; 8 = 20 \text{ cot } z'$$

$$\therefore \text{cot. } z = \frac{3}{10} \text{ and cot. } z' = \frac{2}{5}$$

which gives z and z'.

Again, let α, $180 - \alpha$ be the azimuth angles on each side of the meridian, then from the Δ (z, L, D), $(z' L, D,)$ we get

$$\cos. \alpha = \frac{\cos. D. - \cos. L. \cos. z}{\sin. L. \sin. z} = -\frac{\cos. D - \cos. L. \cos. z'}{\sin. L. \sin. z'}$$

which gives

$$\cos. L = \cos. D \times \frac{\sin. z + \sin. z'}{\sin. z . \sin. z' + \cos. z . \sin. z'}$$

$$= \cos. D \times \frac{\cos. \frac{z-z'}{2}}{\cos. \frac{z+z'}{2}}$$

and therefore the latitude is known.

Hence also α may be found.

905. Let L be the co-latitude, D the co-declination of the sun, α the azimuth, and in the Δ (L, D,) let the $\angle$ opposite L, or that in which the sun is situated be called θ; then

$$\sin. \alpha = \sin. \theta \times \frac{\sin. D}{\sin. L} \propto \sin. \theta$$

$\therefore \alpha =$ maximum, when $\theta = 90°$. Or when the vertical circle passing through the sun touch his diurnal circle. But this contact, when the zenith is between the tropics, can evidently take place on one side of the meridian as well as on the other. Consequently, to persons situated within the tropics, there will be two maximum azimuths daily.

906. Let h be the difference between the times of setting, or the distance between the two declination circles which pass through the stars. Also let D, D′ be their co-declinations, and L the latitude of the place; then from the right-angled △ (L, D), (L, D′), we have

$$\cos. \theta = \tan. L . \cot. D$$
$$\cos. (\theta + h) = \tan. L . \cot. D'$$

θ being the $\angle$ included by the altitude of the pole and D.

$$\therefore \frac{\tan. D}{\tan. D'} = \frac{\cos. \theta + h}{\cos. \theta} = \cos. h - \sin. h . \tan. \theta$$

$$\text{and } \tan. \theta = \cot. h - \frac{\tan. D}{\sin. h \tan. D'}$$

which gives θ; and $\therefore$

$$\tan. L = \cos. \theta . \tan. D$$

also gives L.

907. The moon's greatest declination is evidently = the inclination of her orbit to the equator. Hence, if I be the inclination of her orbit to the ecliptic, the greatest declination is

$$23° \; 28' \mp I = D.$$

Again, let λ be her latitude at this epoch, and L the given longitude of her node; then her longitude is

$$L + \sin.^{-1} (\tan. \lambda \cot. I) = l \; \ldots\ldots \; (1)$$

Now let M (Fig. 117) be the place of the moon, R the intersection of the equator Rm, with the ecliptic Rm', &c., as in the figure. Then from the △ MRm, MRm', we get these three equations, (A being the right ascension of the moon),

$$\tan. D = \tan. (\theta + I') . \sin. A \; \ldots\ldots\ldots \; (2)$$
$$\tan. \lambda = \tan. \theta \sin. l \; \ldots\ldots\ldots\ldots\ldots \; (3)$$
$$\text{and } \cos. RM = \cos. D . \cos. A = \cos. \lambda \cos. l \; \ldots \; (4)$$

Eliminating θ from (2) and (3) we get

$$\frac{\sin. l}{\tan. \lambda} = \frac{\tan. D . \tan. I'}{\tan. D - \tan. I'} + \frac{\sin. A}{\tan. D - \tan. I'}$$

$$= m + n \times \sin. A \; \ldots\ldots\ldots\ldots \; (5)$$

by supposition, m and n being known quantities.

Also, from equation (1), we get

$$\tan. \lambda \,.\, \cot. I = \sin. (l - L)$$

$$\therefore \tan. \lambda = \tan. I \,.\, \sin. (l - L) = \frac{\sin. l}{m + n \sin. A} = \text{also}$$

by (4) $\dfrac{\sqrt{(\cos.^2 l - \cos.^2 D \,.\, \cos.^2 A)}}{\cos. D \,.\, \cos. A}$.

Hence we easily get

$$\cot. l = \cot. L - \frac{\cot. I}{\sin. L} \times \frac{1}{m + n. \sin. A}$$

$$\text{and } \sin.^2 l = \frac{(1 - \cos.^2 D \cos.^2 A) \times (m + n \sin. A)^2}{\cos.^2 D \cos.^2 A + (m + n \sin. A)^2}$$

which *two* equations involving only *two unknown* quantities l and A will give them when reduced and resolved.

The remainder of the investigation being operose rather than intricate, we leave it, as an exercise, to the student.

Having found l, and A, he will find λ from equation (4); l and λ will give the required place of the moon at the time of her greatest declination.

908. Let H be the hour $\angle$ from midnight of rising at one place, whose latitude is L; then the hour $\angle$ at the other whose latitude is $90^\circ - L$ is, by the question

$$H - n \times 15^\circ.$$

Hence, if D be the given co-declination of the sun, from the right-angled $\triangle$, (L, D), (90 − L, D), we have

$$\tan. L = \cos. H \,.\, \tan. D$$

$$\tan. L = \cos. (H - n \times 15^\circ) \times \tan. D$$

$$= \cos. (n \times 15^\circ) \tan. L + \sin. (n \times 15^\circ) \sqrt{(\tan.^2 D - \tan.^2 L)}$$

$$\therefore \tan^2 L \times \tan.^2 \left(\frac{n}{2} \,.\, 15^\circ\right) = \tan.^2 D - \tan.^2 L$$

$$\text{and } \tan. L = \tan. D \,.\, \cos. \frac{n}{2}\, 15^\circ.$$

909. In the vertical passing through the sun and zenith describe a circle whose radius is the length of the stick. Next draw a tangent to this circle to such a point that its inclination to the horizon may be 30°. The distance of the centre of the circle from that intersection will evidently be the longest shadow possible.

Hence the stick must be inclined to the horizon at the angle $90° - 30° = 60°$.

Again, if l be the stick, and x that of its shadow, we have

$$l = x \,.\, \sin. 30° = \frac{x}{2}$$

$$\therefore\ x = 2l.$$

910. Since the day is given, the sun's co-declination (D) is known from the tables, and the hour $\angle$ (H) being known, and the co-latitude of the place (L), from the $\triangle$ $(z, D, L,)$ we get

$$\cos. z = \sin. D \,.\, \sin. L \,.\, \cos. H + \cos. L \,.\, \cos. D$$
$$= \sin. \left(\frac{D+L}{2} + M\right) \sin. \left(\frac{D+L}{2} - M\right)$$

where

$$\sin.^2 M = \sin. D \,.\, \sin. L \,.\, \cos.^2 \frac{H}{2}.$$

Hence the altitude of the sun $90° - z$ is known.

Again, since the rays of the sun are parallel, the shadow of the globe will be a cylinder whose axis is inclined to the horizontal plane at the $\angle$ $(90° - z)$, and the radius of whose base is (r) the radius of the globe. Hence the shadow upon the horizon will be an ellipse, (the section of the cylinder) whose semi-axes may easily be shewn to be { its equation is $y^2 = \cos.^2 z \left(\frac{2r}{\cos. z} x - x^2\right)$ }

$$\frac{r}{\cos. z} \text{ and } r.$$

Again, the length of the shadow is the axis-major, or $\frac{2r}{\cos. z}$, which varies as the secant of the zenith distance.

911. Since the two stars are known, their right ascensions A, A′, and co-declinations D, D′ are given. Let also the given co-declination of the other star be D″, its right ascension θ, and its distances from the two stars x, x'. Then from the △ (D, D″, x), (D′, D″, x'), we have

$$\cos. x = \cos. D . \cos. D'' + \sin. D . \sin. D'' . \cos. (A - \theta)$$

$$\cos. x' = \cos. D'' . \cos. D' + \sin. D' . \sin. D'' . \cos. (\theta - \theta')$$

$$\therefore\ - dx \sin. x = d\theta . \sin. D . \sin. D'' . \sin. (A - \theta)$$

$$dx' \sin. x' = d\theta \sin. D' \sin. D'' . \sin. (\theta - A')$$

But since $x + x' =$ minimum,

$$\therefore\ - dx = dx'$$

$$\therefore\ \frac{\sin. x}{\sin. x'} = \frac{\sin. D}{\sin. D'} \times \frac{\sin. (A - \theta)}{\sin. (\theta - A')}$$

Hence, and by means of the two equations above, x x', and θ, may be found, which will more than resolve the problem.

912. On the longest and shortest day the declination of the sun is equal to the obliquity of the ecliptic, and then the meridian altitude of the sun $=$ co-latitude $\mp$ declination. Consequently if l, l' be the lengths of the shadows, and h the height of the rod, we have

$$h = l . \tan. (\text{co-latitude} - \text{obliquity})$$

$$= l' \tan. (\text{co-latitude} + \text{obliquity}).$$

Let L be the latitude, and O the obliquity; then

$$\cot. (L + O)\ :\ \cot. (L - O) :: l : l' :: n : 1$$

by the question.

$$\therefore \cos. (L+O) . \sin. (L-O)\ :\ \cos. (L-O) \sin. (L+O) :: n : 1$$

$\therefore$ $\cos. (L + O) . \sin. (L - O) + \cos. (L - O) \sin. (L + \theta)$: $\cos (L + O) \sin. (L - O) - \cos (L - O) \sin. (L + O)$ :: $n + 1$: $n - 1$.

$\therefore \sin.(\overline{L+\theta}+\overline{L-\theta}) : \sin.(\overline{L+\theta}-\overline{L-\theta}) :: n+1 : n-1$, or $\sin. 2\times L : \sin. 2\times\theta :: n+1 : n-1$.

Q. E. D.

913. Let D be the co-declination of the star, L the co-latitude, and z, z' the observed zenith distances, h the $\angle$ measuring the interval between the equal azimuths α. Then

$$\cos.\alpha = \frac{\cos. D - \cos. L . \cos. z}{\sin. L . \sin. z}$$

$$= \frac{\cos. D - \cos. L . \cos. z'}{\sin. L . \sin. z'}$$

$$\text{and } \cos. h = \frac{\cos.(z-z') - \cos.^2 D}{\sin.^2 D}$$

which give

$$\cos.^2 D = \frac{\sin. \frac{h+z-z'}{2} \sin. \frac{h+z'-z}{2}}{\sin.^2 \frac{h}{2}}$$

$$\text{and } \cos. D = \cos L \times \frac{\sin.(z-z')}{\sin. z - \sin. z'}$$

$$= \cos. L \times \frac{\cos. \frac{z-z'}{2}}{\cos. \frac{z+z'}{2}}$$

$$\therefore \cos.^2 L = \frac{\cos.^2 \frac{z+z'}{2} \times \sin. \frac{h+z-z'}{2} \sin. \frac{h+z'-z}{2}}{\cos.^2 \frac{z-z'}{2} \times \sin.^2 \frac{h}{2}}$$

which gives the latitude.

914. It is easily shewn that in the given periods the numbers of seconds are

152853, 314022

and the periods are in the sesquiplicate ratio of the mean distances, consequently

$$a : a' :: 1 : \left(\frac{314022}{152858}\right)^{\frac{2}{3}}$$

$$:: 1 : \left(\frac{104674}{50951}\right)^{\frac{2}{3}}$$

$$:: 1 : \left(2+\frac{2\times 1386}{50951}\right)^{\frac{2}{3}}$$

$$:: 1 : 2^{\frac{2}{3}}\left(1+\frac{1}{37.7}\right)^{\frac{2}{3}} \text{ nearly}$$

$$:: 1 : 2^{\frac{2}{3}}\left(1+\frac{2}{3}\cdot\frac{1}{37.7}\right) \text{ nearly.}$$

914. The parallax of the moon diminishes the moon's altitude; the refraction, on the contrary, elevates both the moon and the star; and they take place wholly in vertical circles. Hence, from the $\triangle$ having the same vertical $\angle\, z$ at the zenith, we get

$$\cos. z = \frac{\cos. D - \sin. A . \sin. B}{\cos. A . \cos. B}$$

$$= \frac{\cos. d - \sin. a . \sin. b}{\cos. a . \cos. b}$$

$$\therefore \cos. D = \sin. A . \sin. B + \cos. A . \cos. B \frac{\cos. d - \sin. a . \sin. b}{\cos. a . \cos. b}$$

$$\text{vers. } D = 1 - \sin. A . \sin. B - \frac{\cos. A . \cos. B}{\cos. a . \cos. b} . (1 - \text{vers. } d$$
$$- \sin. a . \sin. b)$$

$$= 1 - \cos. (A - B) + \frac{\cos. A \ \cos. B}{\cos. a . \cos. b} \times (\cos. a \times$$

$\cos. b - 1 + \text{vers. } d + \sin. a . \sin. b) = \text{vers. } (A - B) + \frac{\cos. A . \cos. B}{\cos. a . \cos. b} . (\text{vers. } d - 1 + \cos. \overline{a-b}) = \text{vers. } (A-B) + \frac{\cos. A . \cos. B}{\cos. a . \cos. b} . (\text{vers. } d - \text{vers. } (a - b)).$

Q. E. D.

A more commodious form is this

$$\frac{\cos.(A+B)+\cos.D}{\cos.A.\cos.B} = \frac{2\cos.\frac{a+b+d}{2}\cos.\frac{a+b-d}{2}}{\cos.a.\cos.b}$$

which is also easily deduced.

916. Let L be the latitude of the place; then from the right-angled $\triangle$ (L, 75°) we have (by the question)

$$\tan.L = \cos.4 \times 15^\circ \times \tan.75^\circ$$
$$= \frac{1}{2} \times \frac{\cos.15^\circ}{\sin.15^\circ}$$

$$\text{But } \cos.^2 15^\circ = \frac{\cos.30^\circ + 1}{2} = \frac{2+\sqrt{3}}{4}$$

$$\text{and } \sin.^2 15^\circ = \frac{1-\cos.30^\circ}{2} = \frac{2-\sqrt{3}}{4}$$

$$\therefore \tan.L = \frac{1}{2} \times \frac{\sqrt{(2+\sqrt{3})}}{\sqrt{(2-\sqrt{3})}} = \frac{1}{2}.(2+\sqrt{3})$$

which indicates an error in the enunciation.

917. If ρ be the radius-vector of the comet's orbit, and r its perihelion distance; then the time of its distance (from the sun) increasing from r to ρ is (see 484)

$$t = \sqrt{\frac{2}{3\mu}} \times \sqrt{(\rho - r)} \times (\rho + 2r)$$

μ being the mass of the sun and comet.

In this formula let $\rho = a$, the mean distance of the earth from the sun. Then, by the question,

$$t = \sqrt{\frac{2}{3\mu}} \times \sqrt{\left(a - \frac{a}{3}\right)} \times \left(a + \frac{2a}{3}\right)$$
$$= \frac{10a^{\frac{3}{2}}}{9\sqrt{\mu}}.$$

Again, if T denote the length of the year; then (see 453), we have

$$T = \frac{2\pi a^{\frac{3}{2}}}{\sqrt{\mu}}$$

$$\therefore t : T :: \frac{5}{9} : \pi$$

$$\text{and } t = \frac{5}{9 \times \pi} \times T$$

and the number of days in which the comet is moving within the earth's orbit, is, therefore, expressed by

$$2t = \frac{10}{9 \times \pi} \times T,$$

which gives numerically

$$2t = \frac{10}{9 \times 3.14159} \times 365\frac{1}{4}$$

$$= \frac{3652\frac{1}{2}}{9 \times 3.14159} = \frac{7305}{56.54862}$$

$$= 129 \text{ days} . 4 \text{ hours} . 20 \text{ minutes}.$$

MISCELLANIES.

918. BY 436, Vol. II. we have

$$F = \frac{c^2}{\rho^3} - \frac{c^2}{2} \cdot d\frac{\left(\frac{d\rho^2}{\rho^4 d\theta^2}\right)}{d\rho}.$$

Now in the logarithmic spiral

$$\theta = \log. \rho$$

$$\therefore d\theta = \frac{1}{la} \cdot \frac{d\rho}{\rho}$$

$$\therefore \frac{d\rho}{d\theta} = \rho\, l.a$$

$$\therefore \frac{d\rho^2}{d\theta^2} = \frac{(la)^2}{\rho^2}$$

$$F = \frac{c^2}{\rho^3} + \frac{c^2}{2} \cdot \frac{(la)^2}{\rho^2}$$

$$= \frac{c^2 . \{1 + (la)^2\}}{\rho^3} \propto \frac{1}{\rho^3}.$$

919. The curve required is evidently the orthographic projection of the circular arch upon the surface of the water, and is, therefore, by the theory of projections, an ellipse, whose equation is

$$y^2 = \cot.^2 \mathrm{I} \cdot (r^2 - x^2)$$

and semi-axes

$$r \text{ and } r \cot. \mathrm{I}.$$

r being the radius of the arch, and I the meridian altitude of the sun.

920. Since F $\propto$ distance, let us assume

$$F = \mu\varrho.$$

Then

$$vdv = -F\,d\varrho = -\mu\varrho d\varrho$$
$$\therefore\ v^2 = \mu\,.\,(c - \varrho^2)$$
$$= \mu\,(r^2 - \varrho^2)$$

$r - \varrho$ being equal to AC.

Again the force at A being constant, we have

$$F' = \mu r$$

and $vdv = -\mu r d\rho$

$$\therefore\ v'^2 = 2\mu r\,(r - \rho)$$
$$\therefore\ v : v' :: \sqrt{r^2 - \varrho^2} : \sqrt{2r(r-\varrho)}.$$

But if r be the radius of a circle, and ϱ cosine of an arc α; then

$\sqrt{r^2 - \rho^2} = \sin.\ \alpha$, and we also have

$$\text{ch.}\ \alpha = \sqrt{\sin.^2\alpha + \text{vers.}^2\alpha}$$
$$= \sqrt{r^2 - \varrho^2 + (r-\varrho)^2}$$
$$= \sqrt{2r^2 - 2r\varrho}$$
$$= \sqrt{2r\,.\,(r-\varrho)}.$$

Hence

$$v : v' :: \sin.\ \alpha : \text{ch.}\ \alpha.$$

See also *Principia*.

921. The right-ascension of the star, and time of the year, being given, the angular distances of the star and sun from Aries are known, and, consequently, the angle between them. Let this be denoted by α. Also the co-declinations of the star and sun are known, which call D, and D′ respectively. Then if L be the latitude of the place or altitude of the Pole; from the right-angled spherical triangles, whose hypothenuses are D, D′, we easily get two equations between the two unknown quantities, L and the angular distance of the star from the meridian.

922. At any distance x from the centre of the sphere let the density be

$$Ax^{\alpha},$$

x being variable, and A and α certain constants not yet determined. Then if Q denote the quantity of matter in the sphere, whose radius is x, and M its magnitude, we have

$$d \,.\, Q = Ax^{\alpha} \times dM = Ax^{\alpha} \,.\, d\left(\frac{4}{3}\pi x^{3}\right)$$

$$= 4\,A\pi x^{\alpha+2}dx$$

$$\therefore Q = \frac{4A\pi}{\alpha+3} \,.\, x^{\alpha+3}$$

Q being nothing when $x = 0$.

Now if D be the mean density of a sphere whose radius is r, then by the question $mD =$ the density at its surface; and we have, from the quantities of matter being the same in both hypotheses of density,

$$D \,.\, \frac{4}{3}\pi r^{3} = \frac{4A\pi}{\alpha+3} \,.\, r^{\alpha+3}.$$

But $mD = Ar^{\alpha}$.

$$\therefore \frac{4A\pi r^{\alpha+3}}{3m} = \frac{4A\pi}{\alpha+3} \,.\, r^{\alpha+3}$$

$$\therefore 3m = \alpha + 3$$

and $\alpha = 3 \,.\, (m - 1)$.

Hence

Density $= Ax^{3\,(m-1)}$

$\alpha x^{3\,(m-1)}$.

923. Let G be the magnitude of a guinea; bG represents its weight C. Also let P and G be the magnitudes of the platina and silver requisite to make the coin in question; then by the question we have

$$P + S = G$$

and $aP + cS = bG$.

$\therefore aP + aS = aG$

and $(c - a)\,S = (b - a)\,G$

$$\therefore S = \frac{b-a}{c-a} \cdot G$$

$$\therefore P = \frac{b-a}{c-a} \cdot G$$

and

$$apP + csS = \left(ap \cdot \frac{b-a}{c-a} + cs \cdot \frac{c-b}{c-a}\right) G$$

$\therefore apP + csS : bgG :: ap \cdot (b-a) + cs(c-b) : bg$, the analogy required.

924. First the question requires x, y, z, to be the integers. Then

$$\frac{29}{36} x = z + \frac{y}{3} + \frac{5}{36}$$

$$\therefore 29x = 36z + 12y + 5$$

$$\therefore y = 2x - 3z + \frac{5}{12} x - \frac{5}{12}.$$

Let $\frac{5}{12}(x-1) = w$. Then

$$x - 1 = \frac{12w}{5} = 2w + \frac{2}{5} w,$$

and w must be of the form $5p$.

Hence

$$x = 1 + 12p$$

and $y = 2 + 29p - 3z$,

which give the values required, z being assumed any whatever.

925. The earth being a spheroid, the latitude will be the length of the arc of the generating ellipse measured from the equator to the poles. Calling this arc s, and the distance required ϱ, by the equation to the ellipse we have

$$\varrho^2 = \frac{b^2}{1 - e^2 \cos.^2 \theta}.$$

Also if θ denote the angular distance of ϱ from the equator, and

p the $\perp$ from the centre upon the tangent at the extremity of ρ, we have, from the equation to the ellipse,

$$p^2 = \frac{a^2 b^2}{\rho^2 - a^2 + b^2};$$

and the elemental triangle gives

$$ds = \frac{\rho^2 d\theta}{p}.$$

Hence we easily get

$$d\theta = \frac{-b^2}{e^2 \cos.\theta \sin.\theta \,.\, \rho^3} \,.\, d\rho$$

$$\cos.^2\theta = \frac{\rho^2 - b^2}{e^2\rho^2}$$

$$\sin.^2\theta = \frac{b^2 - (1 - e^2)\rho^2}{e^2\rho^2}, \text{ \&c.}$$

and substituting, we have,

$$ds = a\sqrt{\frac{\rho^2 - a^2 + b^2}{(\rho^2 - b^2)(a^2 - \rho^2)}} \times \rho d\rho.$$

Again, making $a^2 - \rho^2 = u^2$ and substituting, we get

$$ds = a\sqrt{\frac{b^2 - u^2}{a^2 - b^2 - u^2}} \times du$$

which is integrable by the method expounded in pp. 144, Vol. II. and gives ρ in terms of s.

926. The diurnal path of a star is a circle inclined to the horizon of the place by an angle which measures the colatitude. Hence the orthographic projection of the circle upon the horizon is an ellipse, whose semi-axes are

a and a cos. colat. ($= a$ sin. lat.) a being the radius of the star's path.

927. By Wood's Optics, we easily obtain the distance of the focus of refracted rays (q) from that of incidental rays (Q).

Hence if A be the point of incidence on the refractor, the required thickness will be

$$\frac{Aq}{2} = \frac{Qq - QA}{2}$$

which are known quantities.

928. Let a be the side of the isoceles right-angled $\triangle$, whose plane suppose to be inclined to the horizontal surface of the water at the angle $\angle\alpha$.

Then taking any element of the $\triangle$ parallel to the horizon ydx at the depth x sin. α, the pressure on that element acting $\perp$ to the $\triangle$ is

$ydx \times x \sin.\ \alpha = (a - x)\ xdx \sin.\ \alpha$, and the effort of this pressure to turn the $\triangle$ about its side at the surface of the water is

$$(a - x)\ xdx \sin.\ \alpha \times x.$$

Hence the whole force from $x = 0$ to $x = x$, tending to make the $\triangle$ thus revolve, is

$$\left(\frac{ax^3}{3} - \frac{x^4}{4}\right) \sin.\ \alpha$$

and for the whole $\triangle$

$$\left(\frac{a^4}{3} - \frac{a^4}{4}\right) \times \sin.\ \alpha = \frac{a^4 \sin.\ \alpha}{12}.$$

Again, since the whole pressure on the $\triangle$ is

$$\int (a - x)\ xdx \sin.\ \alpha = \left(\frac{ax^2}{2} - \frac{x^3}{3}\right) \sin.\ \alpha$$

$$= \frac{a^3 \sin.\ \alpha}{6}\ (x = a)$$

$\therefore$ if y be the distance of the centre of pressure from the horizontal side of the $\triangle$, the effort of the pressure all applied at this centre is

$$\frac{a^3 \sin.\ \alpha}{6} \times y.$$

But this must equal the effort $\frac{a^4 \sin.\ a}{12}$. Consequently

$$y = \frac{a}{2}.$$

The centre of pressure will also evidently be in the line joining the vertex and bisection of the horizontal base. Therefore the distance required is $\frac{1}{2} \cdot \frac{a}{2} = \frac{a}{4}$.

929. Let $\frac{\text{N}}{\text{D}}$ be the fraction in its lowest terms; then since every circulating decimal is equivalent to the vulgar fraction

$$\frac{\text{A}}{10^n - 1},$$

where n is the number of digits in the period, and A the integer represented by those digits, if $\frac{\text{N}}{\text{D}}$ be a circulating decimal, we have

$$\frac{\text{N}}{\text{D}} = \frac{\text{A}}{10^n - 1}.$$

that is

$$\text{A} = \frac{10^n - 1}{\text{D}} \times \text{N}.$$

But since N is not divisible by D, $10^n - 1$ is divisible by D. Again, $10^n - 1$ is prime to 10; for if not it is of the form $2 \times \text{P}$, $5 \times \text{Q}$ or $10 \times \text{R}$, and its last digit is either *even*, 5 or 0, or 0, whereas the last digit is actually 9.

Hence that $10^n - 1$ may be divisible by D, D must also be prime to 10; that is that $\frac{\text{N}}{\text{D}}$ may produce a circulating decimal, the fraction being in its lowest terms, the denominator must be prime to 10. Hence, conversely, &c.

930. The weight of the bodies are Pm and Qn in vacuo, and when immersed in water

$$\text{P}m - \text{P}, \ \text{Q}n - \text{Q}.$$

Hence, if x, $a - x$; x', $a - x'$ be the distances of the fulcrum from the extremities of the lever in the respective cases, we have

$$Pmx = Qn\,(a - x)$$
$$(Pm - P)\,x' = (Qn - Q)\,(a - x')$$

which give

$$x = \frac{Qna}{Pm + Qn}$$
$$x' = \frac{Q.(n-1)\,a}{P\,(m-1) + Q\,(n-1)}.$$

Hence the alteration required is

$$x - x' = \frac{PQa\,(m-n)}{(Pm+Qn)\,(P.\overline{m-1} + Q.\overline{n-1})}.$$

931. Let l be the length of the cylinder, and M its bulk. Then by the question the weight of the part immersed is

$$\frac{5}{6} \times \left(\frac{M}{2} - M\right) = -\frac{5}{12}M$$

the vertical tendency upwards of that part. The weight of the part not immersed is

$$\frac{M}{6} \times \frac{1}{2}.$$

Hence by the property of the lever, if x be the distance of a fulcrum from the centre of gravity of the part which is out of the water, placed so as to produce an equilibrium between these forces acting in the centres of gravity of the parts $\frac{l}{6}$ and $\frac{5l}{6}$, and in opposite vertical directions, we have

$$\frac{M}{12} \times x = \frac{5}{12}M.\left(\frac{l}{2} + x\right)$$
$$\therefore x = -\frac{5l}{8}.$$

Again the re-action of this fulcrum downwards is

$$-\left(\frac{M}{12} - \frac{5M}{12}\right) = \frac{M}{3},$$

whence the magnitude direction, &c. of the force required.

Paged wrong from here.

APPENDIX.

APPENDIX,

CONTAINING

A SELECTION OF

COLLEGE EXAMINATION PAPERS.

The Student is here presented with a number of such examination papers as have actually been set before the candidates for prizes and honours at the annual and other examinations. By the first fifteen of them Freshmen, at the termination of their noviciate, were examined; the next fifteen were given to such students as had completed their second year, that is, to Junior Sophs; the five following ones to Third-Year Men, or Senior Sophs; the two next to candidates for Scholarships; and the three last were given at Fellowship Examinations. The value of such papers to the aspirants after honours at Cambridge is too evident to be insisted on for a moment. They will also be exceedingly useful to such students as wish to be taught on the Cambridge Plan, without incurring the expenses of an University Education; and it is presumed, there are but few who would not be so instructed. Moreover, these papers, together with the Senate-House Problems, supply the reader with specimens of every kind of Mathematical Examination held at Cambridge. They completely develop the nature of the studies pursued there.

TRINITY COLLEGE.

EUCLID.

---

1. Describe a parallelogram equal to a given rectilinear figure, having an angle equal to a given rectilinear angle.

2. If a straight line be divided into two equal, and also into two unequal, parts; the squares of the two unequal parts are together double of the square of half the line, and of the square of the line between the points of section.

3. If two straight lines within a circle cut one another, the rectangle contained by the segments of one of them is equal to the rectangle contained by the segments of the other.

4. Inscribe an equilateral and equiangular hexagon in a given circle.

5. If any number of magnitudes be proportionals, as one of the antecedents is to its consequent, so shall all the antecedents taken together be to all the consequents.

6. If magnitudes taken jointly be proportionals, they shall also be proportionals when taken separately.

7. If four magnitudes of the same kind be proportionals, the greatest and least of them together are greater than the other two together.

8. Similar triangles are to each other in the duplicate ratio of their homologous sides.

9. In any parallelogram, the sum of the squares of the diagonals is equal to the sum of the squares of the sides.

10. If two circles touch each other externally, any line drawn through the point of contact will cut off similar parts of their circumferences.

11. Determine a point in the base of a triangle, from which if a line be drawn to the vertex, it shall be a mean proportional between the segments of the base.

12. If the sides of a trapezium be bisected, and the points of section be joined, the inscribed figure is a parallelogram, and equal half the area of the trapezium.

13. The sum of the opposite sides of a quadrilateral figure described about a circle are equal.

14. Describe a triangle that shall be equal in area to a given equilateral and equiangular pentagon.

15. Divide a triangle into two equal parts; 1st, by a line drawn parallel to one of its sides; and 2dly, by a line passing through a given point in that side.

16. Through two given points in the diameter of a circle, one of which is the centre, to describe another circle that shall touch it internally.

17. AB, DC, are two diameters of a circle perpendicular to each other; and the arc AEB is described with C as centre, and CA as radius. The lune ADBE equals the triangle ABC.

18. Given the base, the vertical angle, and the sum of the two remaining sides of a triangle; to construct it.

19. The rectangle contained by the diagonals of a quadrilateral figure inscribed in a circle is equal to both the rectangles contained by its opposite sides.

GEOMETRY.

---

1. How many sides has that polygon whose interior angles equal ten right angles?

2. In any right-angled triangle, the square whch is described upon the side subtending the right angle, is equal to the squares described upon the sides which contain the right angle.

3. If E be any point in a rectangle, $AE^2 + ED^2 = EB^2 + EC^2$.

4. Bisect the four sides of the trapezium ABCD in a, b, c, d: join ab, bc, cd, da, and shew that the trapezium $abcd$ equals $\frac{1}{2}$ ABCD.

5. Upon stretching two chains AC, BD, across a field ABCD, I find that BD and AC make equal angles with DC, and that AC makes the same angle with AD that BD does with BC, from these data prove that AB is parallel to CD.

6. To find the side of an equilateral and equiangular dodecagon inscribed in a circle.

7. In equal circles, angles, whether at the centres or circumferences, have the same ratio which the circumferences on which they stand have to one another.

8. Prove that the area of any trapezium, whose opposite sides are parallel, is found by multiplying the arithmetic mean between the two parallel sides into their perpendicular distance.

9. Given the sines, cosines, and tangents of two arcs to find the sines, cosines, and tangents of their sum and of their difference.

10. There is a triangular piece of ground, whose area equals 525 square yards, and two of whose sides measure 30 and 42 yards respectively; required the length of its remaining side.

11. At the top of a tower which is 30 yards high, I find a certain obelisk makes an angle of 20° 55′; also, the foot of the tower is horizontally distant 40 yards from the foot of the obelisk; required the height of the obelisk.

N. B. the side of the tower, as well as of the obelisk, is supposed to be perpendicular to the horizon.

12. What is the perpendicular height of a hill, whose angle of elevation taken at the bottom of it equals 46°, and 200 yards farther upon an horizontal plane equals 31°?

13. Wanting to know my distance from an inaccessible object on the other side of a river, I measure 100 yards from each of two stations, A and B, (which are 500 yards asunder) in a direct line from the object, and placing two marks C and D at the end of the 100 yards measured from A and B *respectively*, I find that from A to D there are 550 yards, and from B to C 560. Required the distance of the object from A and B.

14. Given the perimeter of a triangle and the three angles to construct it.

PROPOSITIONS IN PLANE GEOMETRY.

1816.

1. If the exterior angle of a triangle be bisected by a straight line which also cuts the base produced, the segments between the bisecting line and the extremities of the base, have the same ratio which the other sides of the triangle have to one another. Shew that the converse is also true.

2. Equiangular parallelograms have to one another the ratio which is compounded of the ratios of their sides.

3. The rectangle contained by the diagonals of any quadrilateral figure inscribed in a circle is equal to the sum of the rectangles contained by its opposite sides.

4. If the exterior angle of a triangle be bisected, and also one of the interior and opposite, the angle contained by the bisecting lines is equal to half the other interior and opposite angle of the triangle.

5. (1). If upon the sides BA, CA, of any triangle, any two parallelograms be drawn, and their sides produced to meet in K and KA be joined, then the parallelogram constructed upon the base BC, with one side equal and parallel to KA, will be equal to the sum of the other two.

(2). When the triangle becomes right-angled at A and the parallelograms on the sides become squares, shew that the parallelogram on the base is also a square. (This is = Prop. 47. Eucl. B. I.)

6. The square described upon the side of a regular pentagon inscribed in any circle is equal to the sum of the squares described upon the sides of a regular hexagon and decagon inscribed in the same circle.

7. If a straight line be drawn from C, the point of bisection of a given arc ACB, cutting the chord AB or the chord produced in any point E and the circumference of the circle in D, prove that in each case the rectangle contained by CD and CE is equal to the square described on CB.

8. The greatest of all straight lines passing through either of the points of intersection of two given circles which cut each other, and terminated both ways by the two circumferences, is that which is parallel to the line joining the centres of the two circles.

9. If the sides of a regular polygon of n sides be produced to meet, the sum of the angles made by the lines thus produced at the points of intersection is equal to $2n - 8$ right angles.

10. Represent the arithmetic, geometric, and harmonic means, between two given lines geometrically.

11. The centre of the circle circumscribed about any triangle, the point of intersection of the perpendiculars let fall from the angular points of the same triangle to the opposite sides, and the point of intersection of the lines joining the angular points with the middle of the opposite sides, all lie in the same right line.

12. If four circles touch each either internally or externally, three sides of any quadrilateral figure, the centres of these circles will lie in the circumference of the same circle.

13. Describe a circle passing through a given point which shall touch both a given circle and a given straight line.

14. If from the centre and angular points of a regular hexagon perpendiculars be drawn to any given right line, six times the perpendicular from the centre is equal to the sum of the perpendiculars from the angular points.

EUCLID, Book XI.

1817.

1. Define the five regular solid figures.

2. If a straight line stand at right angles to each of two straight lines in the point of their intersection, it shall also be at right angles to the plane which passes through them.

3. If two straight lines be parallel, and one of them be at right angles to a plane; the other also shall be at right angles to the same plane.

4. Draw a straight line perpendicular to a given plane from a given point without it.

5. Planes to which the same straight line is perpendicular are parallel to each other.

6. Every solid angle is contained by plane angles which together are less than four right angles.

7. The face of a regular octahedron and of a dodecahedron being given, it is required to construct the solids.

ST. JOHN'S COLLEGE.

ALGEBRA.

1. $$\frac{3x+7}{14} - \frac{2x-7}{21} + 2\tfrac{3}{4} = \frac{x-4}{4}$$

2. $$\left.\begin{aligned} 3x + 6y + 1 &= \frac{6x^2+180-24y^2}{2x-4y+3} \\ 8x - \frac{151-16x}{4y-1} &= \frac{9xy-110}{3y-4} \end{aligned}\right\}$$

3. $$\frac{x+2}{x-1} - \frac{4-x}{2x} = 2\tfrac{1}{3}$$

4. $$\left.\begin{aligned} 5y + \frac{\sqrt{x^2-15y-14}}{5} &= \frac{x^2}{3} - 36 \\ \frac{x^2}{8y} + \frac{2x}{3} &= \sqrt{\frac{x^3}{3y} + \frac{x^2}{4}} - \frac{y}{2} \end{aligned}\right\}$$

5. A brewer, from a certain quantity of ingredients which cost £20. brews 500 gallons of ale, (on which there is a duty of 6d. a gallon,) and sells it at 2s. a gallon. Afterwards from the same quantity of ingredients, he brews a certain number of gallons of strong beer, (on which he pays the ale duty,) and the remainder small beer, making together the same number of gallons as before,—when by mixing them together, and selling the mixture as ale, he finds his gains increased in the proportion of 10 : 7. Determine the number of gallons of strong beer, supposing the duty on small beer ¼ of that on ale.

6. The number of deaths in a besieged garrison amounted to 6 daily, and allowing for this diminution, their stock of provisions was sufficient to last for 8 days. But on the evening of the sixth

day 100 men were killed in a sally, and afterwards the mortality increased to 10 daily. Supposing the stock of provisions unconsumed at the end of the 6th day sufficient to support 6 men for 61 days. It is required to find how long it would support the garrison, and the number of men alive when the provisions were exhausted.

7. A man buys a guinea at the market price of standard gold, but an act of parliament passing which makes it illegal to sell the guinea in the same way that he bought it, he privately clips off one twenty-fifth part. He may now legally sell it as a light guinea, and he finds that in consequence of the rise of *pure* gold in the ratio of 239 : 249 he just gains the clippings by his purchase.

It is required to find the ratio of pure gold and alloy in the guinea, and also the relative value of equal quantities of pure gold and alloy, it being known that the sum of the squares of the numbers which express the two ratios, exceeds eleven times their sum by the number $233\frac{2}{121}$.

1815.

1. $$\frac{2x+1}{29} - \frac{402-3x}{12} = 9 - \frac{471-6x}{2}.$$

2. $$\left\{\begin{array}{l} \dfrac{3x-5y}{3} - \dfrac{2x-8y}{12} = \dfrac{y}{2} + \dfrac{1}{3} + \dfrac{1}{4} \\[2ex] \dfrac{x}{7} + \dfrac{y}{4} + 1\frac{1}{3} : 4x - \dfrac{y}{8} - 24 :: 3\frac{1}{3} : \end{array}\right\}$$

3. $$\frac{x+8}{2} + \frac{16-2x}{2x-5} = 5\frac{1}{3}.$$

4. $$\left\{\begin{array}{l} \dfrac{x+y+\sqrt{x^2-y^2}}{x+y-\sqrt{x^2-y^2}} = \dfrac{9}{8y}\cdot(x+y) \\[2ex] (x^2+y)^2+x-y = 2x.(x^2+y)+506 \end{array}\right\}$$

5. At the review of an army, the troops were drawn up in a solid mass, 40 deep; when there were just one-fourth as many men in front as there were spectators. Had the depth however been increased by 5, and the spectators drawn up in the mass with the army, the number of men in front would have been 100 fewer than before. Determine the number of men of which the army consisted.

6. A number of persons purchased a field for £345. The youngest contributed a certain sum, the next £5. more, the third £5. more than the second, and so on to the oldest. For the greater accommodation of the seniors, the field was divided into two parts, the younger half taking a portion proportional to the sum they had subscribed; and in order that each might have an equal share in

this portion, they agreed to equalize their contributions, and each to pay £22. Required the number of persons, and the sums paid by each.

7. A and B travelled on the same road, and at the same rate, from Huntingdon to London. At the 50th mile-stone from London, A overtook a drove of geese, which were proceeding at the rate of three miles in two hours; and two hours afterwards met a stage waggon, which was moving at the rate of nine miles in four hours. B overtook the same drove of geese at the 45th mile-stone, and met the same stage waggon exactly forty minutes before he came to the 31st mile-stone. Where was B when A reached London?

1816.

1. $$\frac{4x-34}{17} - \frac{258-5x}{3} = \frac{69-x}{2}.$$

2. $$\begin{cases} \dfrac{4x-2y+3}{3} - \dfrac{18-x+5y}{7} = \dfrac{x}{4} - \dfrac{y}{5} - \dfrac{1}{7} - 7\dfrac{7}{10} \\ 2x-y+15 : y-2x+15 :: \dfrac{x}{3} - \dfrac{y}{4} + \dfrac{3}{4} : \dfrac{y}{4} - \dfrac{x}{3} + \dfrac{1}{12}. \end{cases}$$

3. $$\frac{2x}{x-4} + \frac{2x-5}{x-3} = 8\tfrac{1}{3}.$$

4. $$\begin{cases} \dfrac{y}{x} - \dfrac{9\sqrt{x}}{y} - \dfrac{81}{xy} = (2y+9)\cdot\dfrac{\sqrt{x}}{y} \\ \dfrac{\sqrt{y}}{x} + 3\sqrt{\dfrac{x}{y}} = \dfrac{9}{x\sqrt{y}} + \sqrt{x}. \end{cases}$$

1. A packet sailing from Dover with a fair wind, arrives at Calais in two hours; and on its return the wind being contrary, it proceeds six miles an hour slower than it went. Now when it is half way over, the wind changing, it sails two miles an hour faster, and reaches Dover sooner than it would have done had the wind not changed, in the proportion of 6 : 7. Required the rates of sailing, and the distance between Dover and Calais.

2. From the middle of a town two streets branched off, and crossed a river that ran in a straight course, by two bridges A and B. From their junction, a sewer equally inclined to both streets led to a point in the river, at the distance of 6 chains from the bridge A, and a distance from B, less by 11 chains than the length of the sewer: the expense of making it amounting to as many pounds per chain as there were chains in the street leading to A. The sewer however being insufficient to carry off the water, an additional drain was made from a point in this street, distant 4

chains from the bridge A, which entered the river at the same point with the sewer, and was equally inclined to the river and sewer. Now it was found that a drain down the middle of each street, at the rate of £9 per chain, would have cost only £54 more than the expense of the sewer. Required the lengths of the streets, and the sewer; and the distance of its mouth from the bridge B.

3. On the institution of Saving Banks, an industrious labourer, with his wife and children, saved each of them a certain number of pence in a decreasing arithmetic progression. The sum saved monthly, was less by 3*s*. 3*d*. than would have purchased one-sixth of as many bushels of wheat, as the séventh child såved pence; the price of wheat being such, that the sum saved by the eldest and the fifth child, augmented by 10*s*. would buy two bushels. But wheat rising 2*s*. per bushel, and work being scarce, the family find the sum saved would not buy as much wheat as their former savings, by two bushels; when it appears that at this rate the sum annually saved would be less by five guineas than by the former.—Now the two youngest dying, it is found that if the remaining members of thé family saved each one shilling less than the oldest child had done before the rise of wheat, their monthly account with the bank would not be affected by the deaths of the two youngest; but if they saved only 2*d*. less than the oldest had done, their monthly account would be 2*s*. 1*d*. more than it was at the first institution. Of how many did the family consist? What were the sums saved by each? and the price of wheat?

JUNE, 1817.

1. $$\left\{\frac{5x-1}{2} - \frac{7x-2}{10} = 6\,\frac{3}{5} - \frac{x}{2}\right\}$$

2. $$\left\{\begin{array}{l} \sqrt{y} - \sqrt{a-x} = \sqrt{y-x} \\ \sqrt{y-x} + \sqrt{a-x} : \sqrt{a-x} :: 5 : 2 \end{array}\right\}$$

3. $$\left\{5 \cdot \frac{3x-1}{1+5\sqrt{x}} + \frac{2}{\sqrt{x}} = 3\sqrt{x}\right\}$$

4. $$\left\{\begin{array}{l} \dfrac{y}{2x} + \dfrac{2}{2} \cdot \dfrac{y - \sqrt{x-1}}{y^2 - 2\sqrt{x^2-1}} = \dfrac{\sqrt{x+1}}{x} \\ \dfrac{1}{4} y^4 = y^2 x - 1 \end{array}\right\}$$

5. A farmer laid up a stock of corn, expecting to sell it in six months at three shillings per bushel more than he gave for it. But the price of corn falling one shilling per bushel, he found that by selling it, he should lose the price of five bushels. He there-

fore kept it till the end of the year, and selling it at two shillings per bushel under prime cost, found his loss to be ten shillings less than his expected gain. Required the quantity of corn laid up, and price per bushel, allowing five per cent. simple interest.

6. A ship and crew of 175 men set sail with a store of water sufficient to last to the end of the voyage. But in thirty days the scurvy made its appearance, and carried off three men every day, and at the same time a storm arose, which protracted the voyage three weeks. They were however just enabled to arrive in port, without any diminution in each man's daily allowance of water. Required the time of the passage, and the number of men alive when the vessel reached harbour.

7. The hold of a vessel partly full of water (which is uniformly increased by a leak) is furnished with two pumps worked by A and B, of whom A takes three strokes to two of B's, but four of B's throw out as much water as five of A's. Now B works for the time in which A alone would have emptied the hold, A then pumps out the remainder, and the hold is cleared in thirteen hours twenty minutes. Had they worked together, the hold would have been emptied in three hours forty-five minutes, and A would have pumped out 100 gallons more than he did. Required the quantity of water in the hold at first, and the horary influx at the leak.

JUNE, 1818.

1. $$\frac{4x-21}{7}+7\frac{5}{6}+\frac{7x-28}{3}=x+3\frac{3}{4}-\frac{9-7x}{8}+\frac{1}{12}.$$

2. $$\left\{\begin{array}{l} \dfrac{3x-2y}{3}+1+\dfrac{11y-10}{8}=\dfrac{4x-3y+5}{7}+\dfrac{45-x}{5} \\ 45-\dfrac{4x-2}{3}=\dfrac{55x+71y+1}{18} \end{array}\right\}$$

3. $$x^{\frac{7}{3}}+\frac{41\sqrt[3]{x}}{x}=\frac{97}{\sqrt[3]{x^2}}+x^{\frac{5}{3}}.$$

4. $$\left\{\begin{array}{l} \dfrac{x^2y^2}{2}+4-40y^2=140-y^2\sqrt{x^2-\dfrac{272}{y^2}} \\ x^2-\dfrac{2}{y}\left(\dfrac{3}{y}+15x\right)=\dfrac{80}{y^2}+\dfrac{5x}{y}. \end{array}\right\}$$

5. On January 1, 1799, a certain beggar received from A as many groats as A was years old, who repeated a similar donation every January for the seven following years, during the last of of which A died, his alms to the poor man having in all amounted to £7 18*s*. 8*d*. Required in what year he was born, and his age at his death.

6. A entered into a canal speculation with fourteen others, and the profits of this concern amounted in all to £595, more than five times the price of an original share. Seven of his former partners in this affair joined in a scheme for navigating the said canals with steam-boats, each venturing a sum of money less than his former gains by £173. But the steam-boats unexpectedly blowing up, A found he had lost £419 by them, for the company not only never recovered the money advanced, but had lost all they had gained by digging the canals and £368 besides. What were the prices of shares in the two concerns originally?

7. A, B, and C were three architects. A and B built four warehouses with flat roofs, each a large one, and each a small one, the linear width of the two large ones being the same, and also that of the two small ones. A built his as long and as high as they were wide, but B made the length and height of his large one equal to the width of his small one, and the length and height of his small one equal to the width of his large one, in such a manner that the difference between the solid content of those built by A and those built by B was 73728 cubic feet. C also built a warehouse upon a square plat of ground which was equal to the difference between the ground-plats occupied by those which A built, and found that it would have just stood on 2688 square feet, if he had added eight times as many square feet to the ground-plat as there were linear feet in its width. How many feet wide were the several buildings erected by A, B, and C?

TRINITY COLLEGE.

QUESTIONS IN ALGEBRA.

1. Investigate the rule for the expansion of a binomial; and shew that when the power is expressed by the integer (m), the number of terms is $(m + 1)$.

2. Find the least common multiple of $\left(\frac{a}{m}\right)$, $\left(\frac{b}{n}\right)$, $\left(\frac{c}{p}\right)$, each being a fraction in its lowest terms: and also the greatest common measure of the two quantities (a) and (mb), when (m) is an integer.

3. Extract the cube root of $a^3 - \frac{1}{216} + \frac{a}{12} - \frac{a^2}{2}$.

4. Solve the following equations:

1. $\dfrac{x-3}{2} - \dfrac{2x+3}{6} = -1.$

2. $\dfrac{2x+5}{2-x} - \dfrac{2}{7} = -1.$

3. $x + \dfrac{3x-5}{2} = 12 - \dfrac{2.\overline{x-2}}{3}.$

4. $\sqrt{5}\,.\,\sqrt{x+2} = 2 + \sqrt{5x}.$

5. In the following equations find the values of x and y:

1. $\dfrac{x-2}{y-3} + \dfrac{y-3}{x-2} = 2.$

   $x + y = 7.$

2. $x^2 + xz = 12.$

   $z^2 + xz = 24.$

3. $x^{\frac{2}{3}} \times y^{\frac{1}{2}} = 2y^2.$

   $8x^{\frac{1}{3}} - y^{\frac{1}{2}} = 14.$

4. $x + y + xy = 11.$

   $\overline{x^2+y^2} \times xy = 78.$

5. $x^2 + y^2 + xy = 19.$

   $x^4 + x^2y^2 + y^4 = 133.$

6. $x^2 + xz = 10.$

   $y^2 + yz = 21.$

   $z^2 + zx = 24.$

6. Solve the following equations:

1. $3x - \dfrac{2}{x} = 5.$

2. $\dfrac{3x^{\frac{4}{3}}}{2} - \dfrac{5x^{\frac{2}{3}}}{4} = -296.$

3. $x^2\,.\,\overline{x+4} + 2x\,.\,\overline{x+4} = 2 - \overline{x+4}.$

7. A has £400 due to him from B, and £520 at the end of twelve years; at what time ought the two sums to be paid together, so that neither person may sustain loss; simple interest being allowed at the rate of £5 per cent.?

8. Find the number of years for which an annuity of £40 may be purchased, by the present payment of £245 simple interest being calculated at £4. per cent.

9. A travels from C, one mile the first day, five the second, nine the third, &c.; B sets out from the same place 4½ days after A, and travels twice as many miles as A does in the last day, except four. In how many days will B overtake A?

10. A's income of £400 is paid in equal parts at the end of 3, 6, 9, 12 months; and his expenses, which are as the numbers 1, 2, 3, 4, are paid at the same time; and the remaining sums being lent at an interest of £5 per cent. amount at the end of the year to £113 15s. Find the amount of his annual expenses.

11. A C . . . . B . . . . . . D

A, C.... B, are $(\overline{n+1})$ stones placed a yard from each other, and D is another assumed station:—Two persons, M and N, set out from A; M to carry the stones separately to A, and N to carry them to D. Find the distance B D, so that N may travel twice as many yards as M.

12. Find the number of permutations of the letters in the word COLLEGE.

13. If the digits of a circulating decimal be a, b, c, its equivalent fraction is, $\dfrac{10^2 a + 10b + c}{999}$.

14. 210 trees are to be planted in rows on a triangular piece of ground containing 9600 square yards; a tree being planted at each angle, and the extreme trees of every row parallel to a side being situated on the two remaining sides, the number of trees in succeeding rows being 2, 3, 4, &c.; find the distance between contiguous trees in the directions of the three sides, supposing them to be as the numbers 3, 4, 5.

1811.

1. Express in general any decimal of n places; a, b, c, &c. being the digits.

2. Prove the rule for finding the greatest common measure.

3. Prove that if a and b be each prime to c, ab is prime to c.

4. Solve the following equations:

1. .... $\dfrac{y}{3} + 18 = 5y.$

2. .... $x - \dfrac{12}{x} = 4.$

3. .... $\dfrac{x^{\frac{1}{3}}}{2} - 3 = \dfrac{7}{2x^{\frac{1}{3}}}$

4. .... $\sqrt{2+x} = \sqrt{2x-1} - \dfrac{1}{\sqrt{2+x}}$

5. .... $\begin{cases} x^4 - x^2y - 6y^2 = 18. \\ x^2 - 3y = 2 \end{cases}$

6. .... $\begin{cases} x + 6x^{\frac{1}{2}} = y^{\frac{2}{3}} \\ y^{\frac{1}{3}} - x^{\frac{1}{2}} = 2 \end{cases}$

[N. B. The result of every equation is not a whole number.]

5. A travels at the rate of seven miles in five hours, B sets off from the same place eight hours after, and travels the same road at the rate of five miles in three hours; how long and how far will A travel before he be overtaken by B?

6. A person lays out a certain sum of money in goods, which he sold again for £24, and gained as much per cent. as the goods cost him. What was the sum laid out?

7. Find the sum of seven terms of the series $\frac{2}{5} - \frac{3}{10} + \frac{9}{40}$ &c. and of $\frac{4}{10} + \frac{5}{10^2} + \frac{2}{10^3} + \frac{5}{10^4} + \frac{2}{10^5}$ &c. ad infinitum. Insert two arithmetic means between 2 and 7; and prove an arithmetic mean greater than a geometric.

8. How many permutations can there be in the letters of the word ALGEBRA?

9. Prove that the series $n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3}$. &c. where n is a whole number, will always be a whole number, whatever be the number of terms.

10. Write down three terms of $\overline{a^2 - x}|^{\frac{1}{7}}$, and the cube of $\overline{x^2+y+z}$.

11. Divide $\overline{a^2-x^2}|^{\frac{1}{3}}$ by $\overline{a+x}|^{\frac{1}{3}}$, and bring the fraction to its lowest terms.

12. Investigate the rule for extracting the square root of a binomial, one of whose factors is a quadratic surd, and the other rational; and apply it to find the root of $7 - \sqrt{13}$.

13. Prove the rule for discovering when the multiplication of two numbers is correct, by dividing the multiplicand, multiplier, and product, by 9.

14. Express 359, when the local value is 6; and prove the rule.

15. Find the greatest and least positive integral values of x and y in the equation $3x + 5y = 68$.

16. What is the present worth of £130, due fourteen months hence, allowing 4 per cent. simple interest?

17. A person purchased an annuity of £100 per annum for ten years, to commence at the end of seven, and payable half yearly. After three of the seven years have expired, what is its value allowing 5 per cent. compound interest? Express the result logarithmically.

1816.

1. Prove that $\frac{a^n - b^n}{a-b}$ = an integer, and shew that $\frac{a^n + b^n}{a+b}$ is an integer or a fraction, according as (n) is odd or even.

2. Expand $\frac{b}{\overline{a - x^2}|^{-n}}$; write down the rth term of $\overline{a - x}|^{-n}$; and extract the cube root of $x^6 - \frac{8}{x^3} + 12 - 6x^3$.

3. Solve the following equations:

(α) $mx + x = 4b + 2x$

(β) $\sqrt{4a + x} = 2\sqrt{b+x} - \sqrt{x}$.

(γ) $\sqrt{x + \sqrt{x}} - \sqrt{x - \sqrt{x}} = \frac{3}{2}\sqrt{\frac{x}{x + \sqrt{x}}}$.

(δ) $\frac{10 - x}{x} + \frac{x}{10 - x} = \frac{17}{4}$.

(ϵ) $\left.\begin{array}{l} 3x - \frac{3x}{y} = y^2 - y \\ y^2 + x = 4 \end{array}\right\}$

(ζ) $\frac{1}{x} + \frac{1}{a} = \sqrt{\frac{1}{a^2} + \sqrt{\frac{4}{a^2x^2} + \frac{9}{x^4}}}$

(η) $a^{(x^2)} = \frac{b^x}{c^3}$.

4. Insert (m) harmonic means between (a) and (b).

5. If, between all the terms of an arithmetical progression, the same number of arithmetic means be inserted, the new series will still form an arithmetical progression.

6. Sum the following series:

$2 + 5 + 8 +$ &c. to (8) terms.

7, 5, 3, &c. to (5) terms

$\frac{1}{1 . 2} - \frac{1}{2 . 3} + \frac{1}{2 . 3^2} - \frac{1}{2 . 3^3} +$ &c. ad infin.

$\frac{1}{4} - \frac{1}{8} + \frac{1}{16} -$ &c. .. to (4) terms.

$\frac{a}{r} + \frac{a+b}{r^2} + \frac{a + 2b}{r^3} +$ &c. ad infin.

$1 + 5 + 13 + 29 + 61 +$ &c. to (n) terms.

7. What is the nnmber of permutations of (n) things, of which there are (p) of one sort, (q) of another, and (r) of a third. And shew that the total number of combinations which (n) things admit of, $= 2^n - 1$.

8. A, who travels only every other day, sets off from a certain place nine days after B, in order to overtake him, but travels four times as fast as B does. When will they come together?

9. Divide the number 35 into two such parts, that the square of the less divided by the difference of the two parts $= 45$.

10. A person owes £150, to be paid at the end of nine months; and £60, at the end of six months. Required the equated time of payment, and investigate the rule.

11. $a : b$ is a ratio where (a) is prime to (b). They are the least in that proportion—and shew that $\overline{a + x}|^m : a^m$ very nearly in the proportion of $a + mx : a$, when (x) is small compared with (a).

12. Required the square root of $5 - 4\sqrt{3}$, and of $9\sqrt{-4} - 3$.

13. Required the discount upon £100, due three months hence, at $4\frac{1}{2}$ per cent. per annum.

14. Find the present worth of an annuity (A) to continue for (n) years at simple interest.

15. What is the present worth of an annuity of £60, to commence in two years, and to continue for ever at $3\frac{1}{2}$ per cent. per annum.

16. Given the log. of $8.1213 = .9096256$. It is required to state the log. of 812.13; and of .81213; and of .081213.

17. Resolve $\sqrt{17}$ into a continued fraction.

18. Required a number such, that, divided by 5, 4, 3, respectively, it may leave remainders 2, 3, 4, respectively.

19. The difference between any number, and that number inverted, is divisible by the *local value* minus *one*.

1817.

1. Find the sum of the coefficients of a binomial raised to the nth power; write down the pth term of the binomial, and deduce from thence the 4th term of $(a^{\frac{2}{3}} - b^{\frac{3}{2}})^{\frac{4}{5}}$.

2. Prove that when four quantities are proportional, the product of the extremes $=$ the product of the means; and convert $\frac{a}{b} = \frac{c}{d}$ into a proportion.

3. Solve the following equations:

$$1. \quad \frac{2\sqrt[3]{7x - 6}}{3} + \frac{3}{4} = \frac{25}{12}$$

$$2. \quad \left.\begin{array}{l} \dfrac{x+y}{3} + \dfrac{x-y}{4} = 59 \\ x : 3y :: 11 : 5. \end{array}\right\}$$

3. $\frac{x^2}{2} - \frac{x}{3} + 7\frac{3}{8} = 8.$

4. $\left.\begin{array}{r} x + y + xy = 19 \\ x^2y + xy^2 = 84 \end{array}\right\}$

5. $x^2 = 21 + \sqrt{x^2 - 9}.$

6. $\left.\begin{array}{r} x^2 - x + y^2 - y = 18 \\ xy + x + y = 19. \end{array}\right\}$

7. $x^2.\overline{x+4} + 2x.\overline{x+4} = 2 - \overline{x+4}$

8. $\overline{x+2}\,^2 + 2\sqrt{x}.\overline{x+2} - 3\sqrt{x} = 46 + 2x.$

4. Find a sum consisting of P pounds, Q shillings, the double of which shall be Q pounds, P shillings.

5. Find two numbers in the proportion of 9 to 7, so that the square of their sum shall equal the cube of their difference.

6. A person being asked the hour of the day, answered thus: if $\frac{3}{8}$ of the number of hours remaining till midnight be multiplied by 4, the product will as much exceed 12 hours, as $\frac{1}{4}$ of the present hour from noon is below 4. What was the hour after noon?

7. What number is that, which being divided into any two parts, x and y, $x^2 + y = y^2 + x$?

8. The sum of 7 numbers in arithmetical progression $= 28$, and the sum of their cubes $= 784$. Determine the progression.

9. Find 4 numbers in arithmetical progression, which being increased by 2, 4, 8, and 15 respectively, the sums shall be in geometrical progression.

10. Which is the greater, a geometrical or an arithmetical mean; and by what quantity does the greater mean exceed the less?

11. Prove that the reciprocals of quantities in harmonical progression are in arithmetical progression, and find a fourth harmonical proportional to 6, 8, and 12.

12. Prove that of n things, r of them being always taken together, the number of permutations $= n.(n-1).(n-2)\ldots(n-r+1)$; and that the number of combinations $=$

$$\frac{n.(n-1).(n-2)\ldots.(n-r+1)}{1 \;.\; 2 \;.\; 3 \;\ldots.\; r};$$

and find the number of permutations which can be made of the letters in the word "examination," two of them being always taken together.

13. Sum the following series:

(1.) $8 + 15 + 22 +$ &c. to 12 terms.

(2.) $116 + 108 + 100 +$ &c. to 10 terms.

(3.) $3 + \frac{1}{2} + \frac{1}{12} +$ &c. ad inf.

(4.) $1 + 2 - \frac{1}{4} + \frac{1}{32} -$ &c. ad. inf.

(5.) $3 + 7 + 15 +$ &c. to n terms.

14. If ab be prime to c, a, and b are each of them prime to c.

15. If the sum of the odd digits of any number N, whose local value is r, be equal to the sum of the even digits, $\frac{N}{r+1}$ is an integer. Required a proof.

16. Find in what time £600 will double itself at simple interest; and also find the difference in the interest of £1200 put out at 5 per cent. for ten years, at simple and at compound interest.

17. The discount of £500 due four years hence : £500 :: 1 : 6; required the rate of interest.

18. £600 is due eight months hence, £500 nine months hence, and £1200 at present. Find the equated time of payment by the common method, and shew wherein the error of the process consists.

19. A person borrowed £4000. In what time will he be out of debt, supposing him to vest £200 at the end of every year, till the whole be paid off; the rate of interest being 5 per cent. *per annum*, simple interest?

20. Extract the square root of $2\sqrt{-3} - 2$, $7 + 4\sqrt{3}$, and $2\sqrt{-1}$.

21. In how many ways can £40 be paid with half-guineas and pistoles; the value of the pistole being 17*s.*?

ALGEBRA. PART II.

1820.

---

1. If (a) be a root of the equation $x^n + px^{n-1} + qx^{n-2} + \ldots\ldots + tx + u = 0$; prove that $(x-a)$ is a divisor of the equation without assuming its resolution into factors.

2. Transform the equation $x^3 - 4x^2 + 5x - 2 = 0$ into one which shall want the third term.

3. If (P) be the greatest negative coefficient of any equation (n) the number of terms preceding the first term whose coefficient is negative, prove that $1 + \sqrt[n]{P}$ is a *superior* limit of the roots of the proposed equation.

4. The roots of the equation $nx^{n-1} + (n-1)\,px^{n-2} + \ldots\ldots = 0$ are limits between the roots of the equation $x^n + px^{n-1} + \ldots\ldots = 0$ when the roots of the latter equation are possible.

5. Solve the following equations:

(1.) $x^3-6x^2+6x+8=0$, one root being $1+\sqrt{3}$.
(2.) $x^3-3x^2-46x-72=0$, by the method of divisors.
(3.) $x^3-18x^2+54x-72=0$, the roots being in harmonical progression.
(4.) $x^4-10x^3+35x^2-50x+24=0$, ——— arithmetical progression.
(5.) $x^4-5x^3-5x^2+45x-36=0$, two roots being of the form $+a$ and $-a$.

6. The roots of every equation of the form $x^{2n}+px^{2n-1}+\ldots\ldots+px+1=0$ may be found by the solution of an equation of half the number of dimensions.

7. Solve the equation $x^6+1=0$
(1.) By means of the preceding question.
(2.) By trigonometrical formulæ.

8. The equation $x^6-2x^5+6x^4-8x^3+12x^2-8x+8=0$, has equal roots. Find them.

9. (1.) Find the roots of the equation $x^3-6x^2+3x-18=0$ by Cardan's method.
(2.) Shew that this method is applicable only when two roots of the proposed cubic are impossible.

10. Give Euler's solution of the biquadratic $x^4+px^2+qx+r=0$.

11. The roots of the equation $x^n+px^{n-1}+qx^{n-2}\ldots\ldots+tx+u=0$ being a, b, c, &c. if

$S_1=a+b+c+\ldots\ldots$
$S_2=a^2+b^2+c^2+\ldots\ldots$
$\ldots=\ldots\ldots\ldots$
$S_m=a^m+b^m+c^m+\ldots\ldots$ {(m) being any integer not $<n$;} prove that
$S_m+pS_{m-1}+qS_{m-2}+\ldots\ldots+uS_{m-n}=0$.

12. The roots of the equation $x^3+px^2+qx+r=0$, are a, b, c, transform it into one whose roots shall be a^3, b^3, c^3.

13. (1.) If (a_1) be an approximate root of the equation
$x^3+px^2+qx=r$,
so that $a_1^3+pa_1^2+qa_1=r_1$;
prove that $x=a_1+\dfrac{a_1.(r-r_1)}{r \text{ or } r_1+2a_1^3+pa_1^2}$ *very nearly*;
r or r_1 being used according as (a_1) is $>$ or <1.
(2.) Approximate by this formula to the value of (x) in the equation $x^3-2x=5$.

TRINITY COLLEGE.

1817.

PLANE TRIGONOMETRY.

1. Trace the signs of the Sine, Cosine, Tangent, and Secant, through the circle.

2. Transform the formula,
$(\cos. A)^m + a. (\cos. A)^n . (\sin. B)^p + b . (\cos. A)^r . (\sin. C)^r + \&c.$
where the radius $= 1$, to an equivalent formula, where the radius $= r$; and prove the rule.

3. Given the sines and cosines of two arcs A and B; it is required to find sin. (A+B) and sin. (A−B).

4. Prove that, $\sin. (A+B) . \sin. (A-B) = (\sin. A)^2 - (\sin. B)^2$
and $\cos. (A+B) . \cos. (A-B) = (\cos. A)^2 - (\sin. B)^2$

5. If $x + \frac{1}{x} = 2 \cos. A$, prove that $x^3 + \frac{1}{x^3} = 2 \cos. 3A$.

6. $\text{Tan.} (45°+A) = \text{Tan.} (45°-A) + 2 \text{Tan.} 2A$. Prove this, and explain what is meant by a Formula of Verification.

7. $\text{Tan.} A + \text{cotan.} A = 2 \text{cosec.} 2A$.
$\text{Tan.} A - \text{cotan.} A = 2 \text{cotan.} 2A$.

8. In any triangle, the sum of any two sides : their difference :: the tangent of half the sum of the angles subtended by those sides : the tangent of half their difference.

9. Given the sine of 1′, shew how the sines of all arcs from 1′ to 90° may be found, rad. = 1.

10. Given two sides of a triangle, and an angle opposite to one of them, solve the triangle; and shew the ambiguity in this case.

11. Given two sides and the included angle; solve the triangle.

12. Explain the method of finding the distance between two visible but inaccessible objects on an horizontal plane; and shew how the requisite triangles are to be solved.

13. Two sides of a triangle and the angle included being given, find the area of the triangle.

14. The perimeter and the three angles of a triangle being given, find each of the sides.

TRINITY COLLEGE.

1816.

SPHERICAL TRIGONOMETRY.

1. Every plane section of a sphere is a circle.

2. The sum of the three angles of a spherical triangle is greater than two, and less than six, right angles.

3. The angles of a spherical triangle are A, B, C; the sides respectively opposite to them, (a), (b), (c); the rad. of the sphere $= 1$. Prove the following theorems:

I. $$\text{Cos. } C = \frac{\cos. c - \cos. a \times \cos. b}{\sin. a \times \sin. b}$$

II. $$\frac{\sin. A}{\sin. a} = \frac{\sin. B}{\sin. b} = \frac{\sin. C}{\sin. c}.$$

III. $$\text{Tang. } \tfrac{1}{2}(A+B) = \text{Cotan. } \tfrac{1}{2} C \,.\, \frac{\cos. \frac{1}{2}(a-b)}{\cos. \frac{1}{2}(a+b)}$$

$$\text{Tang. } \tfrac{1}{2}(A-B) = \text{Cotan. } \tfrac{1}{2} C \,.\, \frac{\sin. \frac{1}{2}(a-b)}{\sin. \frac{1}{2}(a+b)}$$

IV. If $(a+b+c) = S$

$$\sin.^2 \tfrac{1}{2} C = \frac{\sin. \left(\frac{S}{2} - a\right) \times \sin. \left(\frac{S}{2} - b\right)}{\sin. a \times \sin. b}$$

$$\cos.^2 \tfrac{1}{2} C = \frac{\sin. \left(\frac{S}{2}\right) \times \sin. \left(\frac{S}{2} - c\right)}{\sin. a \times \sin. b}.$$

4. What are the general theorems deduced by the application of the formulæ I. and III. to the polar triangle?

5. Prove Napier's rules for that case in which the complement of an angle is the middle part.

6. Given the obliquity of the ecliptic, the right-ascension and declination of a star; find the angle of position in terms of those quantities.

7. In a spherical triangle, two sides and the included angle are given:—Required the third side in a form suited to logarithmic computations?

8. Find the area of a spherical triangle.

SPHERICS.

1820.

1. If all the sides of one spherical triangle be respectively equal to all the sides of another, then all the angles of the one are equal respectively to all the angles of the other.

2. Every section of a sphere is a circle.

3. Deduce an expression for the area of a spherical polygon of n sides.

4. Shew that, in the complete solution of all the cases of a right-angled spherical triangle, there must necessarily be six ambiguous equations: explain the modes by which apparent ambiguities are removed in all the other cases.

5. Shew how Napier's two rules may be applied to quadrantal triangles. Enumerate the circular parts, and write down the ten equations necessary to their solution.

6. In how many equations can we exhibit the solution of all the cases of oblique-angled spherical triangles?

7. In any spherical triangle in which A, B, C, are the angles, and a, b, c, the opposite sides respectively:

(1.) Given $a\,b$ C, required c, in a form for logarithmic computation, without the intervention of A, B.

(2.) Prove that

$$\tan.\frac{A+B}{2}=\frac{\cos.\frac{a-b}{2}}{\cos.\frac{a+b}{2}}\times\cot.\frac{C}{2}.$$

8. If $S=\dfrac{a+b+c}{2}$,

Prove that

$$\log.\sin.\frac{A}{2}=$$

$$\frac{1}{2}\times\{20+\log.\sin.(S-b)+\log.\sin.(S-c)-\log.\sin.b-\log.\sin.c\}$$

ST. JOHN'S COLLEGE.

1820.

DIFFERENTIAL AND INTEGRAL CALCULUS.

---

1. Give a definition of the fluxion, or differential of a quantity, and from that definition shew how the differential of a^x may be found.

2. Differentiate the nth hyp. log. x.

3. Exemplify the infinitesimal analysis, by drawing a tangent to a circle at a given point.

4. Trace the curve whose equation is $x^3 - y^3 = ax^2$, and construct its asymptote.

5. If any number of lines, all terminating in a given point, and situated in the same plane, be given in magnitude and position: it is required to find the position of another line passing through the same point, such that the sum of the projections of all the former lines upon this last shall be a maximum.

6. Draw an asymptote to the spiral whose equation is $\cos. \frac{\theta}{m} = \frac{a}{r}$ and find the angle contained between its apse and asymptote.

7. Find the value of $\dfrac{a - \sqrt{2ax - x^2}}{\sqrt[3]{a^2} - \sqrt[3]{2ax - x^2}}$ when $a = o$, and also of $\dfrac{e^x - e^{\sin. x}}{x - \sin. x}$ when $x = 0$.

8. Integrate the following differentials:

$$\frac{dx}{\sqrt{a + b\sqrt{x}}},\ \frac{dx}{\sqrt[4]{x^4 - 1}},\ dx\sqrt{x + \sqrt{1 + x^2}} \text{ and } nx\frac{d^2y}{dx^2} + \sqrt{1 + \frac{dy^2}{dx^2}} = 0$$

9. Find the point of suspension when the time of an oscillation of a given straight line is the least possible.

10. Determine the centre of gyration of a rectangle revolving in its own plane round an axis given in position.

11. Find the area of the catenary.

12. Shew that $1 + \cos. x + \frac{\cos. 2x}{1 . 2} + \frac{\cos. 3x}{1.2.3} + \&c.$ *ad infinitum* $= e^{\cos. x} . \cos. (\sin. x)$.

TRINITY COLLEGE.

1820.

DIFFERENTIAL AND INTEGRAL CALCULUS.

1. Define the 1st, 2d, 3d, &c., differentials of a function according to Newton's method of limits, and also according to Lagrange; shew that both methods coincide, and exemplify in the case of sin. x.

2. Prove the rules for differentiating,

(1.) $\frac{u}{z}$ where u and z are both functions of x.

(2.) tan. x.

(3.) a^x.

(4.) log. x.

and differentiate.

(1.) $(a + bx^n)^m$;

(2.) $\dfrac{\sqrt{1+x^2} + \sqrt{1-x^2}}{\sqrt{1+x^2} - \sqrt{1-x^2}}$:

(3.) $\log. \left(\dfrac{e^x - 1}{e^x + 1}\right)$

(4.) $\dfrac{2}{\sqrt{(a^2-b^2)}} \times \text{arc} \left(\tan. = \dfrac{\sqrt{(a-b)}\sqrt{(1-\cos. x)}}{\sqrt{(a+b)}\sqrt{(1+\cos. x)}}\right)$.

3. Find the area (1.) of the cycloid, (2.) of the logarithmic curve, (3.) Shew that the surface and solidity of a sphere, are each two thirds of the surface and solidity of the circumscribing cylinder.

4. Deduce Maclaurin's theorem from Taylor's, and apply the former to find the tangent in terms of its arc.

5. Transform the series for log. $(1 + x)$ into one which con-

verges for all values of x; and apply it to compute the modulus of Briggs's system.

6. In a series $A+Bx+Cx^2+Dx^3 \ldots\ldots$ where the ratio which any one coefficient bears to that immediately preceding it, is always finite, such a value may be found for x as will render any one term greater than the sum of all that follow. Required a proof, and determine whether this be possible in the series,

$$1+x^2 x+3^2 x^2+4^2 x^3+ \ldots\ldots$$
$$1+1.2.x+1.2.3\,x^2+1.2.3.4\,x^3 \ldots\ldots$$

7. Shew by the Differential Calculus, or from considerations purely algebraical, that if $a : x :: y : b$; $x+y$ will be a minimum when $x=y$.

8. Required all the values of x which will render y a maximum or minimum in the following equations, and determine those values which give maximum results, those which give minimum, and those which give neither.

(1.) $y = x^5 - 5x^4 + 5x^3 + 1$
(2.) $y^4 - 4a^2xy + x^4 = 0.$

9. Prove that under a given spherical surface, a hemisphere contains the greatest segment of a sphere.

10. Required the least paraboloid that can be circumscribed about a given sphere.

11. Find the value of $\dfrac{x^{\frac{1}{2}} - a^{\frac{1}{2}} + (x-a)^{\frac{1}{2}}}{(x^2-a^2)^{\frac{1}{2}}}$ when $x=a$,

and of $\dfrac{\pi}{4x}\,.\,\tan.\,\dfrac{\pi x}{2}$ when $x=0$.

12. Find the equations to the tangent, and normal of a curve at a given point, and apply them to the parabola.

13. A curve is convex or concave towards the axis of the abscissæ, according as the ordinate and its second differential coefficient have the same or different signs. Required a proof.

14. Trace the curve whose equation is $x^2y^2-a^3x+a^2x^2=0$; find its point of contrary flexure, and the angle at which it cuts the axis.

15. Draw an asymptote to the reciprocal spiral.

16. Investigate the general expression for the radius of curvature in curves referred to an axis, and apply it to the catenary whose equation is $y=a.\ \log.\left(\dfrac{x+a+\sqrt{(2ax+x^2)}}{a}\right)$.

17. Find the differential of the surface generated by the revolution of a curve round its axis.

18. Find the integrals of

(1.) $\dfrac{x^n dx}{a-x}$, (2.) $\dfrac{Ax^2dx - Bxdx}{x^2-px+q}$, (3.) $\dfrac{x^3dx}{\sqrt{(2ax+x^2)}}$

(4.) $\dfrac{dx}{x^5\sqrt{(1-x^2)}}$.

19. Find the content of the solid generated by the revolution of a cycloid round the tangent at its vertex.

20. Prove that the solid generated by the revolution of any regular polygon inscribed in a circle round its diameter, is equal to one-third of the surface generated, multiplied into the perpendicular drawn from the centre on one of its sides.

21. Find the area of the curve whose equation is $y = px^n$, between the values of $x = a$, and $x = b$; and investigate the value of the result in the case of the common hyperbola.

22. Shew that all the curves included under the equation $y = px^n$, are rectifiable when $\dfrac{1}{2(n-1)}$ is a whole number.

23. Find the length of an arc of the spiral of Archimedes contained between two polar distances p and q; and supposing the radius of the circle to be a, prove it equal to an arc of a parabola contained between two ordinates p and q, the latus rectum being $\dfrac{a}{\pi}$.

24. Find the convex surface and solid content of the portion of a right cylinder, which is cut off by a plane passing through the centre of the base, and inclined to it at an angle of 45°.

TRINITY COLLEGE.

MECHANICS.

1. If two forces each $(= A)$ act at an angle (m°) they will compound a force $= 2A.\sqrt{1-s^2}$, where $s =$ sine of $\left(\dfrac{m^\circ}{2}\right)$.

2. A given force is to be resolved into different pairs of forces, whose sum shall bear to it the ratio of 5 : 3. Determine the minor axis of the limiting ellipse.

3. When a piece of timber of unequal dimensions is put on a fulcrum 13 feet from the smaller end, it is in equilibrio; but when put on a fulcrum only 12 feet from the smaller end, it requires a weight of 210 lbs. on that end to support it. Find the weight of the timber?

4. The arms of a false balance are as $m : n$, and a body weighs (a lbs.) at one end, and (b lbs.) at the other; find,

(1.) The true weight of the body.

(2.) The excess of the sum of the two apparent weights above twice the true weight.

(3.) The distance of the point of suspension from the middle of the beam.

5. Suppose a billiard-table to be an irregular hexagon, and (A) and (B) two balls upon it given in position. Determine against what point of any one side the ball (A) must impinge, so that after rebounding from it, and from every other side in succession, it may hit the other ball B.

6. In the time in which a heavy body falls down a well, and its sound on the bottom returns to an ear at the top, a pendulum 61 inches long vibrates eight times:—What is the depth of the well?

7. Find the lines of swiftest and slowest descent from one given circle to another.

8. Determine in what locus a person must be placed, to throw a perfectly elastic ball against a given point in a given vertical plane, so that the ball may each time return to his hand.

9. In the collision of imperfectly-elastic bodies, find the ratio of the relative velocity before impact to the relative velocity after impact; and show that the sum of the products of each body into the square of its velocity before impact, is greater than the sum of the products of each body into the square of its velocity after.

10. What must be the form of a triangle whose centre of gravity is the centre of the circumscribing circle?

11. A and B are hung over a fixed pulley:—Required the ratio of H : L of an inclined plane on which a body shall descend as many feet in a given time as the heavier (A) of the two bodies.

12. A weight (W) is drawn along the horizontal plane (CB) by another weight (P) acting over a pulley fixed at the perpendicular height (BA) above the horizon:—With what force will (W) be accelerated? And what will be its velocity acquired at (B)?

13. In the steelyard, if the weight increase in arithmetic progression, the divisions of the scale will be at equal intervals; and if each of these intervals be equal to the shorter arm, the moveable weight will be equal to the difference of the arithmetical progressions.

14. A body being let fall from the top of a tower, was observed to fall half way in the last second:—What was the tower's height?

15. (AB) and (AC) are two inclined planes of a common height (AD); the length of the plane $AB = a$:—To find what must be the length of the other plane AC, so that a given weight (P) on the plane AB may draw another given weight (W) up the plane AC in the least time possible, (P) and (W) being connected by a string over a pulley at (A).

16. A body is projected at an angle of 45° with a velocity acquired by a heavy body falling down the axis of a cycloid:—Required the ratio of the time of flight to the time of oscillation in the cycloid?

17. There is a mountain on the earth's surface of such a height that a clock (which keeps true time at the bottom) when carried to the top loses two minutes a-day:—What is the altitude of the mountain, supposing the earth's radius to be 6982000 yards?

18. If the axis of a parabola be perpendicular to the horizon, and chords be drawn from the vertex to any point in the curve, compare the times of descent down them by the force of gravity.

19. What is the least velocity with which a body must be projected from the top of an inclined plane, so as just to reach the extremity of the plane?

20. Suppose a body to be projected downwards from a given point (A) with a given velocity (a); and after (n) seconds are elapsed, another body is projected upwards from a given point (B) with a given velocity (b):—Where will they meet?

21. What number of pulleys (in a system where each pulley hangs by a separate string, and all the strings are parallel) must be applied to a weight (= 96 lbs.) so that P (= 1 lb.) may sustain it on an inclined plane whose height is one-third of its length, P being supposed to act parallel to the length?

22. Define the centres of gravity, gyration, oscillation, and percussion; and find with what part of a cylindrical stick 50 inches long must (B), whose arm is twenty inches long, strike (b), so as to give the greatest possible blow.

23. If a chord of a given length be fastened to two hooks, A and B, not in a horizontal line, and a weight (W) slide freely along the chord, find where (W) will rest; and compare the pressure on either hook with the weight (W).

24. A globe weighing 200 lbs. is supported between two inclined planes whose inclinations are 60° and 45° respectively:—What is the weight supported on each plane?

25. Investigate a general Theorem for determining the ratio between (P) acting at any angle on the back of a scalene wedge, and the sum of the resistances (supposed to be wholly effective) acting at different angles on the sides; and apply it to an isosceles wedge, when (P) acts perpendicularly, and the equal resistances act parallel to the back.

ST. JOHN'S COLLEGE.

1819.

MECHANICS AND TRIGONOMETRY.

1. STATE the laws of motion, and the experiments by which the third is proved.

2. Two lines, SP and HP, revolving about the points S and H, represent two forces whose compound force is constant. Find the curve which is the locus of P.

3. Two chords AB, AC, of a circle, represent two forces; one of them, AB, being given, find the position of the other when the compound force is a maximum.

4. When there is an equilibrium on the single moveable pulley whose strings are not parallel; if the whole be put in motion—the velocity of the weight : velocity of the power :: the power : the weight.

5. Find the greatest inclination of a plane upon which a given elliptic cylinder, whose axis is horizontal, can be supported.

6. Two planes have a common base, and are inclined to the horizon at 30°. An inelastic body is projected up one of them with a given velocity, it then descends and oscillates between them. Find the whole space described, and time of motion.

7. Compare the relative velocities before and after impact, in the case of imperfectly elastic bodies.

8. A parabola is placed with its axis vertical. Draw the line of quickest descent from the curve to the focus.

9. Two equal and elastic balls are let fall at the same instant in the same vertical line from two altitudes $9a$ and $4a$ above a horizontal plane. Find the successive points of impact, and the spaces described by each before the return to their original positions.

10. The time of an oscillation in the arc, LZP, of a cycloid, is equal to the time of describing the semicircumference lzp, with the velocity at V continued uniform.

11. Two particles of matter are attached at different points to an inflexible line without gravity, which is suspended by its extremity. Find the time of a small oscillation, supposing one of them to lose all its weight, and the other all its inertia.

12. A given weight, P, is attached to a given cylinder, Q, by means of a string wrapped round its circumference, and passing over the common vertex of two inclined planes. P is drawn up one, while Q descends down the other. Compare the lengths of the planes.

13. Having given the velocity and direction of projection, find

the range on a given inclined plane passing through the point of projection, and the time of flight.

14. Compare the effect of an uniform force acting down and up alternately, in parallel directions, on a crank, with the effect of the same force acting always at right angles in a complete revolution.

15. An uniform chain is coiled on a smooth horizontal plane, a given length being drawn out, is projected along the plane with a given velocity. Find the velocity after the description of any given space.

16. Shew that an angle varies directly as the arc which subtends it, and inversely as the radius.

17. Having given the sines, cosines, and tangents of two arcs; to find the sines, cosines, and tangents of their sum and difference.

18. If an angle be taken whose tangent is to the radius as the greater side of a triangle to the less, the radius will be to the tangent of the excess of this angle above half a right angle, as the tangent of the semi-sum of the angles opposite those sides to the tangent of their semi-difference. Prove this geometrically; and shew the superiority of it as a practical solution, when two sides and the included angle of a triangle are given.

19. The hypothenuse of a right-angled triangle being constant, find the corresponding variations of the sides.

20. Expand A^x, and from thence find the value of x in the equation $A^x = N$.

21. A person at the foot of a hill running east and west observes a tower due north of him, and takes the elevation of it above the hill. He then walks in a direction N.E. till the tower bears due west of him, when he again takes its elevation. Determine from hence the inclination of the hill; and the distance between the points of observation being given, find the height of the tower.

TRINITY COLLEGE.

MECHANICS.

1. A, B, and C, are three bodies whose perpendicular distances from a given plane are d, d', d''; C is on the opposite side to A and B; prove $A \times d + B \times d' - C \times d'' = \{A + B + C\} \times d'''$ where d''' is distance of centre of gravity.

2. BA is perpendicular to the horizon; BDA is a semi-circle, BCG a quadrant; take any plane AC. If a ball is thrown up AC

with velocity acquired down BA, it will describe a space equal to AC + CD in the time of falling through BA.

3. In a straight lever the sum of the products of each body, and its distance from the fulcrum, is equal on both sides.

4. In a bent lever of uniform density and thickness, whose arms are (a) and (a'), (a) being parallel to the horizon, and weighing (b) lbs.; compare P and W, when the inclination of the arms is (θ), and P (acting at the end of arm a) is inclined at $\angle(\theta')$, and W at $\angle(\theta'')$.

5. Prove the general proposition of the wedge; apply the result to the case of an equilateral wedge, where the power on the back acts perpendicularly, and the resistances on the sides are equal, and act perpendicularly to the back.

6. A straight lever is parallel to the horizon; given its length, given a weight P hung at one end; required the variation of the position of the fulcrum, supposing W to vary in arithmetic progression.

7. A body, G, is kept at rest by three forces proportional to AG, BG, CG; G is centre of gravity of the triangle formed by joining A, B, C.

8. If with centre of gravity of any number of bodies as centre, and with any radius, a circle be described, the sum of the products of each body, and the square of its distance from any assumed point in circumference, is constant.

9. Prove that in perfect elasticity $Aa^2 + Bb^2 = Ap + Bq^2$, where a and b are the velocities of A and B before impact, and p and q after. Compare also elasticity and compression when $Aa^n + Bb^n = Ap^n + Bq^n$.

10. In a single moveable pulley, where the strings passing under the moveable pulley are not parallel, compare P and W; first, when the strings are equally, secondly, when they are unequally inclined to the horizon.

11. An imperfectly elastic ball falls perpendicularly from a height (a):—Required whole space described by ball after 5 rebounds, and the greatest height after last rebound.

12. Assuming the time of oscillation to equal the time of describing semi-circle, &c., investigate the actual value of the time of oscillation, and thence, compare it with time down axis.

13. In inclined planes, P : W :: W's velocity : P's.

14. Determine the expressions for range and greatest height, upon a plane passing through point of projection; and compare greatest height of all parabolas with a given velocity to farthest range.

15. AH is a vertical diameter; HBA, CEA, two contiguous circles, touching in A. Prove the time down BC, DE, &c., to be constant.

16. If a body is kept at rest by three forces, and lines be drawn at any equal angles to the directions in which they act, forming a

triangle, the sides of the triangle represent the quantities of the forces.

17. If 3 forces are represented by 3 sides forming the solid angle of a parallelopiped, the resulting force is the diagonal of the parallelopiped.

If the 3 forces are equal, and act in planes perpendicular to each other, compare the compound force with them.

18. A is vertex of triangular pyramid, G is centre of gravity. If upon body at G forces act in directions AG, BG, CG, DG, and proportional to them, it remains at rest.

19. A uniform beam AB, is moveable about fixed point A, and supported by given weight P over fixed pulley C; AC is equal to AB, and parallel to horizon. Required position in which AB rests?

20. Make a body oscillate in a given cycloid.

21. VP $= \frac{1}{2}$ radius; MN perpendicular the diameter. Cycloid area MVN $= \frac{1}{2}$ hexagon inscribed in the circle.

22. Compare times of describing vertical diameter and any other:—Required also that diameter, the time through which $= 2$ time down vertical diameter.

23. If the number of oscillations performed in same time by two pendulums (whose lengths are L and l) be as $m : m + n$, compare force of gravity at the two heights.

24. If one pendulum is at distance of (n) radii from the earth's centre, at what point below the surface must another of equal length be placed to vibrate in same time?

ST. JOHN'S COLLEGE.

1820.

MECHANICS AND TRIGONOMETRY.

1. The moving forces acting on two bodies are reciprocally proportional to the quantities of matter. Compare the velocities generated in any time.

2. If SA, the least distance in an ellipse, be one third of SM the greatest, and SA be taken to represent one force, then SB, the mean distance, will compound with SA, the least possible force; and the compound force will be a mean proportional between SM and SA.

3. When there is an equilibrium on the inclined plane, if motion be communicated, the velocity of the power is to the velocity of the weight as the weight to the power.

4. Find the inclination of a plane, on which a regular figure of n sides will just be supported.

5. A and B are two elastic balls placed on a billiard table, FC is a reflecting cushion. Produce AB to C. Then if A impinge on B and drive it against the cushion, they will meet, after the reflection of B, if the angle of impact equal 45 degrees minus $\frac{1}{2}$ the $\angle$ of impact.

6. If the number of mean proportionals between two balls A and X be increased without limit; A's velocity : that communicated to X :: $\sqrt{X}$: $\sqrt{A}$.

7. A weight is fixed to the lowest point in a circle moveable in a vertical plane about its centre; another equal weight is attached to a string wrapped round the circumference. Find the velocity acquired by the descending weight in any space.

8. If the axis of a parabola be horizontal, and the weight W be supported on the curve, by means of a string passing over a pulley in the focus to which a given weight P is attached; then W will vary as the corresponding abscissa.

9. If a body be projected successively in all possible directions, from the same point, with the same velocity, prove *geometrically*, that the locus of the ultimate intersections of the successive parabolas is a parabola.

10. BLV is a cycloid; DOV the generating circle on the axis. Draw ROL parallel to the base which is horizontal. Then the time down BL : time down DR :: arc DO : chord DO.

11. The velocity of a body in a cycloid varies as the right sine of a circular arc, whose radius equals the length of the arc at the beginning of motion, and versed sine the arc fallen through.

12. Having given the velocity and direction of projection; to find where the body will strike the horizontal plane passing through the point of projection; and the time of flight.

13. Find the least velocity with which a body projected from the top of a tower of given height, after reflection from the horizontal plane, shall strike the top of another tower, whose distance and height are given. Find also the direction of projection and the time of flight.

14. Two equal weights are attached to a string passing over a cycloid whose base is horizontal. Find the whole pressure; and prove that the pressure estimated in a vertical direction is uniform.

15. A sphere and hollow cylinder of equal weight are suspended by a string passing over a solid cylindrical pulley equal in weight to the former. To determine the circumstances of the motion and tension of the string.

16. Given the three sides of a triangle, explain the different methods of obtaining one of its angles, and shew which is to be preferable when that angle is nearly a right angle

17. If A be any angle $\frac{1-\cos. A}{1+\cos. A} = \tan.^2 \frac{1}{2} A$.

18. The tangent + cotangent of an angle equal twice the cosecant of twice the angle.

19. Having given two sides of a triangle and the included angle, shew how the other parts may be found.

20. A on foot, and B on horseback, are travelling together towards the same town; A takes a foot path leaving the road at a given angle, B goes on till he comes to a cross road. They arrive at the same instant. Given their rates of motion and greatest distance of separation, find the distance travelled by each, and the species and dimensions of the included area.

21. If p, q, r, s, be the coefficients of an equation whose roots are the tangents of A, B, C, &c. Then the tangent of A + B + C + &c. $= \dfrac{p - r + t - \&c.}{1 - q + s - \&c.}$

22. A person wishing to know the height of a spire due south of him, observes a small cloud pass behind it, the wind blowing south-west. Soon after the cloud passed over the Moon. He then measures his distance from the foot of the spire; and on return home calculates the Moon's altitude and angular distance from the south at the time of observation. Shew how from these data he may determine the height of the spire.

TRINITY COLLEGE.

1820.

MECHANICS.

STATICS.

1. When forces keep each other in equilibrium round a fixed point, the sum of all their *moments* is $= 0$; those being reckoned negative which tend to turn the system in the opposite direction.

2. Find the resultant of any number of forces in the same plane acting on a point. Apply the formulæ to the following example:—

AB, AC, AD, are three lines making angles of 120° with each other; the point A is acted on by pulling forces in AB and AC, which are as 3 and 4, and by a pushing force DA, which is as 5. Find the force which will keep it at rest.

3. A string fastened at A and passing over a fixed pulley B, has

a known weight W hang by a knot at C; find what weight must be appended at P, that CB may be horizontal.

4. A weight Q hanging freely, supports an equal weight P upon an inclined plane, by means of a string passing over a pulley below the plane: find the position of equilibrium.

5. When a body is sustained upon a curve whose co-ordinates are x and y, by any forces whose components in those directions are X and Y, shew that

$$Xdx + Ydy = 0.$$

Apply the formula to find the position of equilibrium when a weight Q hanging freely, supports a weight P upon a parabola whose axis is *horizontal*, by means of a string passing over the focus.

6. Find the centre of gravity of any number of points in the same plane.

7. The sum of the squares of the distances of three equal bodies from each other, is three times the sum of the squares of their distances from their common centre of gravity.

8. Prove the differential expression for the centre of gravity of any solid of revolution; and find the centre of gravity of a hemisphere.

9. ABCD is a quadrilateral figure, of which the two shorter sides AB, BC are equal, as also the two longer AD, DC; and the angle ABC is a right angle: what is the greatest length of the side AD, that the figure may stand on the base AB on a horizontal plane without oversetting?

10. Given a bent lever with arms of uniform thickness, moveable in a vertical plane about the angular point: find the positions in which it will rest.

11. A given beam considered as a line is supported on two given inclined planes: find the position of equilibrium.

12. Given the pressure upon one of the four legs of a rectangular table of known weight; find the pressures of the other three. Shew that without this *datum* the problem is indeterminate.

13. ABC is a right-angled isosceles triangle, and three *equal* forces act *in* the lines AB, BC, CA. At what point of the plane ABC, produced if necessary, must a force be applied to keep it at rest, and what must be its magnitude and direction?

14. A beam BC hangs by a string AB from a fixed point A, with its lower extremity C upon a horizontal plane: find the position in which it will rest. Also find the horizontal force which must be applied at C to retain it in a given position.

15. A false balance has one of its arms exceeding the other by $\frac{1}{m}$ of the shorter. It is used, the weight being put as often in one scale as the other. What is the shopkeeper's gain or loss *per cent.*?

16. In an arch which is in equilibrium, the weights of the vous-

soirs are as the differences of the tangents of the angles which their joints make with the vertical.

DYNAMICS.

1. A bow is drawn by a force of 50 lbs; the weight of the arrow being $\frac{1}{10}$ lb, compare the force of gravity with the initial accelerating force which the string exerts upon the arrow, when it is let go; neglecting the inertia of the bow.

2. If a, b, be the velocities of two bodies A, B *before* their direct impact; u, v the velocities *after*, α and β the velocities gained and lost respectively, and e the fraction which measures the elasticity;

$$Aa^2 + Bb^2 = Au^2 + \beta v^2 + \frac{1-e}{1+e}(A\alpha^2 + B\beta^2)$$

3. A and B are two given points in the diameter of a circle: find in what direction a perfectly-elastic body must be projected from A, so that after reflection at the circle it may strike B.

4. Prove that if a body be accelerated by a constant force

$$v = ft \text{ and } s = \tfrac{1}{2} ft^2.$$

5. Find the velocity and direction with which a body must be projected from a given point, that it may hit two other given points in the same vertical plane.

6. AB is the vertical diameter of a circle: a perfectly elastic body descends down the chord AC; and being reflected by the plane BC, describes its path as a projectile. Shew that this path strikes the circle at the opposite extremity of the diameter CD.

7. Find the equation to the *cycloid;* and shew that in the same cycloid the oscillations are *isochronous.*

TRINITY COLLEGE.

1820.

HYDROSTATICS.

1. The pressure of a fluid against any surface in a direction perpendicular to it, varies as the area or the surface multiplied

into the depth of its centre of gravity below the surface of the fluid.

2. A hollow cone without a bottom stands on a horizontal plane, and water is poured in at the vertex. The weight of the cone being given, how far may it be filled so as not to run out below?

3. What must be the magnitude and point of application of a single force that will support a sluice-gate in the shape of an inverted parabola?

4. Find the specific gravity of a body which is lighter than the fluid in which it is weighed.

5. If the specific gravity of air be called m, that of water being 1, and if W be the weight of any body in air, and W′ its weight in water, its weight *in vacuo* will be,

$$W + \frac{m}{1 - m}(W - W'.)$$

6. Three globes of the same diameter and of given specific gravities, are placed in the same straight line. How must they be disposed that they may balance on the same point of the line *in vacuo* and in water?

7. If a homogeneous hemisphere, floating in a fluid, be slightly inclined from the position of equilibrium; shew that the *moment* of the fluid to restore it to that position, is not affected by placing any additional weight at its centre.

8. A regular tetrahedron moves with its vertex forwards, in a direction perpendicular to its base. Compare the resistance on the oblique planes with that on the base

9. If the particles of an elastic fluid repel each other with forces varying inversely as the fourth power of their distances, the compressive force on any portion varies as $(\text{density})^2$.

10. Explain the method of measuring altitudes by means of the barometer and thermometer.

11. Two barometers, whose tubes are each l inches long, being imperfectly filled with mercury, are observed to stand at the heights h and h', on one day, and k and k' on another. Find the quantity of air left in each, reducing it to the density when the mercury is at the standard altitude of 30 inches, and supposing the temperature invariable.

12. Construct a common forcing-pump; and shew what is the force requisite to force the piston down.

13. In the common sucking-pump, given the lowest point to which the piston descends, find the length of the stroke that the pump may work.

14. A cylinder which floats upright in a fluid, is pressed down below the position of equilibrium: when it is left to itself, find the time of its vertical oscillations, neglecting the resistance.

15. A vessel generated by the revolution of a portion of a semi-

hyperbola round its *conjugate* axis, is emptied by an orifice at the centre of the hyperbola: find the time.

16. A close vessel is filled with air n times the density of atmospheric air. A small orifice being made, through which the air rushes into a vacuum, find the time elapsed when the density is diminished one half.

17. A tube of uniform diameter consists of two vertical legs connected by a horizontal branch. When it is made to revolve with a given velocity round one of its vertical legs as an axis, find the height to which the water will rise in the other.

18. Let a spherical body descend in a fluid from rest; having given the diameter of the sphere and its specific gravity relatively to that of the fluid, it is required to assign the time in which the sphere describes any given space.

19. If the density of a medium vary inversely as the distance from a center, and the centripetal force inversely as any power of the distance from the same centre, a body may describe a logarithmic spiral about that point.

20. If the resistance on a body which oscillates small arcs in a fluid vary as the n^{th} power of the velocity, the difference of the arcs of descent and ascent will vary as the n^{th} power of the whole arc.

TRINITY COLLEGE.

1820.

OPTICS.

1. Given the distance of the focus of incident rays from the centre of a given spherical reflector; find the distance of the geometrical focus of reflected rays from the centre, when they are incident nearly perpendicularly.

2. Given the position of an object placed between two plane reflectors inclined at a given angle; find the total number of images, and apply it to the case where the angle of inclination is equal to $11^\circ. 15'$.

3. A straight line is placed before a concave spherical reflector, at the distance of one-third of its radius from the surface; find the dimensions of the curve formed by its image, the radius of reflector being 9 inches.

4. Having given the ratio of the sines of incidence and refraction, when a ray passes out of one medium into each of two others, to find the ratio of the sine of incidence to the sine of refraction out of one of the latter mediums into the other.

5. When parallel rays are incident nearly perpendicularly upon a spherical refracting surface, the distance of the intersection of the refracted ray and the axis, from the centre, is the greatest when the arc is evanescent.

6. A pencil of parallel rays passes from water through a globule of air; find the focus after the second refraction.

7. Find the focal length of a compound lens.

8. Explain the construction of Newton's telescope, and shew how it must be adjusted to the eye of a long-sighted person.

9. An object whose *real* depth below the surface of the water is ten feet, is viewed by an eye fifteen feet above the surface. What must be the focal length of a lens through which it is viewed, that its *apparent* depth may be ten feet?

10. If a plane mirror be turned round uniformly, the angular velocity of the image of a given object formed by continual reflection at its surface : angular velocity of reflector :: 2 : 1.

11. The radii of a spherical reflector and sphere of glass of same aperture and power are in the proportion of 3 : 1. Compare the density of rays in sun's image formed by them.

12. Two straight lines are inclined at a given angle, and a point E is given without them, a line E F f moves round the point E, and cuts the given lines in F and f; find the locus of the mirror D, so that f shall always be the image of F.

13. Suppose a mirror M to move in a straight line AB, and an object D in the line AC at right angles to it, and distance between object and mirror to be constant; to determine the locus of an eye, which being always at the same distance as the object from the mirror, shall always see the object.

14. Having given the refracting powers of two mediums, to find the ratio of the focal lengths of a convex and concave lens, formed of these substances, which, when united, produce images nearly free from colour.

15. When a ray of light is incident obliquely upon a spherical reflector, to determine the intersection of the reflected ray and the axis of the pencil to which it belongs, and shew that when the focal length is given, the longitudinal aberration of parallel rays varies as $(\text{lin. apert.})^2$ and lateral aberration varies as $(\text{lin. apert.})^3$.

16. If an object be placed in the principal focus of a double convex lens, the visual angle is the same, whatever be the distance of the eye from the glass.

17. Find the length of a caustic generally, and apply it to the case when the reflecting curve is a semicycloid, rays parallel to axis.

18. Find the density of rays in a caustic, when reflecting surface is a hemisphere, radiating point in surface.

TRINITY COLLEGE.

1816.

ASTRONOMY.

---

1. What are the situations of the Sun, Moon, and Earth at the time of full Moon—of new Moon—of half Moon? What are the situations of the bodies at the time of a Lunar eclipse?

2. The longest day is (n) hours more than a day at the equator: what is the latitude of the place?

3. Given the synodic year of a Planet; compare the radius of its orbit with that of the Earth's.

4. What is the elongation of Venus from the Earth when she will appear stationary? Their mean daily motions being known, compute the time when that appearance will take place.

5. Find the time of shortest twilight for a given north latitude: and shew that then the Sun's azimuths at the beginning and end of twilight are supplements of each other, and that the Sun's declination is south.

6. A Star, as it leaves the prime vertical, is observed to have an increase of azimuth, which is, to the contemporary increase of altitude, :: $a : b$. Required the latitude of the place.

7. The latitudes of two Stars which have the same right-ascension, are given, and also their difference of longitude. Deduce the longitude of each and the right-ascension.

8. A telescope deviates from the plane of the meridian by a small angle: and it is observed, that the time elapsed between the appulses of two known Stars to it is not equal to their difference of right-ascension converted into time. Correct the deviation.

9. The increments of gravity in moving from the equator to the pole, are proportional to the $(\text{sin.})^2$ of lat.; the Earth being supposed a sphere.

10. Given the lengths of two pendulums vibrating seconds in two places of known latitude. Find the eccentricity of the ellipse, by the revolution of which round the polar diameter, the Earth would be generated.

11. The Earth moves in an ellipse round the Sun in the focus: Required the law of the force by which it is retained in the orbit.

12. The periodic times in different ellipses round the same centre of force, are in the sesquiplicate ratio of the major axes.

13. Prove the law of variation of the angular velocities of different bodies, revolving in their orbits round the same centre of force; and find when that part of the equation of time arising from the unequal angular motion of the Sun in the ecliptic is a maximum.

14. Find the true anomaly in terms of the eccentric.

15. Give a formula for clearing the Moon's distance of the effects of refraction and parallax.

16. Correct the meridian line as determined by the method of equal altitudes—the Sun having changed his declination by a given quantity, during the interval of the two observations.

17. What would be the curve of aberration of a Star, if the Earth were to move in an ellipse round the *centre?*

18. Find the effect of aberration on the right-ascensions and declinations of Stars. Investigate, in each case, the position of the Earth when that effect $= 0$.

19. Investigate the dimensions of the Conic section traced out in the course of a given day at a given latitude, by the extremity of the shadow cast upon an horizontal plane, by a style perpendicular to it.

20. Explain the principle of the Sextant: and the division and use of the Vernier.

21. Compare the densities of the Sun and Earth; the periodic times of the Earth and Moon, and the Moon's horizontal parallax being known.

TRINITY COLLEGE.

1817.

ASTRONOMY.

1. By what arguments is it inferred that the Earth revolves about its axis, and about the Sun?

2. Find the right ascension and declination of a comet, when its distance from two known stars is given.

3. Required the time of the Sun's transit over the vertical wires of a telescope, on a given day at a given place.

4. The apparent meridian altitude of the Sun's lower limb $= 53° 40'$, his apparent semi-diameter $= 15° 50'$, his mean refraction $= 29''$, the parallax $= 4''.5$, collimation $= 34''.5$, and declination $= 16° 13'$. Find the latitude of the place of observation.

5. The altitude of the Sun was observed to be half of his declination at 6 o'clock. Prove that twice the sine of the latitude of the place $=$ the secant of the Sun's altitude.

6. There are two places on the same meridian, whose latitudes

are the complements of each other; and on a given day the Sun rises an hour sooner at one place than at the other. Required the latitudes of the two places.

7. If A and a be the altitudes of a star, on the same vertical circle on the same day, (d) the declination of the star, and (l) the latitude of the place, $\frac{\sin. l}{\sin. d} = \frac{\cos. \frac{A \sim a}{2}}{\sin. \frac{A+a}{2}}$; Required a proof.

8. Construct a vertical south-east dial, for a given place.

9. When the Sun is in the equinoctial, the locus of the extremities of the shadow cast by a perpendicular object upon an horizontal plane is a straight line.

10. If a body revolve uniformly in a circular orbit, it is retained in that orbit by a force which tends to the centre of the circle; and if the periodic times in such circle $\propto R^n$, the force $\propto \frac{1}{R^{2n-1}}$.

11. Determine the quantity of refraction by observations made upon the circumpolar stars. When will this method fail?

12. The parallax of a planet in right ascension being given, it is required to find the distance of the planet from the earth's centre, the earth being supposed spherical.

13. The sine of the excentric anomaly : the sine of the true anomaly :: the radius vector : the semi-axis minor.

14. Find the distance of a planet from the Sun.

15. Suppose an eclipse of the Moon to last three hours; to how great a portion of the Earth will some part of it be visible?

16. Prove that when the first point of Aries rises, the ecliptic makes the least angle with the horizon; and explain from thence the phenomenon of the harvest Moon.

17. When will the right ascension and declination of a star be diminished, and when increased, by the retrograde motion of the equinoctial points?

18. The equation of time arising from the obliquity of the ecliptic is a maximum, when the cosine of the Sun's declination is a mean proportional between the radius and the cosine of the obliquity.

19. Find the aberration of a star in latitude on a given day, and also the aberration of a planet in longitude.

20. Explain the method of determining the difference of longitude of two places on the Earth's surface, by means of a chronometer; and state the errors to which this method is liable.

ST. JOHN'S COLLEGE.

JUNE 1820.

ASTRONOMY.

---

1. If two sides of a spherical triangle be equal to a semicircle, the arc drawn from the vertex bisecting the base is a quadrant.

2. If two sides of a spherical triangle be less than a semicircle, the angle opposite the less side is less than 90 degrees.

3. If a great circle be stereographically projected, the radius of the projection is the secant, and the distance of its centre from the centre of the primitive is the tangent of the inclination of the circle to the primitive.

4. Having given the Sun's declination and diameter, and the latitude of the place; find how long the disk is in rising.

5. Find the absolute quantity of refraction for considerable altitudes according to Boscovich's method.

6. On a certain island the Sun was observed to be vertical when on the meridian. The declination was found from the tables to be D degrees; and the time of sunrise at Greenwich to be T, and of sunset t hours before that observed by the chronometer. Find the distance and bearing of the place from Greenwich.

7. Having given the Sun's rising amplitude and altitude when on the prime vertical, determine the latitude of the place.

8. The Sun's declination which is given, is greater than the co-latitude which is also given. Determine the nature and dimensions of the curve, which is traced out by the end of the shadow of a vertical stick on an horizontal plane.

9. Determine, from three observations of a spot on the Sun's disk, the inclination of the Sun's equator to the ecliptic, the longitude of the ascending node, and the time of the Sun's revolution on its axis.

10. When a planet is stationary, the horary change in elongation is to the horary change in annual parallax as the periodic time of the planet to the periodic time of the Earth.

11. Find the shape and dimensions of the Moon's penumbra.

12. Explain how the Sun's parallax may be found by the transit of the inferior planets.

13. By the effect of aberration, a Star's latitude is least when the Sun's place is 90 degrees before the Star's place in the ecliptic, and from that time the increment in latitude varies as the versed sine of the Sun's longitude, reckoned from that point.

14. The longitude is least when the Sun is in syzygy with the

Star, and from that time the increment in longitude will vary as the versed sine of the Sun's longitude reckoned from that point.

15. If a Star be situated very near the pole of the ecliptic, the decrement of the apparent longitude varies as the versed sine of the Sun's longitude, reckoned from a point 90 degrees behind the Star's place in the ecliptic.

TRINITY COLLEGE.

1820.

ASTRONOMY.

1. Explain the construction and use of the Vernier. Within what limits may angles be read off by an instrument of which the arc is subdivided to 20′, and 20 divisions of the Vernier are equal to 19 of the arc?

2. Explain the mode of correcting a small error in the meridian plane by observations made with a transit-instrument on a circumpolar star. Supposing the time between the lower and upper transit $= T$, and between the upper and lower $= T + t$, work out the proper correction.

3. Determine the obliquity of the ecliptic by meridian altitudes, taken on successive days before and after the solstice, and apply the proper corrections.

4. Explain the mode of determining the lengths of a sidereal and solar year.

5. Assuming the length of a solar year to be $365d.\ 5h.\ 48'\ 48''$, determine the correction of the civil year, in order that it may always nearly coincide with the solar.

6. Under what circumstances is a star said to rise or set cosmically, achronically, and heliacally?

7. "Hesiod says that sixty days after the winter solstice the star Arcturus rose at sun-set: from which it follows that Hesiod lived about 100 years after the death of Solomon." (Sir Isaac Newton's Chronology). Exhibit the calculations on which this conclusion is founded.

8. When the Sun is in either of the equinoctial points, determine the locus of the extremities of the shadow of a perpendicular style on a horizontal plane.

9. When the Sun is in either of the equinoctial points, and the

style of the dial is perpendicular to the horizontal plane on which the hour-lines are drawn, determine the construction of the dial.

10. Upon what experiments does it appear that in the passage of a ray of light through a variable medium like our atmosphere, the sine of incidence is to the sine of refraction in a constant ratio?

11. Explain the mode by which Bradley obtained the following formula:

$$\text{Refraction} = \frac{a}{29.6} \times \tan.\,(z - 3r) \times 57'' \times \frac{400}{350 + h}$$

$a =$ altitude of the barometer in inches
$29.6 =$ mean altitude of do.
$z =$ zenith distance
$r = 57'' \times \tan z$
$h =$ height of Fahrenheit's thermometer in inches.

12. Define parallax, and determine the law of its variation for the same body at different altitudes.

13. Explain the mode by which (a) the effect of parallax in right ascension may be observed, and prove that the horizontal

$$\text{parallax} = \frac{15 \times a \times \text{cos. dec}^{\text{n}}.}{\text{cos. lat.} \times \text{sin. hour-angle}}.$$

14. If the velocity of the Earth be in a finite ratio to the velocity of light; (1) find the direction in which a telescope must be held, in order that a given heavenly object may appear in its axis: (2) On the same hypothesis shew that a ray of light proceeding from a heavenly body must strike the retina at a different point from what it would do if the eye of the spectator were at rest; and therefore, by the laws of vision, that the apparent place will differ from the true place of the body. Shew that the quantity and direction of aberration are the same on either of the two preceding considerations.

15. Write down the expressions for the aberration in latitude and longitude, and determine for any given star, when the corrections are positive, and when negative.

16. Bradley observed,

(1) "That each star was farthest north when it came to the meridian about six o'clock in the evening, and farthest south when it came about six in the morning."

(2) "That in both stars the apparent difference of declination from the maxima, was always nearly proportional to the versed sine of the Sun's distance from the equinoctial points."

Confirm these observations, and shew that they only apply to stars situated near the solstitial colure.

17. Prove that in elliptical orbits of small eccentricity, the greatest equation to the centre is twice the eccentricity.

18. Explain the mode of correctly determining the longitude of

the Earth's apogee: and state at what era in the history of mankind the line of the apsides coincided with the line of the equinoxes.

19. Given the place of a planet as seen from the Earth, find its place as seen from the Sun, exhibiting the formulas of heliocentric latitude and longitude.

20. Account for the Moon's vibration in longitude.

21. Find the lunar and solar ecliptic limits; and thence determine the greatest and least number of eclipses, of either kind, that can happen in one year.

22. Suppose the Moon's right ascension to be exactly known, when the Sun is on the meridian; determine when the Moon's centre will be on the meridian.

23. Determine the difference of the longitudes of two places on the Earth's surface, by observations on the passage of the Moon's centre over the meridian.

24. If a small error be made in the assumed distance between two meridians, shew how that error may be corrected by observations on the occultation of a fixed star.

TRINITY COLLEGE.

1820.

NEWTON AND CONICS.

1. Explain by short examples, the method of exhaustions, of indivisibles, and of prime and ultimate ratios.

2. Prove that if a radius vector be drawn bisecting any arc, it must ultimately bisect the chord.

3. If a straight line EDA make with the curve CBA a given angle at the point A, and the ordinates CE, BD be drawn; the triangles ACE ABD, are ultimately in the duplicate ratio of the sides.

4. Let AB be the subtense of the arc, AD the tangent, BD the subtense of the angle of contact perpendicular to the tangent, as in the 11th lemma: then let a series of curves be drawn in which $DB \propto AD^4$, AD^5, AD^6, &c., the angle of contact in each succeeding case will be infinitely less than in the preceding.

5. If the areas described by the radius vector are not propor-

tional to the times, the revolving body is not acted on solely by a force towards a fixed centre.

6. If a body be acted on by a given force and revolve in a circle, the arc described in any given time is a mean proportional between the diameter of the circle and the space through which a body would descend in the same time from rest if acted on by the same force.

7. The velocity at any point of a curve described round a centre of force = the velocity which a body, acted on by the given force at that point, would acquire by descending through $\frac{1}{4}$ part of the chord of curvature.

8. Given the force of gravity = 32 feet, and the radius of the earth = 4000 miles; deduce a numerical comparison between the force of gravity and the centrifugal force at the equator.

9. If a heavy body be whirled round in a vertical plane, and the centrifugal force at the top just keep the string extended; what will be the tension of the string at the lowest point of rotation?

10. In any orbit, let x = dist. p = perpendicular on the tangent: centripetal force $\propto \frac{dp}{p^3 dx}$. Apply this expression to determine the law of the force in an ellipse round the centre, and in a circle with the centre of force in the circumference.

11. Deduce expressions for the chord of curvature passing through the focus, and the diameter of the curvature at any point of an ellipse.

12. All parallelograms described about any conjugate diameters of a given ellipse or hyperbola are of equal area.

13. Compare the centripetal and centrifugal forces at any point of an orbit; prove that in an ellipse round the centre, there are four points where these forces are equal.

14. Prove (Newton, Prop. XI.) that $Gv \times vP : Qv^2 :: CP^2 : CD^2$.

15. The perpendicular from the focus of a parabola upon the tangent is a mean proportional between the focal distances of the point of contact and the vertex.

16. Prove that the force tending to the focus of a parabola $\propto \frac{1}{D^2}$.

17. The velocity of a body revolving in a parabola round the focus = the velocity of a body revolving in a circle at half the distance.

18. If two bodies revolve in an ellipse in the same periodic time; one about the focus, and the other about the centre; compare the forces towards these centres at the extremities of the major axis, and find the distance from the centres at which the forces are equal.

19. If the force $\propto \frac{1}{D^2}$ and a body be projected in any direction, except directly to or from the centre of force; prove that it will describe a conic section, and point out the relation between the velocity of projection and the particular curve described.

TRINITY COLLEGE.

1820.

NEWTON'S PRINCIPIA. Book I.

---

1. (1) The centripetal force (F) in any curve $= Q.\frac{dp}{p^3dx}$, (p) being the perpendicular from the centre of force on the tangent, at distance (x). Determine Q.
   (2) Find the value of (F) in the ellipse—the force tending to the centre.

2. If a body be acted on by two forces tending to two fixed centres, it will describe, about the straight line joining those centres, equal solids in equal times.

3. A body describes a parabola about a centre of force situated in the focus:
   (1) Find its position at any assigned time.
   (2) Given two distances from the focus, and the difference of anomalies. Find the true anomaly.

4. The time of a body's descent, in a right line, towards a given centre of force varies as $\frac{1}{(\text{dist.})^2}$ from that centre. Required the law of the variation of the force.

5. A body at P is urged by an uniformly-accelerating force in the direction PS, and at the same time is impelled in the opposite direction by a force varying as $\frac{1}{(\text{dist.})}$ from S. Find its velocity at any point N.

6. In the logarithmic spiral find an expression for the time of a body's descent from a given point to the centre, and prove that the times of *successive* revolutions are in geometrical progression.

7. A body acted on by a force varying as $\frac{1}{(\text{dist.})^n}$ from the centre, is projected from a given point, in a given direction, and with a given velocity.
   (1) Find the equation to the trajectory described.
   (2) Determine in what cases the body will fall into the centre, or go off to infinity.

8. The force varying as $\frac{1}{(\text{dist.})^5}$, shew under what restrictions of the velocity of projection, the body's approach towards the

centre, and its recess towards infinity, will be limited by *asymtotic circles.*

9. The difference of the forces by which a body may be made to move in the quiescent and in the moveable orbit varies as $\frac{1}{(\text{dist.})^3}$ from the centre.

10. (1) Deduce the equation to the orbit in fixed space.
(2) Shew that when any one of Cotes's three last spirals is made the moveable orbit, the orbit in fixed space will be one of the same species.

11. Why are the principles of the 9th section inapplicable to the *complete* explanation of the planetary motions?

12. Make a body oscillate in a given hypocycloid.

13. Given the position of a body on a rigid logarithmic spiral which it is made to describe by a force varying as $\frac{1}{(\text{dist.})^3}$ from the pole, find
(1) The point where the body will leave the spiral.
(2) The time of arriving at that point.
(3) The elements of the orbit which it will then describe.

14. Demonstrate the 66th Proposition.

15. Find expressions for the disturbing forces on P when at a given distance from quadratures.

16. If ST and the absolute force of S be changed, the periodic linear errors of P vary as $\frac{1}{(\text{Period of T})^2}$. Pr. 66. Cor. 14.

17. Prove that the mean disturbing force on the moon in a whole revolution $= -\frac{1}{2}$ the mean addititious force.

18. When the force varies as the (dist.), shew that there will be no disturbance.
(1) Had this law pervaded the universe, what would have been the consequence?
(2) On what circumstances in the *variations* of the elements of the orbits does the stability of the planetary system depend?

19. S is the centre, SA the radius of a sphere, each of whose particles has an attractive force varying as $\frac{1}{(\text{dist.})^n}$. Having assumed in SA produced any point P, and having taken SP : SA :: SA : SI, find the ratio of the attractions which the whole sphere exerts on equal corpuscles placed at P and I.

20. The attractions of ellipsoids upon particles placed on the surface urging them in directions perpendicular to any principal section are proportional to the distances of the particles from that section.

21. Prove that a shell of homogeneous matter contained between

two concentric spherical surfaces, will attract a particle placed without it in the same manner as if all the matter were collected in the centre, in the following cases:

(1) When the law of attraction is that which obtains in nature.
(2) When it varies as the distance.
(3) When it $= Ax + \frac{B}{x^2}$, (x) being the distance.

ST. JOHN'S COLLEGE.

JUNE, 1820.

NEWTON.

---

1. EXPLAIN what is meant by continued finite curvature. Shew that if QP be any arc of a curve, and QR a subtense perpendicular to the tangent, limit $\frac{QP^2}{QR}$ = diameter of curvature at the point P; and apply this expression for finding the diameter of curvature at the vertex of a cycloid.

2. Let AB be any arc of a curve of finite curvature, AK, BK normals at A and B meeting in K, and BG perpendicular to the chord AB meeting AK in G; prove that in the limit AK : AG :: 1 : 2.

3. Investigate the relation between the centripetal and centrifugal forces at any point in any orbit: the equation to the curve in which they are equal; and the law of the force by which it will be described.

4. If the force $\propto \frac{1}{D^2}$ and a body descend in a straight line; find the velocity and time corresponding to any given space by Newton, Prop. 39. Cor. 2, 3.

5. If a body be projected in any direction from a given point above a given plane, and be acted upon by a force perpendicular to the plane, and varying inversely as the nth power of the distance from it; find the equation to the trajectory, and shew for what values of n, that equation will be expressed in finite terms.

6. A body begins to fall from A to a centre of force S, varying inversely as the cube of the distance; find the nature of the curve

AP, when the time down AN is equal to the time of describing NP with the velocity acquired at N.

7. Find generally the equation to the orbit in fixed space in Sec. 9, and from that equation, shew that the difference of force in the fixed and moveable orbit varies as $\frac{1}{D^3}$.

8. If the force vary as A^n, shew that the angle between the apsides in orbits nearly circular $= \frac{\pi}{\sqrt{n+3}}$ *nearly*, and when $n = 1$ explain the reason why we obtain an accurate result.

9. Find the nature of the curve which by its rotation round its axis will generate a surface, in which the times of revolution in circles parallel to the horizon shall be equal at all altitudes.

10. If a string will just bear (p) pounds; through what angle must it be made to oscillate with a weight (q) less than (p) at its extremity so that it may all but break?

11. Investigate an expression for the tangential force in P's orbit supposed circular, and find the velocity generated by it from quadrature to syzygy.

12. Find those positions of the apse of P's orbit where the excentricity is a maximum and minimum, and explain fully Newton's reasoning in Cor. 9. Pr. 66.

QUEEN'S COLLEGE.

MAY, 1819.

MISCELLANEOUS PROBLEMS.

1. GIVE a definition of force; and distinguish between accelerating and moving force.

2. The sum of two forces is to the compound force, when they act at an angle (θ) as (m) to (n), shew that the angle (ι), which the compound force makes with either of the other two, may be determined from the equation

$$\cos.\left(\iota - \frac{\theta}{2}\right) = \frac{m}{n} \cos. \frac{\theta}{2}.$$

3. Two weights P and Q are supported upon two inclined planes AB, BC, of given elevation, by a string passing over a pulley at D.

Find the position of equilibrium, and shew that if the system be put in motion P : Q :: velʸ of Q : velʸ of P; the velocities being measured in a direction perpendicular to the horizon.

4. AD is horizontal, DC vertical, Q a weight connected with one extremity of a beam AB, by a string passing over a pulley at C, in such a manner that CB is vertical. Find Q when there is an equilibrium, and shew that the equilibrium will be maintained, whatever be the position of the beam AB; CB remaining vertical.

5. AB is a vertical line. Find the nature of the curve traced out by C, such, that the square of the time down AC + the square of the velʸ down AC is a constant quantity.

6. P and Q are two weights connected by a string passing over a fixed pulley, whereof P is the greater: at the end of (t'') an additional weight (q) is annexed to Q. Find the velocity of P, after any assigned time.

7. If a ball whose elasticity is to perfect elasticity as (n) to (1), impinge upon a perfectly hard plane; shew that tan. I : tan. R :: n : 1, I and R being the angles of incidence and reflexion.

8. If (θ) be the angular distance of a body from the lowest point in a circular arc; shew that the force in the direction of the arc is to that in the direction of the chord as 2 cos. $\frac{\theta}{2}$: 1.

9. If a ball whose elasticity is to perfect elasticity as (n) to (1) be projected in vacuo at an angle (θ) with velʸ (V); prove that the sum of all the ranges will be expressed by $\frac{V^2 \sin. 2\theta}{g \cdot (1-n)}$ g being taken to represent gravity, or $32\frac{1}{6}$ feet.

---

1. Divide a given cylinder, containing two fluids of different specific gravities, which will not mix, into two parts, such that the pressures on each shall be equal, the depth of each fluid being given.

2. If a plane be immersed vertically in a fluid of which the density varies as the nth power of the depth; find the pressure upon the whole plane.

3. Having given the specific gravities of wood and water, find the specific gravity of brass when (a) cubic inches of brass connected with (b) cubic inches of wood will just float.

4. Explain the reason why a cannon ball will fly farther if a motion of rotation be impressed upon it in a direction perpendicular to the line of its motion, than if that rotatory motion took place in the direction of its motion.

5. How long will a cylinder be in emptying itself by means of two orifices of given dimensions; one in the bottom, and another at a given distance from the bottom.

6. Find the diameter of a capillary tube.

7. If altitudes be taken above the earth's surface in arithmetical progression, prove that the altitude of the barometrical column will decrease in geometrical progression.

8. There are two barometers, in one of which was left (m) inches, and in the other (n) inches of air. In consequence of a change in the state of the atmosphere, the former fell (d) inches. Find the corresponding variation in the latter.

9. If a body descend in a medium whereof the resistance varies as the (vel^{y}); find the space described, corresponding to any assigned time.

---

1. A ray of light incident upon the concave surface of a spherical reflector, in a direction parallel to the axis, after two reflections intersects the incident ray in a given point; find the angle of incidence.

2. In what point of the horizontal line AB will a given line (C), in the same plane with AB, appear the greatest?

3. If (R) and (r) be the radii of the surfaces of a double convex lens, what alteration must be made in (r), so that the focal length may be increased (p) times?

4. Find the focal length of a double concave lens from these data:—the distance of the image from the lens, and the ratio of the object to the image.

5. A cylindrical vessel of given dimensions is so situated, that an eye can just see the farther extremity of the diameter. How much water must be poured in, so that the eye may just see a (cth) part of it?

6. If a quadrant be immersed vertically in a fluid, one of its radii being coincident with the surface; find the equation to the image.

7. Determine the vertical angle of an isoceles glass prism, such that rays incident perpendicular to one side may just be reflected by the base perpendicular to the other.

8. If the Sun's light be transmitted through a prism, a coloured image of the Sun will be formed on the opposite wall; and if the prism be turned round its axis, this image will ascend and descend alternately. Shew that the refractions on each side of the refracting angle are equal at the instant the image ceases to ascend, and begins to descend.

TRINITY COLLEGE.

1820.

PROBLEMS.

1. What sum of money must be laid out in the 3 per cent. consols at $63\frac{3}{8}$ per cent. to produce an income of £400 a year?

2. The sine of any angle of a plane triangle has to the opposite side a constant ratio.—What is this ratio?

3. Find by the method of continued fractions a series of fractions converging to $\sqrt{19}$.

4. Given the three sides of a plane triangle, find

(1) Its area,

(2) The radius of the inscribed circle.

5. Differentiate the following quantities,

$$(1)\quad u = 1\sqrt{\frac{1+\sin. x}{1-\sin. x}};$$

$$(2)\quad u = \cfrac{x^2}{1+\cfrac{x^2}{1+\ \&c.\ ad\ inf.}}$$

6. Find a number which being divided by 2, 3, 5, shall leave for remainders 1, 2, 3, respectively.

7. The angles of any plane triangle being A, B, C, prove that [to radius (1),]

$$4 \sin. A . \sin. B . \sin. C = \sin. 2A + \sin. 2B + \sin. 2C.$$

8. (1) Find the locus of the vertices of all the triangles described on the same base, when one of the angles at the base is always double of the other.

(2) Hence trisect a given angle.

9. Find the radius of curvature at any point of the common cycloid.

10. In any spherical triangle, the arcs of great circles drawn from the three angular points perpendicular to the opposite sides intersect in the same point.

11. Sum the following series:

$$(1)\quad \frac{5}{1.2.3.4} + \frac{7}{2.3.4.5} + \frac{9}{3.4.5.6} + \ldots \text{to}(n)\text{terms}.$$

$$(2)\quad \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots \text{ad inf.}$$

(3) $\dfrac{1}{1.2.3} + \dfrac{1}{2.3.4} \cdot \dfrac{1}{2} + \dfrac{1}{3.4.5} \cdot \dfrac{1}{2^2} + \ldots$ ad inf.

(4) $\text{Sin.}\,\theta + x \sin. 2\theta + x^2 \sin. 3\theta + \ldots\ldots\ldots$ ad inf.

12. A sphere of given diameter descends in a fluid, from rest, by the action of gravity; find the greatest velocity it can acquire, its specific gravity being (n) times that of the fluid.

13. (1) Of all quadrilateral figures contained by four given right lines the greatest is that which is inscriptible in a circle.

(2) If a, b, c, d, be the sides of this quadrilateral, S its semi-perimeter, shew that its

$$\text{area} = \sqrt{\{(S-a)(S-b)(S-c)(S-d)\}}.$$

14. Find the centre of gyration of a given sphere.

15. Any two right lines intersect each other in space; having given their separate inclinations to three rectangular co-ordinates passing through the point of intersection: find their inclinations to each other.

16. (1) Trace the curve whose equation is $y^2(c-x) = x^3 + bx^2$, and find its area when $b = 0$.

(2) The equation to a curve is $y^3 - axy + x^3 = 0$; find the value of the ordinate when a maximum, and the corresponding value of the abscissa. Shew also that it *is* a maximum and not a minimum.

17. State the *principle of virtual velocities*; and hence shew that if any system in equilibrium, acted on by gravity alone, have an indefinitely small motion communicated to its parts, its centre of gravity will neither ascend nor descend.

18. Integrate (1) $\dfrac{dx}{\sqrt{A + Bx + Cx^2}}$. (2) $\dfrac{dx}{1 + x^3}$. (3) $\dfrac{d\theta}{(\cos.\theta)^2}$.

and find the relation of (x) to (y) in the equations

(1) $xdy - ydx = ydx \log. \dfrac{y}{x}$.

(2) $dx + x^3dx = dy + ydx$.

19. If two weights acting upon a wheel and axle put the machine in motion, find the pressure upon the axis without taking into account the machine's inertia.

20. If (a) and (b) denote the semi-axis of an ellipse, (θ) the angle at which the radius of curvature (r) at any point cuts the axis, prove that

$$r = \frac{a^2b^2}{(a^2 \cos.^2 \theta + b^2 \sin.^2 \theta)^{\frac{3}{2}}}.$$

21. The roots of the equation $x^n - px^{n-1} + qx^{n-2} - \&c. = 0$, being α, β, γ, &c. find the value as $\alpha^m + \beta^m + \gamma^m + \ldots\ldots$ in terms of the coefficients p, q, r, &c.

22. AP is any arc of a parabola whose vertex is A and focus S;

let N be the intersection of a perpendicular from S on the tangent at P with the perpendicular to the axis from A.

Then if AS $= a$, $\angle$ASN $= \phi$.

Shew that arc AP $-$ PN $= a \,.\, \text{l. tan.} \left(\frac{\pi}{4} + \frac{\phi}{2}\right)$.

23. If a circle whose diameter is equal to the whole tide in any given latitude be placed vertically, and so as to have the lower extremity of its diameter coincident with the level of low water, prove that the tide will rise or fall over equal arcs in equal times.

TRINITY COLLEGE.

FOR SCHOLARSHIPS.

---

1. Find the value of £3.869, and 365.24215 days.

——— square root of 0.676, and 6.76.

——— amount of $\frac{2}{3}$, $\frac{1}{5}$, $\frac{9}{7}$, and $25\frac{6}{11}$.

2. If twenty men in digging a canal must pump out six tons of water daily, in order to excavate 160 cubic yards in a week, how many cubic yards can thirty men excavate in a week, supposing them to be obliged to pump out eight tons of water daily?

3. If S denote the sum of the even terms, and S′ the sum of the odd terms, of the expansion of $(a+b)^n$; then $S^2 \backsim S'^2 = (a^2 \backsim b^2)^n$.

4. Solve the equations, $\frac{6}{x+1} + \frac{2}{x} = 3$ (A)

$x^3 - 12x + 6 = 0$ (B)

$x^3 - 2x^2 - 13x + 20 = 0$ (C)

$x^4 - 4x^3 + 5x^2 - 4x + 1 = 0$ (D)

5. Find the roots of an equation of this form by construction, viz. $x^3 - 3r^2x - q = 0$.

6. Compute the numerical value of the side of a regular decagon inscribed in a circle, whose radius is ten inches.

7. The three angles of a plane triangle being given, and the distances of the three angular points from a given point within the triangle, to find the sides.

8. The complement of the hypothenuse of a right-angled spherical triangle cannot exceed the complement of either of the other sides.

9. If about the three angular points of a spherical triangle three great circles be described, the triangle formed by these latter will have its sides measures of the angles of the original triangle, or of the supplements of those angles.

10. In a given parabola, to draw a diameter which shall make with its ordinates an angle equal to a given rectilineal angle.

11. Two vertical straight lines being given, to place them at such a distance asunder in the same horizontal plane, that a heavy body shall be as long in falling down the greater, and then moving with its acquired velocity to the less, as it would be in falling down the less vertical, and moving with its acquired velocity to the greater.

12. When a radiant sphere shines upon an opaque sphere, the breadth of the illuminating portion of the former has for its measure the same number of degrees as the dark portion of the latter.

13. The distance being given to which a fluid spouts from a given orifice in the side of a cylindrical vessel, to find by a geometrical construction the height of the fluid's surface in the vessel.

14. The right ascensions and declinations of three stars being given, and the times between their passages over the same vertical wire of a telescope, to find the latitude of the place; one of the stars being supposed in the equator.

15. The altitude of the Sun when due west, and also at six o'clock P.M. being given, find the latitude and the Sun's declination.

16. Having the focus of incident rays upon a medium terminated by two plane sides inclined at a given angle, find the focus of emergent rays.

17. The exact quantity of the year being 365.24215 days, explain the reason of the corrections in the civil year introduced by Julius Cæsar and Pope Gregory.

18. Give the theory of the Trade Winds.

19. Prove that part of the equation of time which arises from the obliquity of the ecliptic to be a maximum when the longitude of the Sun equals the complement of its right ascension.

20. Compare the surface of a sphere with the area of its great circle, and its magnitude with that of its circumscribing cylinder.

TRINITY COLLEGE.

FOR SCHOLARSHIPS.

---

1. To find the locus of the extremities of all the straight lines that can be drawn from the circumference of a given circle, toward the same parts, each of them equal and parallel to a given finite straight line.

2. To find the centre of a given ellipse.

3. To construct the curve of which the equation is $ax^2 + ay^2 + bx + cy + d = 0$.

4. If the product of any two given numbers be a square, each of the two given numbers is the product of two factors, such that the four factors are proportionals.

5. Solve the following equations:

(A.) $\frac{12+2x}{x+3} + \frac{4x-3}{2x+1} - \frac{4x-1}{x-1} = 0$.

(B.) $a^{n-\frac{r}{s}} - b^{m-p} \times c^{qn} = 0$.

(C.) $2xy+x+y-195 = 0$, (to find the integral values of x and y).

(D.) $\left\{\begin{matrix} y^2 - x^2 - 90000 = 0 \\ yx - 300y - 125x = 0 \end{matrix}\right\}$.

6. If none of the coefficients of the equation $x^n+ax^{n-1}+bx^{n-2}+$ &c. $+ g = 0$ be fractional, it cannot have a fractional root.

7. To compare the chance of throwing 7, with the chance of throwing 8, at one throw, with three common dice.

8. Sum the following series:

(A.) $1+3+27+64+$ &c. (to n terms.)

(B.) $\frac{1}{3} + \frac{4}{9} + \frac{9}{27} + \frac{16}{81} +$ &c. (*ad infinitum.*)

(C.) $\frac{5}{1.2.3.4} + \frac{7}{2.3.4.5} + \frac{9}{3.4.5.6} +$ &c. (to n terms.)

9. (A.) Find the fluent of $\frac{x\dot{x}}{\sqrt{(x^2-a^2).(b^2-x^2)}}$; of $\frac{\dot{z}}{x^2\sqrt{a^2-v^2}}$, z being a circular arch, of which x is the cosine, and v the tangent; and of $\frac{x^{\frac{1}{2}}\dot{x}}{a+bx^2}$.

(B.) Solve the fluxional equation $x\dot{y}^2 - y\dot{x}\dot{y} - \frac{y^2}{a}\dot{x} = 0$.

10. If A + B be less than a semicircle,

$$\sin.\frac{A \backsim B}{2} : \frac{\text{ver. sin. } A \backsim \text{ver. sin. } B}{2} :: \text{rad.} : \sin.\frac{A+B}{2}.$$

11. If P be put for the semi-perimeter of a spherical triangle, the sides of which are denoted by a, b, c, and the opposite angles by A, B, C,

$$\cos.^2 \tfrac{1}{2} A = \frac{\sin. P \sin. (P - a)}{\sin. b \sin. c}.$$

12. The upper extremity of an inclined plane being given, to determine its position, so that the time shall be a minimum, in which a body falls down it, and afterwards moves to a given point in the horizontal plane, with that part of its acquired velocity, which is not destroyed by its impact on the horizontal plane.

13. A given sphere, and its circumscribing cylinder, of the same uniform density, being supposed to revolve round their axes, with equal angular velocities, to compare their momenta.

14. A hollow sphere is to be formed of a substance, the specific gravity of which is greater than that of air, in the ratio of n to 1, and is afterwards to be filled with gas, the specific gravity of which is less than that of air, in the ratio of 1 to m; the thickness of the shell being given, to find its diameter so that it may float in the air.

15. To describe the construction, and determine the magnifying power of a Compound Microscope.

16. To describe the construction of an Achromatic Lens, and explain the reasons of that construction.

17. To determine the Sun's parallax, from observations made on the transit of Venus.

18. The times of a star's transit over the meridian, and over two vertical circles at given distances from the meridian, having been observed, to compute the latitude of the place of observation, in terms of the azimuths, and hour-angles thus given.

19. To determine under what circumstances of the velocity of projection, a body, projected from a given point, in a given direction, and acted upon by a force inversely proportional to the nth power of the distance from the centre, will come to the centre, or to an apse.

20. If a body, acted upon by the constant force of gravity, fall down the concave side of a circular arch, the tangent of which, where the body begins to fall, is perpendicular to the horizon, to find the point where the pressure on the curve shall be equal to an nth part of the weight of the body.

TRINITY COLLEGE.

FOR FELLOWSHIPS.

---

1. SOLD one-half of a bale of goods for £50. and by so doing *lost* five per cent. At what price must the other half be sold, so that five per cent. shall be *gained* on the whole?

2. Solve the equation $\frac{x^2}{2} - \frac{x}{3} + 20\frac{1}{2} = 42\frac{2}{3}$.

3. Similar triangles are in the duplicate ratio of their homologous sides.

4. Let α, β, γ, be the respective angles, which a line makes with three rectangular co-ordinates; shew that these angles are so connected that

$$\cos.^2 \alpha + \cos.^2 \beta + \cos.^2 \gamma = 1.$$

5. In a plane triangle, whose angles are A, B, C, and the opposite sides a, b, c, the angle A may be expressed by the series

$$m \,.\, \sin.\, C + \frac{m^2}{2} \sin.\, 2C + \frac{m_3}{3} \,.\, \sin.\, 3C, \, \&c.$$

m being the value of the fraction $\frac{a}{b}$.

6. Find an expression for the excess of the sum of the three angles of a spherical triangle above two right angles.

7. In the revolution of a body round the focus of an ellipse of small excentricity, show that the supposition of the angular velocity round the other focus being uniform is not strictly accurate.

8. Find the centre of gravity of the surface generated by a common semicycloid revolving round the tangent at the vertex as an axis.

9. Find the sum of series $\frac{1}{r} + \frac{4}{r^2} + \frac{9}{r^3} + \frac{16}{r^4}$ to n terms; and of the series $\frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} -$ &c. *ad infinitum.*

10. Find the fluent of $\frac{\dot{x}}{\sqrt{A + Bx - Cx^2}}$. And solve the equation

$$x\dot{x} + y\dot{y} - y\dot{x} + x\dot{y} = 0.$$

11. AP is a portion of the common parabola, PT a tangent at the point P, PB the normal, and TO a perpendicular to the axis

from the intersection of the tangent. Show that if BP be produced to meet TO in O, BO is equal to the radius of curvature at the point P.

12. In throwing *cross* and *pile*, it is known that the piece has a tendency one way, estimated at $\frac{1}{20}$. Required the probability of throwing *cross* twice together.

13. Find the angle between the true and apparent path of a Star affected by refraction during the time of an observation.

14. Find the position, in which a given isosceles prism will float in a fluid, the vertex being immersed.

15. The oscillations of a pendulum in a medium when the resistance varies as the velocity are isochronous. Compare the time of an oscillation with the time of an oscillation of the same pendulum *in vacuo*.

16. Construct the curve whose equation if $y^4 - 96\,a^2y^2 + 100\,a^2x^2 - x^4 = 0$. Show whether there be any points of inflexion, or any asymptotes, and find the greatest values of x and y.

17. Trisect a circular arc, and construct the equation either by the conchoid of Nicomedes, or by means of the hyperbola.

18. Show that the sine of incidence is to the sine of refraction in a given ratio, both in Huygens' hypothesis of Undulations, and in Newton's of Material Particles attracted by the medium through which the rays of light pass.

Investigate the velocities of the ray before and after refraction on both hypotheses.

19. Explain the method of Exhaustions, by which the ancients were enabled to measure the periphery of the circle.

20. Find the horary motion of the Moon's nodes in a circular orbit, (Newton, B. III. Prop. 30.)

TRINITY COLLEGE.

1819.

FOR FELLOWSHIPS.

---

1. A Grocer had 150lbs. of tea, of which he sold 50lbs at 9*s*. per pound, and found that he was then gaining only $7\frac{1}{2}$ per cent. But he wished to gain 10 per cent. on the whole. At what rate must the remaining 100lbs. be sold that he may attain his wishes?

2. Solve the following equations:

$$(1.)\quad \left.\begin{aligned} x + y + \frac{y^2}{x} &= 20 \\ x^2 + y^2 + \frac{y^4}{x^2} &= 140 \end{aligned}\right\}$$

$$(2.)\quad \left.\begin{aligned} x - \sqrt{x} &= 3 - y \\ y - \sqrt{y} &= 4 - x \end{aligned}\right\}$$

(3.) $x^4 - 5x^3 + 7x^2 - 5x - 6 = 0.$

(4.) $x^x = a$ (by approximation).

3. In what time will a sum of money placed at compound interest double itself at $4\frac{1}{2}$ per cent.?

4. Required the value of the infinite series $\sqrt[4]{18 + 189\sqrt[4]{18 + 189\sqrt[4]{18 + \&c.}}}$

5. Required the value of y in the following equations, when $x = 1$.

(1.) $y = (1 - x) . \tan. \frac{\pi x}{2}$ (π = semi-circumference of a circle, rad. $= 1$).

(2.) $y = \frac{x^x - x}{1 - x + \log. x}$.

6. Transform the equation $x^3 - 5x^2 + 8x - 5 = 0$ into one whose roots are the squares of the differences of every pair of roots: and show the mode of determining, from the transformed equation, the impossible roots in the original equation.

7. From a bag containing four white and eight black balls, three persons (A, B, and C.) take each a ball in turn, *viz.* A first, then B, then C, and so on in succession, until the person, who first draws a white ball, wins. What are their respective chances?

8. In a spherical triangle, the two sides and the angle between them being given, find the base.

9. The vertical angle of an isosceles spherical triangle is always greater than the angle included between the chords of the equal sides.

10. Sum the following series:

(1.) $2 + 6 + 12 + 20$ to n terms.

(2.) $\frac{1}{1.2} + \frac{1}{4.5} + \frac{1}{7.8} + \frac{1}{10.11}$. &c. *ad infin.*

(3.) $\text{Cos.} A + \frac{1}{2} \cos 2A + \frac{1}{3} \cos. 3A +$ &c. to n terms.

11. Sum the series, $\sin. \phi \times \sin. \theta + \sin. 2\phi \times \sin. 2\theta$ &c. to n

terms: and show that when n is infinite, the sum $=0$ whatever be the ratio of ϕ to θ, except that of equality.

12. A body attracted towards a centre by a force varying inversely as the square of the distance from the centre, meets at a given point of its rectilinear descent with a plane inclined at an angle of 45°. Required the time from the beginning of motion to its reaching the centre.

13. A cylindrical wheel, whose weight is P, unwinds itself from a string passing round its circumference, what weight (W) attached to the extremity of the string will be kept at rest on a plane of given elevation as P descends vertically?

14. A sphere and its circumscribing cylinder revolve round their common axis. Required the ratio of the *momenta* generated in a given time.

15. An homogeneous circular wheel vibrates edgeways being suspended from a point in the circumference. Required its centre of oscillation.

16. Find the centre of gravity of the area included by the arc of a cycloid, by a tangent at the vertex, and by two rectangular ordinates equi-distant from the vertex.

17. In the common parabola the radius of curvature is equal to the cube of the normal divided by the square of the semi-parameter.

18. Trace the curve whose equation is $y = \pm \dfrac{ax - x^2}{\sqrt{2ax - x^2}}$, and find the angles at which it cuts the line of the abscissæ.

19. O is the centre of the circular arc AB, OBT is the secant. The exterior part BT is continually bisected in P. Required the area traced out by OP.

20. Two balls (A and B) are previous to motion at a given distance from each other in the same vertical line: from what height above the horizontal plane must A be let fall—so that B, which is perfectly elastic, may after reflection meet A at a given distance above the plane?

21. Two balls lying on an horizontal plane are connected by a string of unlimited length which passes through a ring in the plane. One of the balls is projected in a given direction with a given velocity and draws the other towards the ring. Required the curve which the projected ball describes.

22. Find the following fluents:

$$\int \dot{z} . \cos. z \;.\; \cos. nz, \quad \int \dot{z} \;.\; \cos.^3 z \quad \int \frac{\dot{x}}{\sqrt{1 - \varepsilon^{-x}}},$$

construct the fluent $\displaystyle\int \frac{\dot{z}}{z + 2az^3 + z^5}$, when a is less than 1;

and find the relation of x to y in the equation

$$x\dot{y} - y\dot{x} = \dot{x}\sqrt{x^2 - y^2}.$$

23. In any curve referred to an axis, the ordinate is a maximum or a minimum, when in the equation $y = fx$, an odd number of differential coefficients becoming $= 0$ the differential coefficient of the succeeding order is negative or positive. And there is a point of contrary flexure when an even number of differential coefficients becoming $= 0$, the differential coefficient of the succeeding order is real and finite.

24. A body urged by a constant force in an uniform resisting medium is projected in a direction contrary to the action of the force with a certain velocity; it is required to determine the velocity at any point of the ascent, the resistance being supposed proportional to the square of the velocity. Find also the greatest height to which the body will ascend.

TRINITY COLLEGE.

FOR FELLOWSHIPS.

1. In a plane triangle the vertical angle, the perpendicular and the rectangle under the segments of the base being given, it is required to construct the triangle.

2. Solve the equation $x^3 - \frac{4x^2}{3} - \frac{17x}{3} + 2 = 0$, two roots being of the form a and $\frac{1}{a}$.—And find the number of all the possible values in integer numbers of x, y and z in the equation

$$5x + 7y + 11z = 224.$$

3. What are the dimensions of the strongest rectangular beam, that can be made out of a given cylinder, when placed to the most advantage? and what is its strength, compared with that of the greatest square beam cut out of the same cylinder?

4. In the wheel and axle (the inertia of which may be neglected) required the ratio between the radii, when a weight (Φ) acting at the circumference of the wheel generates in a given time the greatest momentum in a weight (W) attached to the circumference of the axle.

5. Tangent of half the spherical excess =

$$\frac{\tan.\frac{1}{2}b \times \tan.\frac{1}{2}c \times \sin. A}{1 + \tan.\frac{1}{2}b \times \tan.\frac{1}{2}c \times \cos. A},$$

where b, c, and A are the two sides and included angles of a spherical triangle.

6. The excess of the Sun's longitude above its right ascension may be found by the equation

$$\tan. (L - R.A.) = \frac{\tan.^2\frac{1}{2}\Upsilon \times \sin. 2L.}{1 + \tan.^2\frac{1}{2}\Upsilon \times \cos. 2L}.$$

7. Find an expression from which the effect of parallax upon the horary angle may be accurately calculated; the horizontal parallax, the polar distance of the heavenly body and the time before or after transit, being the only given elements.

8. If an orifice were opened half way to the centre of the earth, what would be the altitude of the mercury in a barometer at the bottom of it, when the altitude at the surface is 30 inches?

9. A vessel formed by the revolution of a parabola round its axis is placed with its vertex downwards, in which there is an orifice one inch in diameter. A stream of water runs into the vessel through a pipe of two inches diameter at the uniform rate of eight feet per second. What will be the greatest quantity of water in the vessel, supposing the latus rectum to be six feet?

10. Find the present worth of the reversion of a freehold estate after the death of a person now sixty years of age, the rate of interest being given?

11. When a ray of light passes out of one medium into another, as the angle of incidence increases, the angle of deviation also increases.

12. To find the least velocity, with which a body projected at a given angle of elevation will not return to the earth's surface.—To find also the latus rectum of the orbit described, and the position of the axis.

13. Supposing the Moon's orbit at present to be circular, what would be the excentricity of it and the periodic time, if the attraction of the earth were diminished $\frac{1}{n}$ part?

14. Find the sum of the following series:

(1.) $\sin. (A)+\sin. (A+B)+\sin. (A+2B)$ &c. *ad infin.*

(2.) $\cos. A + \cos. 3A + \cos. 5A$ *ad infin.*

(3.) $\frac{1^2}{1.2.3.4} + \frac{3^2}{2.3.4.5} + \frac{5^2}{3.4.5.6}$, &c. to n terms,

by the method of increments.

15. Find the following fluents:

(1.) $\int \frac{\dot{x}}{x^3(a+bx^2)}$ (2.) $\int \frac{x\dot{x}}{\sqrt{a+bx+cx^2}}$.

(3.) $\int \sqrt{a^2 - x^2} \times x\dot{x}$, between the values $x = 0$, and $x = a$; and solve the fluctional equation $\frac{(\dot{x}^2 + \dot{y}^2)^{\frac{3}{2}}}{\dot{x}\ddot{y}} = a$.

16. A and B put down equal stakes,—A has m chances of success, and B n chances. There are, moreover, p chances, which entitle both parties to withdraw their stakes,—to find the gain of A.

17. Two equal weights are placed at a given distance from each other on a straight rod supposed to be without weight. Find the point of suspension, so that the pendulum may vibrate seconds.

18. Construct the curve whose equation is $(a - x)^3 \cdot (a + x) = x^2y^2$, and shew whether there are any cusps.

19. Invenire incrementum horarium areæ, quam luna, radio ad terram ducto, in orbe circulari describit.

20. Let the force to a centre vary as the distance, to find all the various curves, along which a body may move, so that its oscillations may be isochronous.

21. Given the diameters of two planets, and the periodic times and distances of their respective satellites; compare their densities and quantities of matter.

INDEX.

INDEX

REFERRING

TO THE SOLUTIONS.

The column marked *p* gives the page in the volume of questions; the column *n* gives the number of the question in that page, and the column *N* refers to its solution, the first number indicating the volume, the second the number in that volume of the solution. When the question is "book-work," or to be found in any Author of established reputation at the University, the column *N* refers to that Author by the following abbreviations:

B. for Bridge	N. for Newton	w. for Wood
C. Coddington	V. Vince	W. N. Wright's Commentary on Newton's Principia.
E. Euclid	W. Woodhouse	

---

p	n	N
p 5	n 3	N 1. 79 1. 80
6	4	W.
	5	2. 324
	6	w.
	7	w.
	8	w.
	9	V.
	10	w.
	11	W. & V.
	12	2. 438
	1	1. 181
	2	2. 616
7	3	2. 372
	4	V. Flux.
	5	2. 694
	6	C. p. 77
	7	Smith
	8	2. 829
	9	2. 830
	10	V. Flux.
	11	1. 681
	12	2. 160
8	13	2. 439
	14	2. 440
	15	2. 126
	16	W. N.
	17	1. 2
	18	2. 441
9	19	2. 552
	20	1. 483
	21	W. N.
	22	2. 65
	23	2. 791 2. 615
	24	2. 553
10	1	2. 270
	2	2. 643
	3	2. 768
	4	2. 831
	5	2. 832
	6	1. 603
	7	2. 161
11	8	2. 435
	9	Cotes
	10	W. N.
	11	2. 1
	12	1. 523 1. 537
p 11	n 13	N w.
	14	W. N.
	1	1. 240
	2	it $=\sqrt{4970.25}$
12	3	$x=\frac{a}{\sqrt{3}}$
	4	1. 132
	5	1. 447
	6	2. 66
	7	W.
	8	2. 271
	9	2. 373
	10	2. 416
	11	2. 695
13	12	V.
	13	2. 770
	14	2. 771
	15	B.
	16	V.
	17	1. 484
	18	1. 683
	19	2. 833
	20	2. 841
	21	w.
14	22	2. 442
	23	2. 443
	24	2. 443
	1	1. 133
	2	1. 266
	3	1. 658
	4	2. 273
15	5	2. 772
	6	2. 696
	7	2. 842
	8	W.
	9	2. 554
	10	1. 351
	11	2. 127
16	12	2. 2
	13	1. 684
	14	1. 316
	15	1. 485
	16	V.
17	17	W. N.
	18	2. 374
	19	2. 773
	20	2. 843
	21	2. 3
	22	1. 794

p 152	*n* 1	*N* 1. 165
	2	2. 846
	3	2. 842
	4	2. 798
153	5	1. 523
	6	2. 733
	7	1. 353
	8	2. 487
	9	2. 837
	10	2. 583
154	11	W.
	12	2. 138
	13	2. 584
	14	N.
155	15	1. 665
	16	1. 615
	17	1. 680
	18	1. 802
	19	1. 652
	20	2. 18
	21	W. N.
156	22	1. 718
	23	2. 284
157	1	1. 420
	2	1. 317
	3	V. W.
	4	B.
	5	2. 124
	6	2. 708
158	7	W.
	8	2. 19
	9	1. 653
	10	2. 585
	11	1. 719
	12	1. 524
	13	1. 803
	1	1. 248
159	2	1. 275
	3	2. 285
	4	W.
	5	2. 626
	6	1. 308
	7	V.
	8	2. 488
	9	2. 437
	10	1. 683
160	11	1. 468

p 160	*n* 12	*N* 2. 139
	13	B.
	14	2. 551
	1	1. 63
	2	1. 166
	3	1. 106
	4	1. 167
	5	1. 421
161	6	W.
	7	2. 343
	8	2. 395
	9	2. 664
	10	2. 799
	11	1. 846
	12	1. 526
	13	1. 720
	14	2. 437
162	15	2. 439
	16	N.
	17	B.
	18	W.
	1	1. 276
	2	Vince, 4to.
	3	2. 628
163	4	W. N.
	5	2. 841
	6	1. 804
	7	W.
	8	Cotes.
	9	2. 489
	10	2. 868 V. Flux.
	11	1. 527
164	12	B. Whewell
	13	V.
	14	1. 666
	15	2. 216
	16	2. 665
	17	1. 667
165	18	1. 721
	19	2. 624
	20	1. 356
	21	W. N.
	22	2. 490
	23	1. 341
	1	1. 168
166	2	W.

CORRECTIONS.

VOL. I.

Page 24, line 11, dele "*evidently and.*"

72, 8, *for* $\begin{matrix} x^n \\ (2a)^n \end{matrix}$ *read* $\frac{x^n}{(2a)^n}$

76, art. 195. This article is wrong. The true solution is,

The number of combinations of the letters in the word *Baccalaureus* is evidently the same as that of the divisors in the product

$$B \times a^3 \times c^2 \times l \times u^2 \times r \times e \times s,$$

or it is (*Barlow's Theory of Numbers,*)

$$(1+1)\times(3+1)\times(2+1)\times(1+1)\times(2+1)\times(1+1)\times(1+1)$$
$$\times(1+1)-5 \text{ or } 1147.$$

Page 121 line 15 *for* is transform *read* is to transform.

168, 1, *for* $\sqrt[2]{1}$. *read* $\sqrt[n]{1}$.

3, *for* x^2 *read* x^n.

169, 5, *for* sign *read* sine.

179, Art. 364. A better solution of this problem is to be found in *Emerson's Increments.*

230, Dele the paragraph beginning with "This method would be found," &c.

274, line 5. The constants here inadvertently omitted, are thus supplied :—

$$\int dx = x + c_1$$

$$\int dx \int dx = \frac{x^2}{2} + c_1 x + c_2$$

$$\int dx \int dx \int dx = \frac{x^3}{2.3} + c_1 . \frac{x}{2} + c_2 x + c_3$$

$$\&c. = \&c.$$

and, generally,

$$\int dx \int dx \,\ldots\ldots\, n \text{ factors} = \frac{x^n}{2.3.4. \ldots . n} + c_1 . \frac{x^{n-1}}{2.3. \ldots . n-1}$$

$$+ c_2 . \frac{x^{n-2}}{2.3. \ldots . n-2} + \&c. + c_{n-1} x + c^n.$$

Page 336 line 13, *for* $\frac{2p-1}{u}$ *read* $\frac{2p-1}{n}$.

CORRECTIONS.

Page 381, last line, *for* $\sqrt{(^2 + \frac{3}{4})}$ *read* $\sqrt{(v^2 + \frac{3}{4})}$

412, omit art. 611. The enunciation is absurd, the two members of the equation being heterogeneous.

Page 416, line 8, *for* c' *read* C′.

497, 16, *for* 776 *read* 771.

500, 5, *for* x^{-} *read* $x^{-\frac{1}{2}}$.

534, 15, *for* 1080 *read* 2700.

569, 4, *for* 706 *read* 702.

574, 3, *for* 704 *read* 681.

607, 18, *for* 777 *read* 722.

CORRECTIONS.

VOL. II.

Page 211, last line, *for* $\frac{y}{z}$ *read* $\frac{y}{2}$.

232, line 15, *for* terms *read* times.

17, *for* m *read* n.

18, *for* m *read* n.

245, 15, *for* $\frac{g}{n}$ *read* $\frac{g}{2}$.

17, *for* $\sqrt{n-1}$ *read* $\sqrt{n}-1$.

329, 12, *for* $^{\frac{3}{2}}$ *read* $a^{\frac{3}{2}}$

363, 3 from bottom, *for* $^{n-1}$ *read* a^{n-1}.

last line, *for* $y^{n\ 1}$ *read* y^{n-1}.

388, 14, *for* 522 *read* 524.

389, 5, *for* $P = R \times \frac{2\sqrt{6}}{5}$ *read*

$\sqrt{R^2-P^2} = R\,.\frac{2\sqrt{6}}{5}$ and consequently

CORRECTIONS.

Page 426, line 15, *for* $v'^2 = \dfrac{2g\,P\,(l-\lambda)}{P' + P\,(l + l' - \lambda)^{-2} \times \text{ \&c.}}$ *read*

$$v'^2 = \frac{2g\,P\,(l-\lambda)}{P' + Pm^2}$$

436, 17, *for* $P \times a = M \times \dfrac{a}{n+1}$ *read*

$$Pa = M\,.\,\frac{na}{n+1}, \text{ and conseq.}$$

476, 2, *for* Laplaca *read* Laplace.

From 513 to page 532, for the running title HYDRODYNAMICS *read* DISCHARGE OF FLUIDS.

---

p xi (Index) *for*	590	*read*	9
	1		590
	2		1
	2		2

---

p 721 (Last Index)	*for*	58	1		*read*	53	1	Ap. to Lacroix.	
728	*for*	152	10		*read*	152	10	2	918
732	*for*	192	6	2	*read*	192	6	2	919
736	*for*	256	18	1	*read*	256	18	Whewell's Dyn.	
739	*for*	303	24	2	*read*	303	24	2	920
739	*for*	306	14	2	*read*	306	14	2	11
740	*for*	325	15	2	*read*	325	15	2	927
742	*for*	348	20	2	*read*	348	20	2	928
743	*for*	365	13	2	*read*	365	13	2	921
743	*for*	367	18	2	*read*	367	18	2	922
744	*for*	377	7	2	*read*	377	7	2	923
744	*for*	378	12	1	*read*	377	12	2	924
744	*for*	390	16	2	*read*	390	16	W. N.	
745	*for*	392	3	1	*read*	392	3	2	929
745	*for*	407	7	2	*read*	407	7	2	925
746	*for*	408	6	2	*read*	408	6	2	926
746	*for*	413	24	2	*read*	413	24	2	591
746	*for*	419	6	2	*read*	419	6	2	930
747	*for*	421	6	2	*read*	421	6	2	931
747	*for*	422	7	2	*read*	422	7	Whewell's Dyn.	

BOOKS

PUBLISHED BY

BLACK, YOUNG, AND YOUNG,

No. 2,

TAVISTOCK-STREET, COVENT GARDEN.

	£.	s.	d.
A History of the North American Indians, by James Buchanan, Esq., H. M. Consul at New York, 8vo.	0	10	6
Baretti's Dictionary of the English and Italian Languages; to which is prefixed, an Italian and English Grammar, founded on the Rules of Dr. Johnson and Buomattei, seventh edition, revised and augmented, 2 vols. 8vo.	1	1	0
Bibliographical Register; or, a Catalogue of English Books, published since the year 1800, systematically arranged with their sizes, prices, and dates, in 8vo., in the Press			
De Lolme (J. L.) on the Constitution of England, 8vo., *boards, new edition*	0	8	0
Demosthenis Opera Omnia Græce et Latine. A new Edition of the complete Works of Demosthenes, comprising the Notes of Taylor, Reiske, Wolff, and other Commentators, together with the Apparatus Criticus of Reiske. A new Text after Bekker, Harless, Rüdiger, and Reiske, has been introduced, and the whole carefully revised and edited by Professor Schæfer of Leipsic.			
Vol. I. to IV. containing the Text, Variæ Lectiones, Indices, and the first volume of the Apparatus Criticus, price to Subscribers	2	5	6
———, *fine paper*	3	0	0
———, *vellum paper*	4	11	0
The fifth volume will appear in April.			
The first three volumes, containing the Text, Indices, and Variæ Lectiones, may be purchased separately, *Common paper*	1	16	0
Fine paper	2	8	0
Evans (T. E.) Constitutions of the Nations of Europe and North America, with a concise History of the Liberties and Political Institutions of Modern Europe, from the French of MM. P. A. Dufau, J. B. Duvergier, and J. Guadet, 8vo. part I.	0	9	0
Part II.—The Netherlands	0	6	0
Jenks' (B.) Prayers and Offices of Devotion, by the Rev. Charles Simeon, 12mo. *bound*, 4s.—8vo. *boards*	0	6	0
Jolliffe's (Rev. T. R.) Letters from Palestine, descriptive of a Tour through Galilee and Judea, with some Account of the Dead Sea and of the present State of Jerusalem; to which are added, Letters from Egypt, 2 vols. 8vo. *boards, 3rd edition*	1	1	0
——— Phædo, a Dialogue on the Immortality of the Soul, translated from the Greek of Plato, 8vo	0	10	6
Jones' (Sir W.) Grammar of the Persian Language; *the 8th edition*, with considerable Additions and Improvements, by the Rev. S. Lee, M.A., D.D. 4to. *boards*	1	1	0
Morell's Studies in History, containing the History of England from the period of the Ancient Britons to the Death of George III., in a Series of Essays, accompanied with Reflections, References to Original Authorities, and Historical Questions, 2 vols. 8vo. *boards*	1	4	0

Books published by Black, Young, and Young.

	£.	s.	d.
Morell's History of England, *for the use of Schools*, 2 vols. 12mo. *bds. 4th edition*	0	11	0
———— Rome, from its Earliest Records to the Death of Constantine; *with a Map of the Roman Empire*, 8vo. *boards*	0	10	6
————, *for the use of Schools, with the Map, boards, 5th edition*, 12mo	0	5	0
———— Greece, *with a Map of Ancient Greece, bds.* 8vo.	0	10	6
———— *for the use of Schools, boards, 5th edition*, 12mo.	0	5	0
Morellii Thesaurus Græco Prosodiacum, demy 4to, *a new edition, by Dr. Maltby*	3	3	0
Nicholas's Narrative of a Voyage to New Zealand, in the Years 1814-15, in company with the Rev. Samuel Marsden, 2 vols. 8vo.	1	4	0
Newton on the Prophecies, 2 vols. 8vo. *new edition*	0	16	0
Pestalozzi's Grammar, or the Mother's Book; exemplifying Pestalozzi's Plan of Awakening the Understanding of Children in Language, Drawing, Geometry, and Numbers, with a complete English Grammar, and Guide to Parsing, by P. H. Pullen, 12mo. *boards, 2nd edition*	0	6	0
———— Intellectual or Intuitive Arithmetic, exemplifying the Doctrine of Mental Calculation by means of Tables, *with three large plates*, by P. H. Pullen, 12mo. *boards*	0	7	0
———— Table, exemplifying the properties of Numbers, *on a sheet*	0	3	0
———— *on a roller*	0	5	6
———— System of Practical Geography, Sacred, Ancient, and Modern: with Rules for the Construction of Maps, Measuring Distances, &c., to which is added an Index, containing names of about 3000 Places recorded in the Bible, and in Greek and Latin Authors, by P. H. Pullen, 12mo. *boards*	0	6	0
Pursh's Flora Americana; or, a Systematic Arrangement and Description of the Plants of North America, containing many new rare Species, 2 vols. 8vo.	1	16	4
———— *with the plates coloured*	2	12	6
Pignotti's History of Tuscany, from the earliest period to the Establishment of the Grand Duchy, with occasional Essays on the Progress of Italian Literature, translated from the Italian by J. Browning, Esq. 4 vols. 8vo. *boards*	2	8	0
Robertson's Grammar of the Modern Greek Language, to which are added Phrases and Dialogues on the most important subjects, with Extracts from Romaic Authors, 12mo. *boards*	0	4	6
Sappho, a Tragedy, translated from the German of Grillparzer, by John Bramsen, 8vo. *sewed*	0	4	6
Shakspeare's Dramatic Works, printed from the text of Johnson and Stevens, *in 1 pocket volume, with a portrait, boards, new edition*	0	15	0
Smith's (Dr. J. Pye) Manual of Latin Grammar, *for Schools and Private Study*, 12mo.	0	2	6
———— (Mrs.) Abridgment of the Prophecies as connected with Profane History, both Ancient and Modern, from the best Authors, *2nd edition*, 12mo.	0	5	0
Taylor's Aristotle.—The Rhetoric, Poetic, and Nicomachean Ethics of Aristotle, translated from the Greek, by Thomas Taylor, 2 vols. 8vo.	1	1	0
———— Translation of Jamblichus, 8vo.	0	12	0
———— Select Works of Plotinus, 8vo.	0	18	0
———— Systematic Arithmetic, shewing the Manner in which the Ancients reasoned on Numbers, 8vo.	0	10	0
Taylor's Mystical Hymns of Orpheus, 8vo. *second edition*	0	10	6
———— Political Fragments of the Ancient Pythagoreans, 8vo.	0	6	0
Tiberius Rhetor de Figuris Altera Parto Auctior, una cum Rufi Arte Rhetorica, edidit J. F. Boissonade, 8vo. *boards*	0	6	0

Fig. 1.

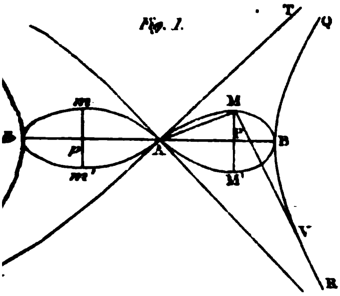
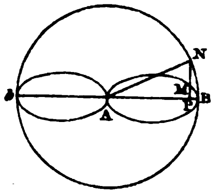


Fig. 2.



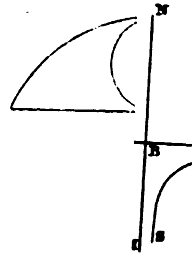


Fig. 7.

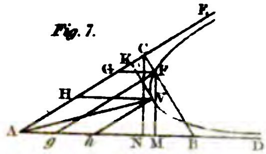


Fig. 6.

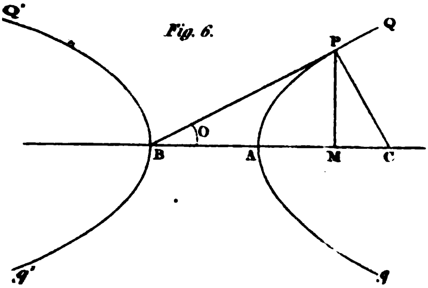


Fig. 11.

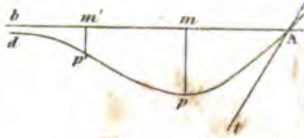


Fig. 10 A.

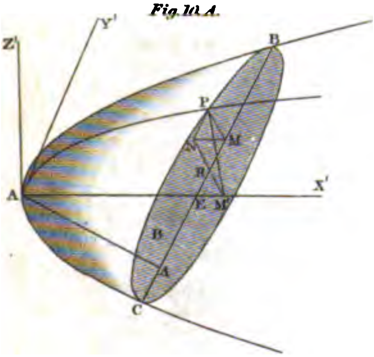


Fig. 13.

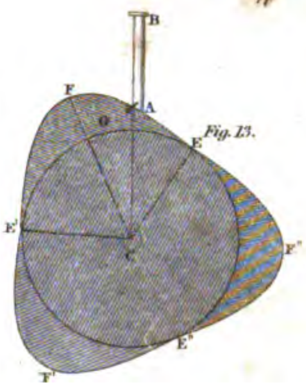


Fig. 16

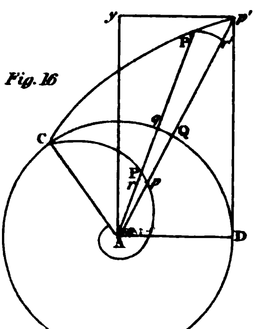


Fig. 17.

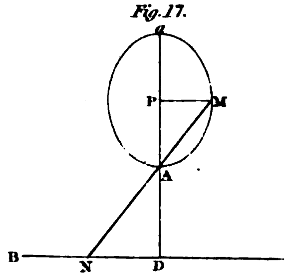


Fig. 20.

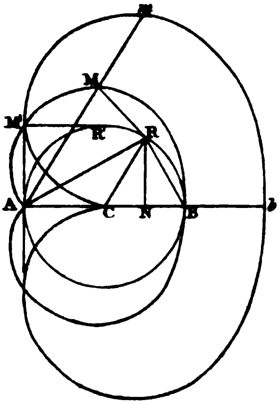


Fig. 21.

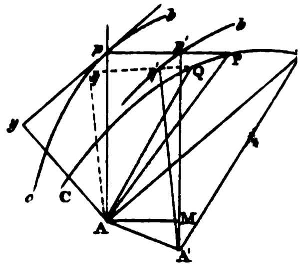


Fig. 24.

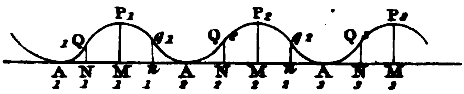


Fig.

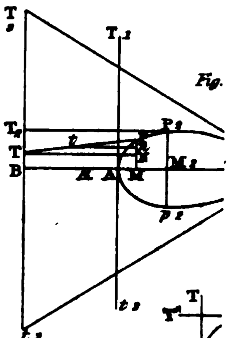
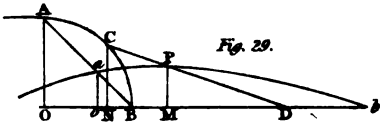


Fig. 29.



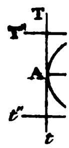


Fig. 34.

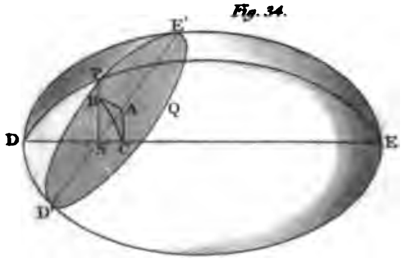


Fig. 34 a.

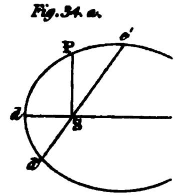
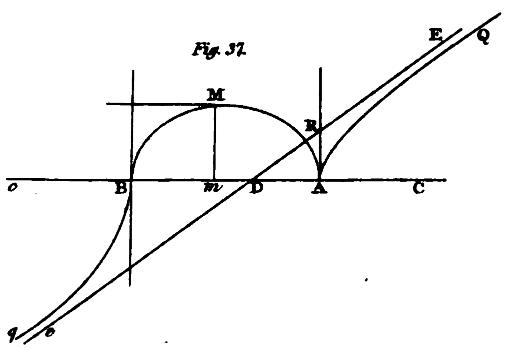


Fig. 37.



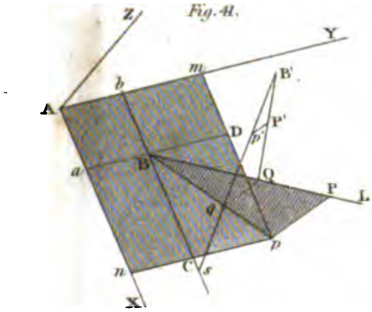


Fig. 41.

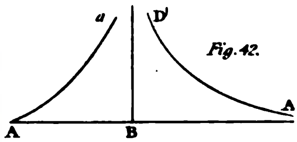


Fig. 42.

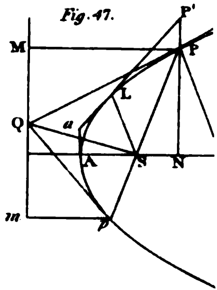


Fig. 47.

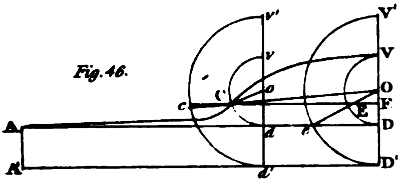


Fig. 46.

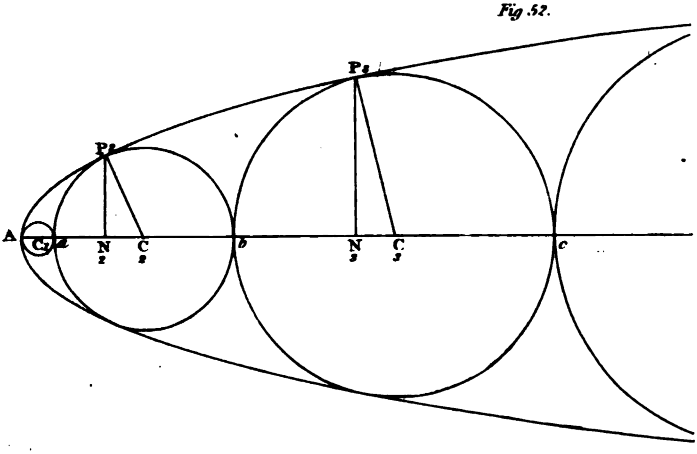


Fig. 52.

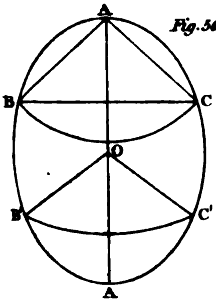


Fig. 56.

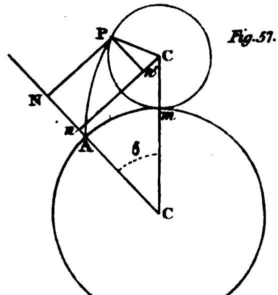


Fig. 57.

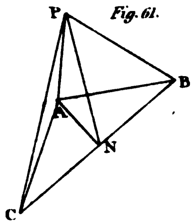


Fig. 61.

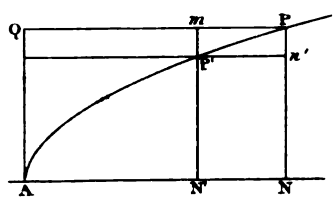


Fig. 62.

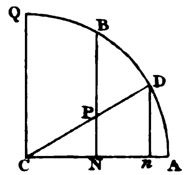


Fig. 66.

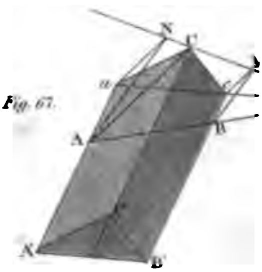


Fig. 67.

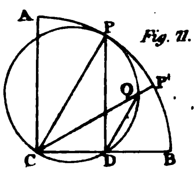


Fig. 71.

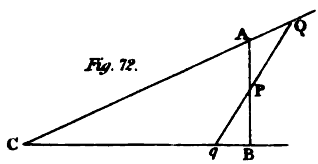


Fig. 72.

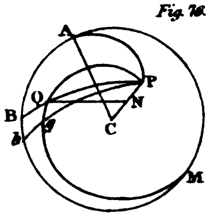


Fig. 76.

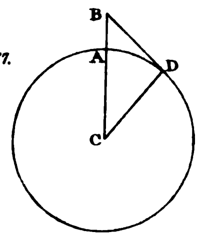


Fig. 77.

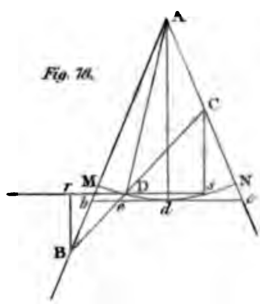


Fig. 78.

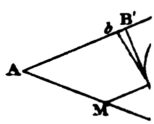
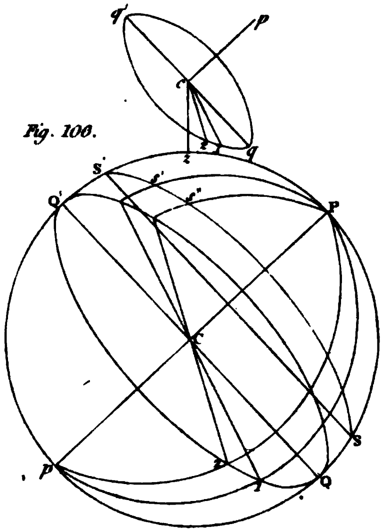


Fig. 106.



ig. 105.

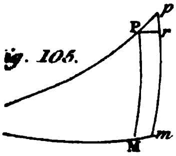
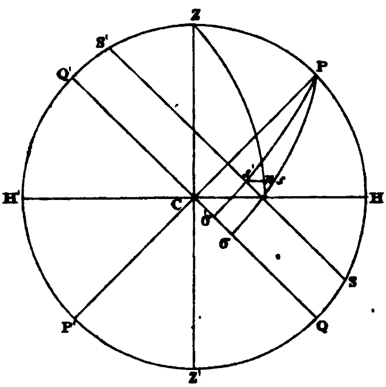


Fig. 110.



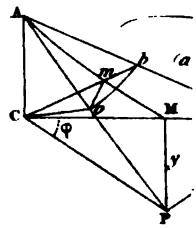


Fig. 114.